A NOTE ON THE PERTURBATION THEORY FOR DISTORTED STELLAR CONFIGURATIONS

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ABSTRACT

In this paper we relate the general perturbation theory developed in the preceding paper with the earlier general discussion of distorted equilibrum configurations.

1. In the preceding paper[1] one of us has developed a general theory of perturbations for describing stellar configurations distorted by tidal and (or) centrifugal forces. The general method consists of simply expressing the changes in the physical parameters caused by the perturbing forces in terms of the density and pressure distributions in an undistorted configuration with the same central density. Thus, considering the purely tidal problem, for example, it is found that the pressure distribution can be expressed in the form

$$\eta = \eta_0 + \chi_0'' \sum_{j=z}^{4} a_{1,j}\, \eta_{1,j}^{*} P_j(\mu) \ , \tag{1}$$

where η denotes the pressure (expressed in units of the central pressure) and the $\eta_{1,j}^{*}$'s are solutions of the differential equations (cf. *op. cit.*, eq. [140])

$$\frac{1}{\xi^2}\left\{\frac{d}{d\xi}\left(\xi^2 \frac{d}{d\xi}\left[\frac{\eta_{1,j}^{*}}{\zeta_0}\right]\right) - j(j+1)\frac{\eta_{1,j}^{*}}{\zeta_0}\right\} = -\eta_{1,j}^{*}\,\frac{\dfrac{d\zeta_0}{d\xi}}{\dfrac{d\eta_0}{d\xi}} \qquad (j=2,3,4) \tag{2}$$

together with the boundary conditions

$$\eta_{1,j}^{*} = \xi^j + O(\xi^{j+2}) \qquad (\xi \to 0) . \tag{3}$$

Further, in equations (1) and (2) η_0 and ζ_0 correspond to solutions for the pressure and density distributions in the corresponding undisturbed configuration, i.e., they satisfy the differential equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\frac{\xi^2}{\zeta_0}\frac{d\eta_0}{d\zeta}\right) = -\zeta_0 \, . \tag{4}$$

The $a_{1,j}$'s in equation (1) are certain numbers defined in terms of the values which $\eta_{1,j}^{*}$ and $d/d\xi(\eta_{1,j}^{*}/\zeta_0)$ take at the boundary of the configuration.[2] Finally,

$$\chi_0'' = \frac{M''}{4\pi \rho_c a^2 R} , \tag{5}$$

where M'' denotes the mass of the secondary, R the distance between the centers of gravity of the two stars, ρ_c the central density of the primary, and a the chosen unit of distance. The solutions for the rotational and the combined rotational and tidal problems take similar forms.

[1] W. Krogdahl, *Ap. J.*, **96**, 124, 1942.

[2] Cf. *op. cit.*, eq. (135).

2. Now there exists a general theory of distorted equilibrium configurations in which the emphasis is on the variation in the forms of the surfaces of constant pressure (or density) through the star.[3] On this theory the isobaric surfaces are written in the form

$$\xi = \bar{\xi} \left\{ 1 + \sum_{j=2}^{4} Y_j(\xi) P_j(\mu) \right\}, \tag{6}$$

where $\bar{\xi}$ is the mean value of ξ for a given value of the pressure; and it is shown that the functions S_j ($j = 2, 3, 4$), defined as

$$S_j = \xi \frac{d \log Y_j}{d\xi}, \tag{7}$$

are solutions of the first-order equation

$$\xi \frac{dS_j}{d\xi} + S_j^2 - S_j - j(j+1) + \frac{6\rho}{\bar{\rho}}(S_j + 1) = 0, \tag{8}$$

with the boundary conditions

$$S_j = j - 2 \qquad (\xi = 0). \tag{9}$$

In equation (8) $\bar{\rho}$ is the mean density interior to ξ.

While the objectives of this theory are more limited than those of the general perturbation theory of the preceding paper, it is clear that equation (8) must be a simple mathematical consequence of equation (2). In this note we shall show that this is actually the case and thus establish the formal equivalence of the two theories.

3. From equation (1) it readily follows that the equation of the isobaric surfaces can be written in the form

$$\xi(\zeta_0) = \bar{\xi}(\zeta_0) \left[1 - \chi_0'' \sum_{j=2}^{4} a_{1,j} \frac{\overset{*}{\eta}_{1,j}}{\xi \frac{d\eta_0}{d\xi}} P_j(\mu) \right]. \tag{10}$$

Comparing equations (6) and (10), we conclude that

$$Y_j = -\chi_0'' a_{1,j} \frac{\overset{*}{\eta}_{1,j}}{\xi \frac{d\eta_0}{d\xi}}. \tag{11}$$

Hence,

$$S_j = \xi \frac{d \log Y_j}{d\xi} = -1 + \xi \frac{\overset{*}{\eta}{}'_{1,j}}{\overset{*}{\eta}_{1,j}} - \xi \frac{\eta_0''}{\eta_0'}, \tag{12}$$

where we have used primes to denote differentiation with respect to ξ.

We have to show that S_j satisfies the differential equation (8) in virtue of its definition and the differential equation (2) which $\overset{*}{\eta}_{1,j}$ satisfies. To show this, we shall first transform equation (2) by introducing the variable

$$\psi_j = \xi \frac{\overset{*}{\eta}_{1,j}}{\zeta_0}, \tag{13}$$

[3] H. Jeffreys, *The Earth*, chap. xiii, Cambridge, England, 1929; T. E. Sterne, *M.N.*, **99**, 451, 1939.

whence equation (2) becomes

$$\frac{\psi_j''}{\psi_j}-\frac{j(j+1)}{\xi^2}+\zeta_0\frac{\zeta_0'}{\eta_0'}=0\,. \tag{14}$$

Now, let

$$\varphi_j=\xi\,\frac{\psi_j'}{\psi_j}\,. \tag{15}$$

Then

$$\frac{1}{\psi_j}\frac{d}{d\xi}\left(\frac{\varphi_j\psi_j}{\xi}\right)-\frac{j(j+1)}{\xi^2}+\zeta_0\frac{\zeta_0'}{\eta_0'}=0\,; \tag{16}$$

or, since

$$\frac{d}{d\xi}\left(\frac{\varphi_j\psi_j}{\xi}\right)=-\frac{\varphi_j\psi_j}{\xi^2}+\frac{\psi_j}{\xi}\varphi_j'+\frac{\varphi_j}{\xi}\psi_j'\,, \tag{17}$$

equation (16) becomes, after some further reductions,

$$\xi\frac{d\varphi_j}{d\xi}+\varphi_j^2-\varphi_j-j(j+1)+\zeta_0\xi^2\frac{\zeta_0'}{\eta_0'}=0\,. \tag{18}$$

4. We shall now express S_j in terms of φ_j. According to equations (13) and (15),

$$\varphi_j=\xi\,\frac{d\log\psi_j}{d\xi}=1+\xi\,\frac{\eta_{1,j}^{*\prime}}{\eta_{1,j}^{*}}-\xi\,\frac{\zeta_0'}{\zeta_0}\,. \tag{19}$$

Hence, combining equations (12) and (19), we have

$$\varphi_j=2+\xi\left(\frac{\eta_0''}{\eta_0'}-\frac{\zeta_0'}{\zeta_0}\right)+S_j\,. \tag{20}$$

We can eliminate η_0'' from the foregoing equation by using equation (4); according to this equation,

$$2\xi\,\frac{\eta_0'}{\zeta_0}+\xi^2\,\frac{\eta_0''}{\zeta_0}-\xi^2\,\frac{\eta_0'\zeta_0'}{\zeta_0^2}=-\xi^2\zeta_0\,, \tag{21}$$

or

$$\xi\left(\frac{\eta_0''}{\eta_0'}-\frac{\zeta_0'}{\zeta_0}\right)=-2-\xi\,\frac{\zeta_0^2}{\eta_0'}\,. \tag{22}$$

We can therefore re-write equation (20) as

$$\varphi_j=S_j-\xi\,\frac{\zeta_0^2}{\eta_0'}\,. \tag{23}$$

We can express the foregoing relation somewhat differently, using the following relation between the actual and the mean densities:

$$\frac{\rho(\xi)}{\bar{\rho}(\xi)}=-\frac{1}{3}\,\xi\,\frac{\zeta_0^2}{\eta_0'}\,. \tag{24}$$

Thus,

$$\varphi_j = S_j + 3\,\frac{\rho(\xi)}{\bar{\rho}(\xi)}\,. \tag{25}$$

We now substitute the foregoing relation in equation (18). We find

$$\xi\,\frac{d}{d\xi}\left(S_j - \xi\,\frac{\zeta_0^2}{\eta_0'}\right) + \left(S_j + 3\,\frac{\rho}{\bar{\rho}}\right)^2 - S_j - 3\,\frac{\rho}{\bar{\rho}} - j(j+1) + \zeta_0\xi^2\,\frac{\zeta_0'}{\eta_0'} = 0 \tag{26}$$

or, after some reductions,

$$\begin{aligned} &\xi\,\frac{dS_j}{d\xi} + S_j^2 - S_j - j(j+1) + \frac{6\rho}{\bar{\rho}}\,(S_j+1) \\ &\qquad = \xi\,\frac{d}{d\xi}\left(\xi\,\frac{\zeta_0^2}{\eta_0'}\right) + 9\,\frac{\rho}{\bar{\rho}} - 9\left(\frac{\rho}{\bar{\rho}}\right)^2 - \zeta_0\xi^2\,\frac{\zeta_0'}{\eta_0'}\,. \end{aligned} \tag{27}$$

Expanding the right-hand side of equation (27) and using the relation (24) we readily obtain

$$\begin{aligned} &\xi\,\frac{\zeta_0^2}{\eta_0'} + 2\xi^2\,\frac{\zeta_0\zeta_0'}{\eta_0'} - \xi^2\,\frac{\zeta_0^2}{\eta_0'^2}\,\eta_0'' - 3\xi\,\frac{\zeta_0^2}{\eta_0'} - \xi^2\,\frac{\zeta_0^4}{\eta_0'^2} - \xi^2\,\frac{\zeta_0'\zeta_0}{\eta_0'} \\ &\qquad = -2\xi\,\frac{\zeta_0^2}{\eta_0'} - \xi^2\,\frac{\zeta_0^4}{\eta_0'^2} - \xi^2\,\frac{\zeta_0^2}{\eta_0'}\left(\frac{\eta_0''}{\eta_0'} - \frac{\zeta_0'}{\zeta_0}\right), \end{aligned} \tag{28}$$

which vanishes identically in virtue of equation (22). Hence,

$$\xi\,\frac{dS_j}{d\xi} + S_j^2 - S_j - j(j+1) + 6\,\frac{\rho}{\bar{\rho}}(S_j+1) = 0\,. \tag{29}$$

Finally, we verify that, according to equations (3), (13), (15), and (23),

$$S_j = j - 2 \qquad \text{at} \qquad \xi = 0\,. \tag{30}$$

This proves the formal equivalence of the two perturbation theories now available.

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