STELLAR MODELS WITH ISOTHERMAL CORES

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ABSTRACT

This paper is devoted to the study of stellar models with isothermal cores. Two types of such configurations have been studied: (1) models with isothermal cores and polytropic envelopes ($n = 3$) and (2) models with isothermal cores and radiative point-source envelopes with a law of opacity $\kappa = \kappa_0 \rho T^{-3.5}$. The most important characteristic of these models is the existence of an upper limit to the fraction ν of the total mass which can be contained in the core. For models of type 2, $\nu_{\max} \sim 35\%$. Also it appears that, as ν increases, the radius of the star first decreases to a minimum value and then increases. Further, the luminosity of the star is found to increase by about a factor 3 from the stage when it has no isothermal core to the stage when the core contains the maximum possible mass.

1. *Introduction.*—The possible physical importance of stellar models with isothermal cores was first indicated by Gamow[1] who suggested that these models may have their counterparts in nature if resonance penetration of charged particles into nuclei should become the main source of energy. For under such circumstances the energy may be thought of as being generated in a spherical shell, in which case the regions interior to the shell would be isothermal. Again, if, following Gamow and Teller,[2] we suppose that the proton disintegration of the light-nuclei (D, Li, Be, and B) provides the energy source for the giants, it is conceivable that the available element at a particular time, lithium say, becomes exhausted in the central regions; also the physical conditions may be such that, before the temperature rises sufficiently for the disintegration of the next element, beryllium, to become effective, a situation may arise when the disintegration of lithium in the outer parts becomes the primary source of energy. Under these circumstances, also, the stellar configurations will have isothermal cores. A situation similar to what we have described may prevail quite generally with the exhaustion of hydrogen in the central regions of stars, during the course of their normal evolution. A study of the physical characteristics of stellar models with isothermal cores becomes, therefore, a matter of some interest. A first attempt in this direction has already been made by Critchfield and Gamow.[3] But the essential peculiarities of the model arising from the isothermal nature of the core has been overlooked by these authors.[4] In this paper we therefore propose to study these models under varying conditions to elucidate their physical characteristics.

2. *The equations of the isothermal core.*—We shall consider first the equilibrium of the isothermal core. In the core we can write

$$P = K_2 \rho + D \,, \tag{1}$$

where

$$K_2 = \frac{k}{\mu H} T_c \,; \qquad D = \tfrac{1}{3} a T_c^4 \,, \tag{2}$$

[1] *Ap. J.*, **87**, 206, 1938; *Phys. Rev.*, **53**, 595, 1938.

[2] *Phys. Rev.*, **53**, 608, 1938.

[3] *Ap. J.*, **89**, 244, 1939.

[4] Thus Critchfield and Gamow assume series expansions for $M(r)$, P, etc., which are valid in the immediate neighborhood of the center, no matter what the equation of state is (cf. eqs. [7] and [8] in the paper referred to in n. 3). Further, it appears that the parts of the isothermal function which are necessary to describe the core cannot be satisfactorily expressed by any kind of series expansion (see §§ 3, 4, and 5 in the present paper).

where K_2 and D are constants. The reduction to the isothermal equation is made by the substitutions[5]

$$\rho = \lambda_2 e^{-\psi} ; \qquad P = K_2 \lambda_2 e^{-\psi} + D , \tag{3}$$

and

$$r = \left(\frac{K_2}{4\pi G \lambda_2} \right)^{1/2} \xi . \tag{4}$$

Further, we have the mass relation

$$M(\xi) = 4\pi \left(\frac{K_2}{4\pi G} \right)^{3/2} \lambda_2^{-1/2} \xi^2 \frac{d\psi}{d\xi} . \tag{5}$$

3. *Stellar models with isothermal cores and polytropic* ($n = 3$) *envelopes.*—As a first example of stellar models with isothermal cores we shall consider the case where the structure of the envelope is governed by the isothermal equation of index $n = 3$. Physically, this assumption implies that in the envelope we have the standard model approximation "$\kappa\eta$ = constant." Under these circumstances we can write

$$P = \left[\left(\frac{k}{\mu H} \right)^4 \frac{3}{a} \frac{1 - \beta}{\beta^4} \right]^{1/3} \rho^{4/3} = K_1 \rho^{4/3} , \tag{6}$$

where K_1 is a constant. The reduction to the polytropic equation is made by the substitutions

$$\left. \begin{aligned} \rho = \lambda_1 \theta^3 ; \qquad P = K_1 \lambda_1^{4/3} \theta^4 ; \\ r = \left(\frac{K_1}{\pi G} \right)^{1/2} \lambda_1^{-1/3} \eta . \end{aligned} \right\} \tag{7}$$

Also, we have the relation

$$M(\eta) = -4\pi \left(\frac{K_1}{\pi G} \right)^{3/2} \eta^2 \frac{d\theta}{d\eta} . \tag{8}$$

The mass M of the whole configuration is given by

$$M = 4\pi \left(\frac{K_1}{\pi G} \right)^{3/2} \omega_3 , \tag{9}$$

where

$$\omega_3 = -\left(\eta^2 \frac{d\theta}{d\eta} \right)_1 , \tag{10}$$

the subscript 1 indicating that the quantity in parenthesis is evaluated at the point where θ has its zero. It may be noted that ω_3 is a homology-invariant constant.[6]

Now, at the interface where the isothermal core joins the polytropic envelope the values of P, ρ, r, and $M(r)$, given by the two sets of formulae (3), (4) and (5), and (7)

[5] See S. Chandrasekhar, *An Introduction to the Study of Stellar Structure*, p. 155, Chicago, 1939.

[6] See *ibid.*, p. 149.

and (8) should be identical. The resulting four equations of fit can be reduced to two equations involving only the homology-invariant combinations

$$u_\infty = \frac{\xi e^{-\psi}}{\psi'}\,; \qquad v_\infty = \xi\,\frac{d\psi}{d\xi}\,; \tag{11}$$

and

$$u_3 = -\frac{\eta\theta^3}{\theta'}\,; \qquad v_3 = -\frac{\eta\theta'}{\theta}\,. \tag{12}$$

We find[7]

$$\left.\begin{aligned} u_\infty(\xi_i) &= u_3(\eta_i)\,, \\ \tfrac{1}{4}\beta v_\infty(\xi_i) &= v_3(\eta_i)\,, \end{aligned}\right\} \tag{13}$$

where the subscript i denotes that the respective quantities are evaluated at the interface. According to equations (13) every intersection of a (u_3, v_3)-curve with the $(u_\infty, \frac{1}{4}\beta v_\infty)$-curve derived from the complete isothermal function gives a solution of the equations of fit and corresponds to a definite configuration of the type we are looking for. An examination of the general arrangement of the (u, v)-curves for $n = 3$ and $n = \infty$ readily shows that solutions for equations (13) exist only for (u_3, v_3)-curves derived from M-solutions;[8] however, it may be noted that not all M-solutions provide solutions to equations (13).

According to the views expressed in § 1, in considering stellar models with isothermal cores we are primarily interested in the changes which occur in the parameters describing a star, as the isothermal core at some fixed temperature "grows" at the expense of the envelope. We shall now obtain the relations necessary for this purpose.

Suppose that a (u_3, v_3)-curve labeled by a certain value for the homology-invariant constant ω_3 intersects the $(u_\infty, \frac{1}{4}\beta v_\infty)$-curve derived from an E-solution of the isothermal equation at a point where $\xi = \xi_i$ and $\eta = \eta_i$. At this point the equations of fit (13) are therefore satisfied. The fraction q of the radius R occupied by the core is clearly given by

$$q = \frac{\eta_i}{\eta_1}\,, \tag{14}$$

where $\eta = \eta_1$ defines the boundary of the particular solution $\theta(\eta, \omega_3)$. The fraction ν of the mass M contained in the core is also readily found. We have

$$\nu = \frac{M(\eta_i)}{M(\eta_1)} = -\frac{(\eta^2\theta')_i}{\omega_3}\,, \tag{15}$$

where the subscript i indicates that the quantity in parenthesis is evaluated at the interface. Using the definitions of u_3 and v_3 we can re-write equation (15) more conveniently as

$$\nu = \frac{(u_3 v_3^3)_i^{1/2}}{\omega_3}\,. \tag{16}$$

[7] See *ibid.*, pp. 170–76.

[8] For the classification of the solutions of the Lane-Emden equation see *ibid.*, chap. iv.

The ratio of the central to the mean density is given by[9]

$$\frac{\rho_c}{\bar{\rho}} = \frac{\lambda_2}{\left(\frac{M}{\frac{4}{3}\pi R^3}\right)} = -\frac{\lambda_2}{3\lambda_1\left(\frac{1}{\eta}\frac{d\theta}{d\eta}\right)_1}\;; \tag{17}$$

or, using the formula

$$\lambda_2 e^{-\psi_i} = \lambda_1 \theta_i^3\,, \tag{18}$$

which expresses the equality of the density at the interface, we have

$$\frac{\rho_c}{\bar{\rho}} = -\frac{\theta_i^3 e^{\psi_i}}{3\left(\frac{1}{\eta}\frac{d\theta}{d\eta}\right)_1} = -\frac{\theta_i^3\eta_i^3 e^{\psi_i}}{3\left(\eta^2\frac{d\theta}{d\eta}\right)_1\left(\frac{\eta_i}{\eta_1}\right)^3}\,. \tag{19}$$

Hence,

$$\frac{\rho_c}{\bar{\rho}} = \frac{(u_3 v_3)_i^{3/2} e^{\psi_i}}{3\omega_3 q^3}\,. \tag{20}$$

Finally, to determine the $R(q)$ relation for a given mass and T_c, we start from the relation

$$K_2 = \beta K_1 \lambda_1^{1/3}\theta_i\,, \tag{21}$$

expressing the equality of p_{gas}/ρ on the two sides of the interface, and eliminate λ_1 from the equation (cf. eq. [7])

$$R = \left(\frac{K_1}{\pi G}\right)^{1/2}\lambda_1^{-1/3}\eta_1\,. \tag{22}$$

We obtain

$$R = \pi G\beta\left(\frac{K_1}{\pi G}\right)^{3/2}\frac{1}{K_2}\left(\frac{\eta_1}{\eta_i}\right)\eta_i\theta_i\,, \tag{23}$$

or, using equations (2), (9), and (12), we have

$$R = Q(q)\beta\,\frac{\mu H}{k}\,\frac{GM}{T_c}\,, \tag{24}$$

where we have written

$$Q(q) = \frac{(u_3 v_3)_i^{1/2}}{4\omega_3 q}\,. \tag{25}$$

Equation (24) is an important relation which determines the dependence of R on q for a configuration of a given mass and fixed central temperature.

Restricting ourselves to the most important case of negligible radiation pressure and putting $\beta = 1$, four solutions of the equations of fit (13) were obtained, using the two

[9] In writing these equations we have assumed that the particular solution of the isothermal equation used is the one for which $\psi = 0$ at $\xi = 0$, i.e., the solution commonly denoted by $\Psi(\xi)$ (see *ibid.*, p. 156).

M-solutions ($\omega_3 = 1.90$ and 1.50) integrated by Fairclough.[10] The results of the fitting are summarized in Table 1. Further, in Figure 1 we have illustrated the ($M(\text{core})/M, q$) and the (R, q) relations. We shall return to the physical meanings to be attached to these relationships in § 5.

4. *Stellar models with isothermal cores and point-source envelopes with the law of opacity* $\kappa = \kappa_0 \rho T^{-3.5}$.—The standard model approximation for the envelopes which we have considered in § 3, while giving an insight into the general behavior of these models, is not in strict conformity with the physical circumstances under which we might expect isothermal cores. For, consistent with the views expressed in § 1, we should rather suppose that the energy is generated in a thin spherical shell (of thickness Δr_i, say) at the interface between the isothermal core and the outer envelope. Under these circumstances the luminosity of the star will be given by

TABLE 1

STELLAR MODELS WITH ISOTHERMAL CORES AND $n=3$ ENVELOPES

ω_3	q	ν	$\rho_c/\bar{\rho}$	$Q(q)$
2.018	0	0	54.2	0.854
1.90	0.151	0.180	101	0.767
1.50	.148	.349	708	0.954
1.50	.094	.209	6.7×10^5	1.207
1.50	.104	.235	2.9×10^6	1.158
1.42(?)*	0.100	0.250	∞	1.250

* The figures in this row are not reliable. They give very rough estimates of the points about which the respective curves spiral.

$$L = 4\pi r_i^2 \rho_i \Delta r_i \epsilon_0 \,, \tag{26}$$

where ϵ_0 denotes the rate of generation of energy per gram of the material. Accordingly, the regions of the star outside $r = r_i$ will be governed by the same equations as those for the point-source model. The equations of equilibrium for these regions are, therefore,

$$\frac{d}{dr}\left(\frac{k}{\mu H}\rho T + \tfrac{1}{3}aT^4\right) = -\frac{GM(r)}{r^2}\rho \tag{27}$$

and

$$\frac{d}{dr}\left(\tfrac{1}{3}aT^4\right) = -\frac{\kappa_0 L}{4\pi c r^2}\frac{\rho^2}{T^{3.5}}\,, \tag{28}$$ [11]

where we have assumed for the coefficient of opacity the law

$$\kappa = \kappa_0 \frac{\rho}{T^{3.5}}\,. \tag{29}$$

[10] *M.N.*, **93**, 40, 1932.

[11] It is conceivable that circumstances may arise which require the replacement of equation (28), valid under conditions of radiative equilibrium, by another equation, valid under conditions of convective equilibrium. However, in the models we shall be primarily concerned with, this is not of much significance. Actually, apart from one possible exception, in the models considered the conditions for the validity of radiative equilibrium are not violated.

In equation (29), κ_0 (which is a constant throughout the configuration) may depend on the chemical composition (in particular on the hydrogen and helium abundances).

We shall now consider the method of fitting an isothermal core to a solution of equations (27) and (28): At the interface the quantities ρ, P, $M(r)$, and r as known along a solution of equations (27) and (28) must join continuously with the respective quantities

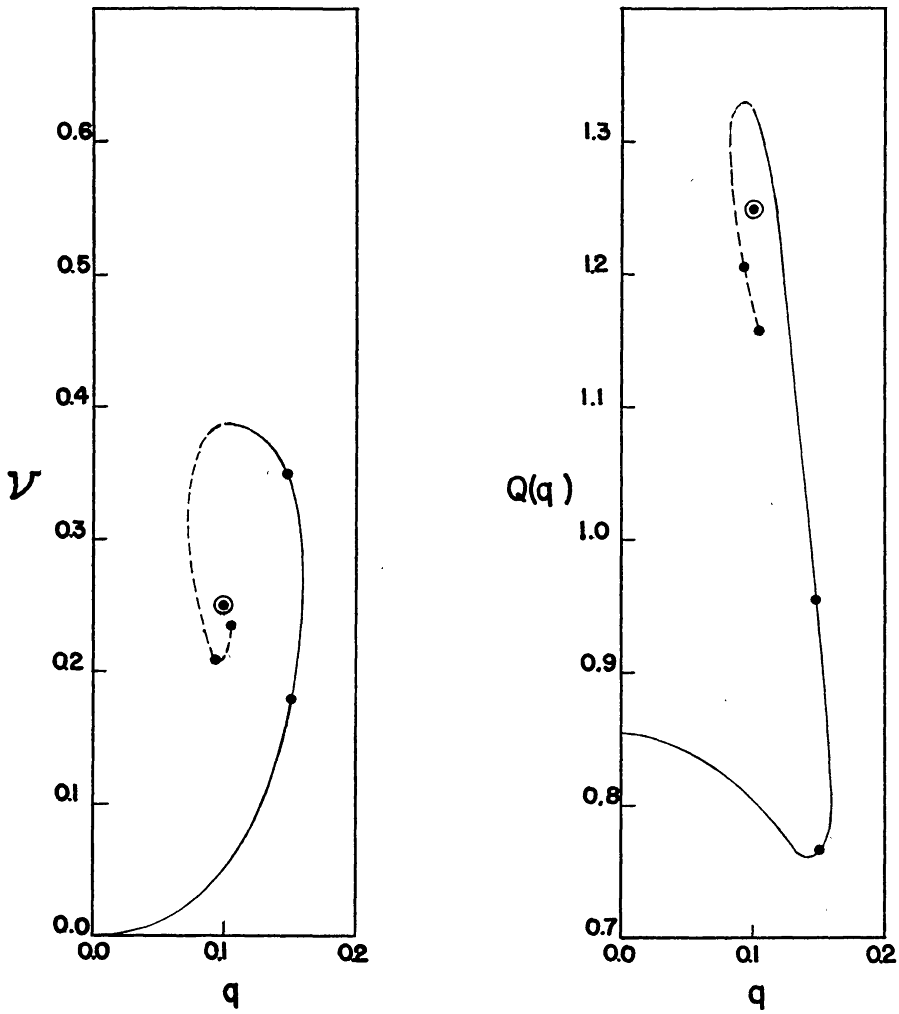

FIG. 1

determined by equations (3), (4), and (5) in terms of an appropriate E-solution of the isothermal equation.

According to equations (3), (4), and (5),

$$u_\infty = \frac{\xi e^{-\psi}}{\psi'} = 4\pi \frac{r^3 \rho(r)}{M(r)}, \tag{30}$$

and

$$v_\infty = \xi\psi' = \frac{G}{\beta(r)} \frac{\rho M(r)}{rP(r)} = \frac{\mu H}{k} \frac{GM(r)}{rT(r)}. \tag{31}$$

Hence, an isothermal core can be fitted to a solution of equations (27) and (28) whenever the curve

$$\left[4\pi \frac{r^3\rho(r)}{M(r)}\,; \qquad \frac{\mu H}{k}\,\frac{GM(r)}{rT(r)}\right] \tag{32}$$

derived from such a solution intersects the (u_∞, v_∞)-curve associated with an E-solution of the isothermal equation. At such an intersection

$$\left.\begin{aligned} u_\infty(\xi_i) &= 4\pi\left(\frac{r^3\rho}{M(r)}\right)_{r=r_i}, \\ v_\infty(\xi_i) &= \frac{\mu H}{k}\,G\left(\frac{M(r)}{rT}\right)_{r=r_i}, \end{aligned}\right\} \tag{33}$$

and the values of r/R and $M(r)/M$ at this point will determine, at once, the ratios q and ν of the radius and the mass of the core to the radius and the mass of the star, respectively. Further, according to the second of equations (33),

$$T_c = Q(q)\,\frac{\mu H}{k}\,\frac{GM}{R}\,; \qquad Q = \frac{\nu}{v_\infty(\xi_i)q}\,. \tag{34}$$

The ratio of the mean to the central density is also readily found. We have (cf. eq. [17])

$$\frac{\rho_c}{\bar{\rho}} = \frac{\lambda_2}{\dfrac{M}{\frac{4}{3}\pi R^3}}\,. \tag{35}$$

On the other hand, according to equations (4) and (5), we have identically

$$\lambda_2 = \frac{M(r_i)}{4\pi r_i^3}\,\frac{\xi_i}{\psi_i'}\,. \tag{36}$$

Hence, combining equations (35) and (36),

$$\frac{\rho_c}{\bar{\rho}} = \frac{\nu\xi_i}{3q^3\psi_i'} = \frac{\nu u_\infty(\xi_i)}{3q^3}\,e^{\psi_i}\,. \tag{37}$$

Now, there exist five integrations of equations (27) and (28) which can be used for our present purposes. Three of these integrations (due to Miss I. Nielsen[12]) are for the solar values of L, M, and R, with $\mu = 1$ and for values of $\log \kappa_0 = 24.792$, 24.892, and 24.992. Further, in these integrations of Miss Nielsen the radiation pressure as a factor in the equation of hydrostatic equilibrium has been ignored. The two other integrations (which were found to give solutions for the equations of fit) are due to Strömgren.[13] These integrations also refer to the solar values of L, M, and R, but with $\mu = 2.2$ and $\log \kappa_0 = 27.4$ and 27.8. Further, in these integrations the effect of the term $aT^4/3$ in equation (27) has also been taken into account.

[12] Under the supervision of B. Strömgren.

[13] *Zs. f. Ap.*, **2**, 345, 1931.

The results of fitting isothermal cores to the five integrations of equations (27) and (28) referred to, are summarized in Table 2.

It is known that the point-source model with negligible radiation pressure and with a law of opacity of the form $\kappa = \kappa_0 \rho^n T^m$ is a homology-invariant configuration.[14] It follows, therefore, that among the models with negligible radiation pressure, which consist of isothermal cores and point-source envelopes, those with a constant q form a homologous family. Hence, the physical relations derived from the three integrations I of Table 2 are invariant to homologous transformations. In particular the relations

$$R(q) = Q(q) \frac{\mu H}{k} \frac{GM}{T_c} ; \qquad \nu(q) = \frac{M(\text{core})}{M} , \tag{38}$$

will be valid for all stars. In practice, however, the foregoing relations will give sufficient accuracy only for stars of mass less than, say, $5\odot$. This is confirmed, for example,

TABLE 2

STELLAR MODELS WITH ISOTHERMAL CORES AND POINT-SOURCE ENVELOPES

	Integration of Equations (27) and (28)	q	ν	$Q(q)$	$\rho_c/\bar{\rho}$	$\frac{L_0(q)}{[Q(q)]^{1/2}} \times 10^{-24}$	Δr_i /R in an Arbitrary Scale	$1-\beta_i$	Remarks
	$\log \kappa_0 = 24.735$; $\mu = 1$	0	0	0.900	37.0	5.73	....	....	Cowling model
I	$\log \kappa_0 = 24.792$; $\mu = 1$	0.139	0.103	0.791	54	6.97	1.00	0.0043	Inger Nielsen's integrations for the point-source envelope. Radiation pressure neglected in equation (27)
	$\log \kappa_0 = 24.892$; $\mu = 1$	.158	.176	0.778	80	8.85	0.98	.0040	
	$\log \kappa_0 = 24.992$; $\mu = 1$	.166	.231	0.788	115	11.1	1.11	.0042	
II	$\log \kappa_0 = 27.4$; $\mu = 2.2$	.158	.239	0.776	160	7.71	0.77	.078	Stromgren's integrations for the point-source envelope. Radiation pressure accurately taken into account. The results of fitting valid for configurations having a mass $4.84\ \mu^{-2}\ \odot$
	$\log \kappa_0 = 27.8$; $\mu = 2.2$	.119	.319	1.07	175	16.5	1.51	.103	
	$\log \kappa_0 = 27.8$; $\mu = 2.2$	0.081	0.224	1.33	635	14.8	1.70	0.115	

by the results of the last three rows of Table 2: These have been derived for the solar mass with $\mu = 2.2$, taking full account of the radiation pressure in the equation of hydrostatic equilibrium. It is, however, clear that we shall obtain the same results for a star of mass $M = (2.2)^2\odot = 4.84\odot$ and $\mu = 1$. From the column "$1 - \beta_i$" in Table 2 we notice that the radiation pressure, while it is appreciable in these models II, is still not of primary importance. This is reflected, for instance, in the fact that $Q(q)$ and $\nu(q)$ for these models fall roughly on the same curve as those for the three other cases in which the radiation pressure in equation (27) has been treated as negligible (see Fig. 2 where the models I are indicated by dots and models II by crosses).

5. *The physical characteristics of stellar models with isothermal cores.*—An examination of the results of §§ 3 and 4 (particularly Tables 1 and 2 and Figs. 1 and 2) brings out the following essential features of these models:

a) At a fixed central temperature, the radius R of the star first decreases as q increases from $q = 0$. For a value of $q \sim 0.15$–0.16 the radius passes through a minimum. Further, there exists also a maximum possible value for q ($q_{max} \sim 0.16$–0.17). As q decreases after passing through q_{max}, R increases very rapidly, reaches a maximum, and begins spiraling about a determinate point.

b) Again, at a fixed central temperature, the fraction of the total mass, ν, contained in the core increases slowly at first and soon very rapidly as q approaches q_{max}. How-

[14] See Chandrasekhar, *op. cit.*, pp. 234–39.

ever, this increase of ν does not continue indefinitely; ν soon attains a maximum value ν_{max}. There exists, therefore, an upper limit to the mass which can be contained in the isothermal core. For the models with point-source envelopes and inappreciable radiation pressure, $\nu_{max} \sim 0.32$ and occurs for $q \sim 0.12$. The curve $\nu(q)$ also shows the spiraling characteristic.[15]

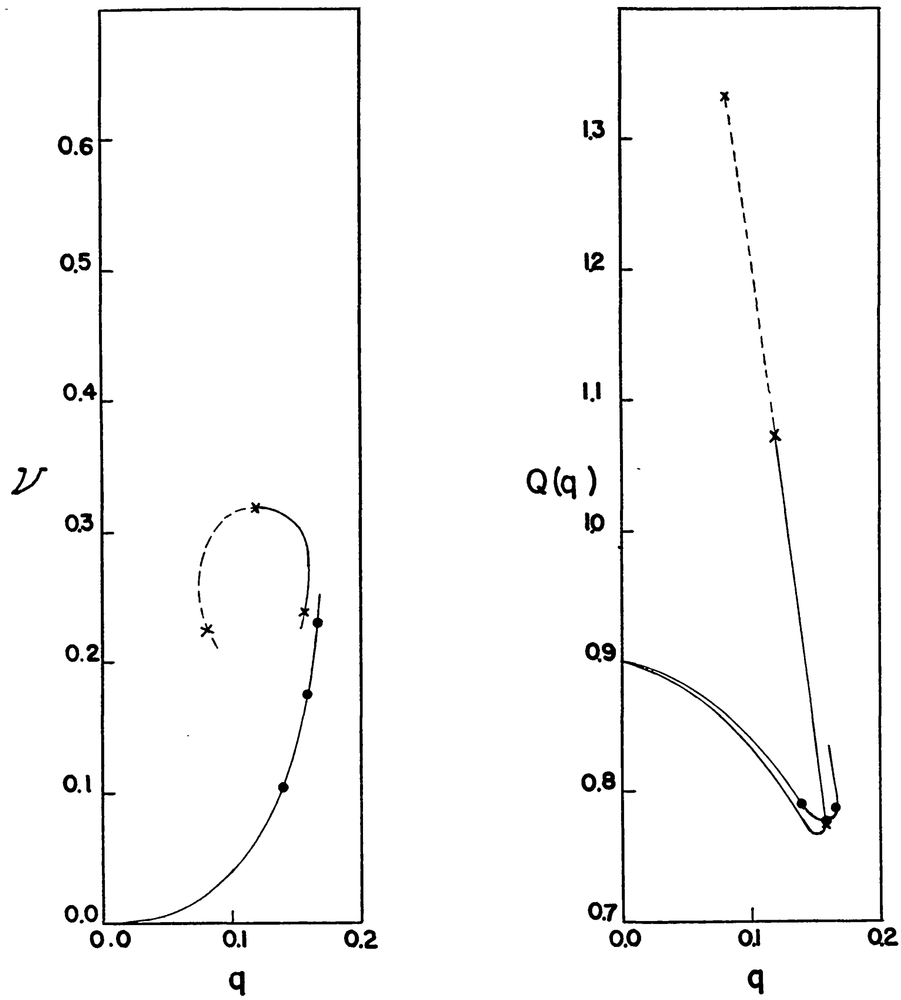

FIG. 2

A result of some importance in the present connection is the variation of the luminosity as q varies at constant central temperature. We shall discuss this variation of the luminosity on the basis of the models considered in § 4.

We may first recall that for a homologous family of stellar configurations derived on the basis of the law of opacity (29) there exists a luminosity formula of the form

$$L = \text{constant}\ \frac{M^{5.5}}{\kappa_0 R^{0.5}}\ \mu^{7.5}\ , \tag{39}$$

[15] This phenomenon of spiraling need not cause any particular surprise. It arises essentially from the oscillatory behavior of the solutions of the isothermal equation as $\xi \to \infty$ (see *ibid.*, pp. 163–66). Situations similar to these described in the text have also been encountered in other connections (see, e.g., S. Chandrasekhar, *Ap. J.*, **87**, 535, 1938; and *M.N.*, **99**, 673, 1939).

where the constant is a characteristic of the family. Consequently, for stellar models with isothermal cores and negligible radiation pressure, we must have a formula of the form

$$L = L_0(q) \frac{1}{\kappa_0} \frac{M^{5.5}}{R^{0.5}} \mu^{7.5} , \tag{40}$$

where, as the notation implies, L_0 depends on q only. Remembering that in obtaining the models I of Table 2 we have used integrations of the point-source envelope computed for the solar values of L, M, and R with $\mu = 1$, it follows that

$$L_0(q) = (\kappa_0)_{\text{integration}} , \tag{41}$$

if we suppose that in equation (40) L, M, and R are expressed in solar units. Similarly, for the solutions II (see Table 2)

$$L_0(q) = \frac{(\kappa_0)_{\text{integration}}}{(2.2)^{7.5}} . \tag{42}$$

Consider now the variation in L as q varies at constant T_c, μ, M, and κ_0. During such a change, R will alter according to (cf. eq. [38])

$$R(q) = Q(q) \frac{\mu H}{k} \frac{GM}{T_c} . \tag{43}$$

Eliminating R between equations (40) and (43),

$$L = \frac{L_0(q)}{[Q(q)]^{1/2}} \left(\frac{k}{GH}\right)^{1/2} \frac{1}{\kappa_0} M^5 T_c^{0.5} \mu^7 . \tag{44}$$

Hence, the variation in the luminosity is governed by the factor

$$\frac{L_0(q)}{[Q(q)]^{1/2}} . \tag{45}$$

If we now consider the more general case in which the radiation pressure in equation (27) is taken into account, it is clear that we can still construct a homologous sequence of configurations. But a homologous family is now determined by two parameters: q and $M\mu^2/\odot$. However, as long as we are interested only in the changes in the luminosity occurring in a star of given M and μ, we can always write down a relation of the form (44). Moreover, any such relation will be valid for a sequence of configurations of constant $M\mu^2$.

The factor (45) governing the variation of L for constant M, T_c, κ_0, and μ is tabulated in Table 2. According to the values given in this table, the luminosity increases by a factor of about 3 from the stage where there is no isothermal core to the stage where the core contains the maximum possible mass.

The variation in the luminosity predicted by equation (44) implies a corresponding variation in the thickness of the energy-generating shell, for, according to equations (26) and (44),

$$4\pi r_i^2 \rho_i \Delta r_i \epsilon_0 = \frac{L_0(q)}{[Q(q)]^{1/2}} \frac{1}{\kappa_0} \left(\frac{k}{HG}\right)^{1/2} M^5 T_c^{0.5} \mu^7 . \tag{46}$$

The foregoing equation can be simplified by using equation (30). We find

$$\frac{\Delta r_i \epsilon_0}{r_i} = \frac{L_0(q)}{[Q(q)]^{1/2} u_\infty(\xi_i)\nu} \frac{1}{\kappa_0} \left(\frac{k}{HG}\right)^{1/2} M^4 T_c^{0.5} \mu^7 , \tag{47}$$

or

$$\frac{\Delta r_i}{R} \propto \frac{q L_0(q)}{[Q(q)]^{1/2} u_\infty(\xi_i)\nu} . \tag{48}$$

The quantity on the right-hand side (apart from a constant factor) is tabulated in Table 2. We notice that the variation in the thickness of the shell is not very marked.

6. *General remarks.*—We shall now consider briefly the bearing of the results summarized in § 5 on the physical problems outlined in § 1 and in particular the implication for the Gamow-Teller theory of the energy production in giants. Suppose that to begin with a star has a central temperature $T_c(\sim 10^{6\circ})$ at which the disintegration of lithium can provide for an adequate source of energy. Under these circumstances the star will approximate to the Cowling model which has a convective core occupying 17 per cent of the radius and containing 15 per cent of the mass of the star. Suppose now that the lithium in the central regions is exhausted and that the process of the diffusion of elements does not take place rapidly enough for the restoration of adequate amounts of lithium to the center. We shall then have a shell-source model. In the early stages ($\nu < 0.15$) the star will consist of an isothermal core, a convective fringe, and a point-source radiative envelope. However, very soon (i.e., when $\nu > 0.15$) the star will consist only of an isothermal core and a radiative envelope. It is now clear that energy production from the disintegration of lithium can continue only as long as the mass in the isothermal core increases. But we have seen that ν cannot increase beyond a certain maximum value $\nu_{\max}(\sim 35$ per cent). When this happens the liberation of energy from the process considered will cease. The star must then readjust itself to a contractive model ($\epsilon \propto T$) and evolve according to the Helmholtz-Kelvin time scale. This will continue till the central temperature increases sufficiently for the liberation of nuclear energy from the disintegration of the next element, beryllium, to become effective. The whole cycle of changes will now be repeated.

In considering the course of changes we have described in the foregoing paragraph, it is of interest to trace the track of evolution in the Hertzsprung-Russell diagram. To illustrate this we have plotted

$$\log \frac{L_0(q)}{[Q(q)]^{1/2}} \tag{49}$$

against

$$\log \frac{L_0(q)}{[Q(q)]^{1/2}} - 2 \log Q(q) \tag{50}$$

in Figure 3. According to our earlier remarks, an evolution of the kind we are considering must cease when the luminosity has reached about its maximum value (cf. Table 2). We may note at this point that at no stage during such an evolution does the isothermal core occupy a large fraction of the radius; indeed, it is always less than about 17 per cent.

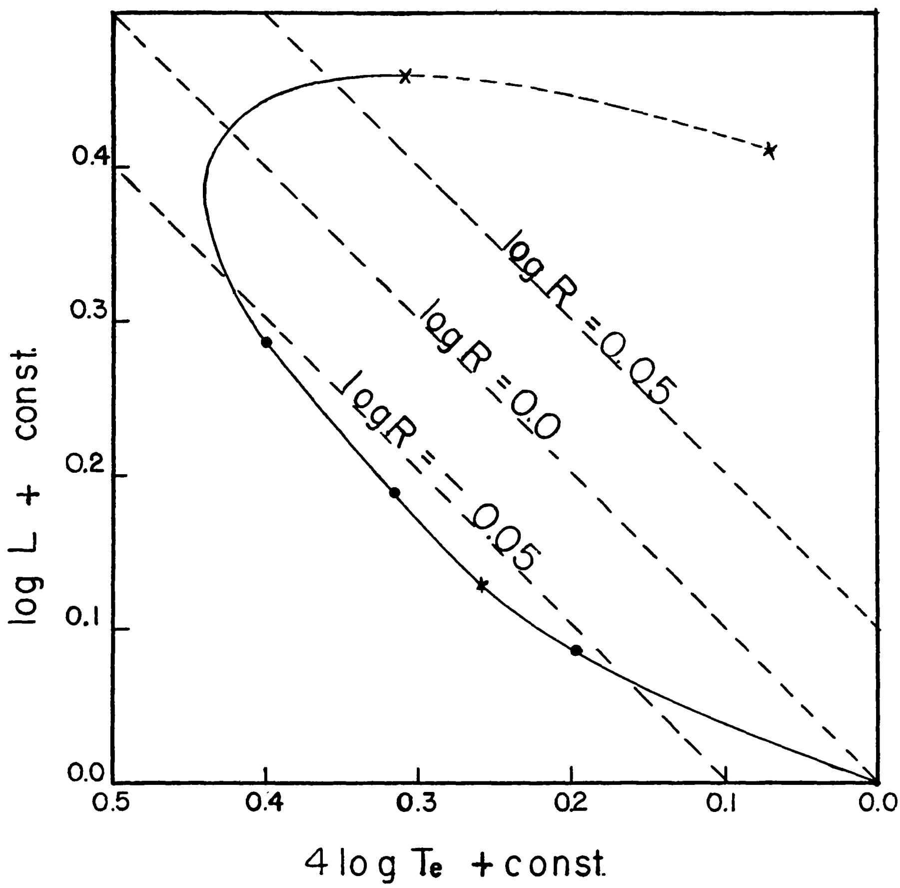

FIG. 3

Consequently, we cannot expect any significant changes in the stability of a star along such a course.

Finally, we may remark that with suitable modifications we may similarly follow the eventual course of evolution of a star as the hydrogen in the central regions becomes exhausted.

YERKES OBSERVATORY
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