THE TIME OF RELAXATION OF STELLAR SYSTEMS. III

S. CHANDRASEKHAR

ABSTRACT

Certain difficulties of principle which confront a rigorous and satisfactory analysis of stellar encounters within the framework of a theory which idealizes them, individually, as two-body problems are pointed out.

1. *Introduction.*—In the two preceding papers[1] the time of relaxation of a stellar system has been considered from two distinct points of view. In both cases the problem has a twofold character: first, to determine the effectiveness of stellar encounters in influencing the motions of the individual stars in a stellar system and, second, to estimate the rate at which a stellar system may be expected to approach the final state of "thermodynamic" equilibrium. The clearest formulation of this latter aspect of the ideas underlying the problem of the time of relaxation of a stellar system is due to Rosseland,[2] who has pointed out that after a sufficient length of time (long enough for a star to have suffered a large number of encounters) successive values of ΔE may be expected to be independent. Consequently, we can expect the ΔE's to combine according to the Gaussian error law. Thus, the probability that, after a time t, ΔE may have a value in the range $\{\Delta E, \Delta E + d(\Delta E)\}$ will be given by an expression of the form (cf. Rosseland, *op. cit.*, eq. [1])

$$W_t(\Delta E)d(\Delta E) = \frac{1}{\sqrt{2\pi\sum_0^t \Delta E^2}}\, e^{-\left(\Delta E - \sum_0^t \Delta E\right)^2 \Big/ 2\sum_0^t \Delta E^2} d(\Delta E)\,. \tag{1}$$

In other words, $\sum_0^t \Delta E^2$ is a measure of the *dispersion* in ΔE to be expected after a given length of time. It is primarily for this reason that the quantity $\Sigma\Delta E^2$ was evaluated in Part I. (Similar arguments apply to the consideration of the sum $\Sigma \sin^2 2\Psi$ where $(\pi - 2\Psi)$ is the true deflection of a star with respect to fixed frame of reference.) Thus, at first sight, it would appear that, by combining the methods of Parts I and II with a law of the form (1) we should be able to make some progress toward a *statistical theory* of stellar encounters. Such a theory would clearly be of the greatest importance, particularly in an analysis of the problem of clusters. However, and it is the object of this note to point out, that a description of stellar encounters from a very much more fundamental standpoint than has been adopted in Parts I and II is necessary before a really satisfactory start can be made toward a statistical theory of stellar encounters.

2. *The evaluation of the sum* $\Sigma\Delta$E.—It is clear that in order that we may be able to use equation (1) it is necessary to evaluate not only the sum $\Sigma\Delta E^2$ but also the sum $\Sigma\Delta E$, and it is in this connection that we encounter certain fundamental difficulties.

Let us suppose that, as in Parts I and II, we can idealize stellar encounters, individually, as two-body problems. As we have seen in Part I, § 3, the parameters defining

[1] Referred to as "I" and "II."

[2] *M.N.*, **88**, 208, 1928.

such an encounter are v_1, θ, φ, D, and Θ.[3] Consider, then, a $(v_1, \theta, \varphi, D, \Theta)$ encounter. The exchange of energy ΔE resulting from such an encounter is (cf. I, eq. [16])

$$\Delta E = -\frac{2m_1 m_2}{m_1 + m_2} V_g V \cos(\phi - \psi) \cos\psi \cos i \,. \tag{2}$$

The contribution to $\Sigma\Delta E$ by all the $(v_1, \theta, \varphi, D, \Theta)$ encounters can therefore be written as (cf. I, eq. [19])

$$\Sigma\Delta E_{(v_1, \theta, \varphi, D, \Theta)} = 2\pi N(v_1, \theta, \varphi)\Delta E V D dD \frac{d\Theta}{2\pi} dv_1 d\theta d\varphi dt \,, \tag{3}$$

or, using equation (2), as

$$\left.\begin{aligned} \Sigma\Delta E_{(v_1, \theta, \varphi, D, \Theta)} = & -4\pi N(v_1, \theta, \varphi) \\ & \times \frac{m_1 m_2}{m_1 + m_2} V_g V^2 \cos i \cos(\phi - \psi) \cos\psi \, D dD \frac{d\Theta}{2\pi} dv_1 d\theta d\varphi dt \,. \end{aligned}\right\} \tag{4}$$

If we used ψ as the variable instead of D, then equation (4) becomes (cf. I, eq. [22])

$$\left.\begin{aligned} \Sigma\Delta E_{(v_1, \theta, \varphi, \psi, \Theta)} = & -4\pi N(v_1, \theta, \varphi) G^2 m_1 m_2 (m_1 + m_2) \frac{V_g}{V^2} \cos i \\ & \times \frac{\cos(\phi - \psi) \sin\psi}{\cos^2\psi} d\psi \frac{d\Theta}{2\pi} dv_1 d\theta d\varphi dt \,. \end{aligned}\right\} \tag{5}$$

On integrating the foregoing equation over ψ, we obtain

$$\left.\begin{aligned} \Sigma\Delta E_{(v_1, \theta, \varphi, \Theta)} = & -4\pi N(v_1, \theta, \varphi) G^2 m_1 m_2 (m_1 + m_2) \frac{V_g}{V^2} \cos i \\ & \times \int \frac{\cos(\phi - \psi) \sin\psi}{\cos^2\psi} d\psi \frac{d\Theta}{2\pi} dv_1 d\theta d\varphi dt \,. \end{aligned}\right\} \tag{6}$$

In equation (6) the integration will have to be extended over the relevant range in ψ. Now, the integral occurring in equation (6) diverges at $\psi = \pi/2$. This, in itself, is not surprising. Actually, the corresponding integrals which occur in the evaluation of either of the sums $\Sigma\Delta E^2$ or $\Sigma \sin^2 2\Psi$ (cf. I, eq. [24], and II, eq. [39]) also diverge at $\psi = \pi/2$. But there is one important difference: the integral in equation (6) diverges to a higher order in D than in either of the two previous cases. To examine explicitly the nature of this higher order of divergence, let us extend the range of integration to $0 \leqslant \psi \leqslant \psi_0$ and put $\psi_0 = \pi/2$ in all terms except those which diverge at $\psi = \pi/2$. Using this method, we readily find that

$$\int \frac{\cos(\phi - \psi) \sin\psi}{\cos^2\psi} d\psi = (-\log\cos\psi_0)\cos\phi + \left(\tan\psi_0 - \frac{\pi}{2}\right)\sin\phi \,. \tag{7}$$

[3] The notation is the same as in I and II.

Combining equations (6) and (7), we find that

$$\left.\begin{aligned} \Sigma\Delta E_{(v_1,\,\theta,\,\varphi,\,\Theta)} &= -4\pi N(v_1,\,\theta,\,\varphi)G^2 m_1 m_2(m_1+m_2)\frac{V_g}{V^2}\cos i\,\frac{d\Theta}{2\pi}\,dv_1 d\theta d\varphi dt \\ &\times\left[\frac{1}{2}\left\{\log\left(1+\frac{D_0^2V^4}{G^2(m_1+m_2)^2}\right)\right\}\cos\phi+\left\{\frac{D_0V^2}{G(m_1+m_2)}-\frac{\pi}{2}\right\}\sin\phi\right]. \end{aligned}\right\} \tag{8}$$

Using the relations (27) of Part I, we can re-write the foregoing equation more conveniently in the form

$$\left.\begin{aligned} \Sigma\Delta E_{(v_1,\,\theta,\,\varphi,\,\Theta)} &= -4\pi N(v_1,\,\theta,\,\varphi)G^2 m_1 m_2(m_1+m_2)\frac{V_g}{V^2}\cdot\frac{d\Theta}{2\pi}\,dv_1 d\theta d\varphi dt \\ &\times\left[\frac{1}{2}\left\{\log\left(1+\frac{D_0^2V^4}{G^2(m_1+m_2)^2}\right)\right\}\cos\Phi+\left\{\frac{D_0V^2}{G(m_1+m_2)}-\frac{\pi}{2}\right\}\sin\Phi\cos\Theta\right]. \end{aligned}\right\} \tag{9}$$

Now, the dominant term in the foregoing expression is the one which involves $[D_0V^2/G(m_1+m_2)]$, and this is seen to be very much more important than the non-dominant terms, for (cf. I, eq. [56])

$$\frac{D_0V^2}{G(m_1+m_2)} = 9.31\times 10^4\,\frac{[D_0/\text{parsec}]}{[(m_1+m_2)/\odot]}\,[V/20\text{ km/sec}^{-1}]^2\,. \tag{10}$$

Consequently, the term $[D_0V^2/G(m_1+m_2)]$ is generally $10^{3.5}$ to $10^{4.5}$ times as large as the logarithmic term, $\log[D_0V^2/G(m_1+m_2)]$. However, the dominant term occurs with $\cos\Theta$ as a factor, and it would appear that the averaging over Θ would eliminate the dominant term from equation (9) and retain only the logarithmically diverging term. But it is now clear that the $[D_0V^2/G(m_1+m_2)]$ term can be ignored only if the average value of $\cos\Theta$ for $0\leqslant\Theta\leqslant 2\pi$ under the conditions of the *physical problem* is less than 10^{-4}. This is hardly likely to be the case, since any fluctuation giving even the slightest preference to a particular value of Θ will make the $[D_0V^2/G(m_1+m_2)]$ term far more important than any of the other terms which would normally be retained. We cannot thus ignore the dominant term in equation (9), for the *formal* reason that the average value of $\cos\Theta$ in the range $0\leqslant\Theta\leqslant 2\pi$ vanishes.[4] At the same time it would be difficult

[4] A recent paper by L. Spitzer (*M.N.*, **100**, 387, 1940) contains an evaluation of the sum $\Sigma\Delta E$. It will be noted that Spitzer retains only the logarithmic term in his expressions. For reasons which are stated in the text the evaluation of $\Sigma\Delta E$ which ignores the dominant term $[D_0V^2/G(m_1+m_2)]$ is likely to underestimate it by a factor of the order of 100 or more when any particular star is being followed during its motion. However, as Dr. Spitzer has pointed out to the writer, if one averages over a large number of "incident" stars, a more complete cancelation of the $[D_0V^2/G(m_1+m_2)]$ term may be expected. If this be assumed, the averaging of equation (9) over Θ will give

$$\left.\begin{aligned} \Sigma\Delta E_{(v_1,\theta,\varphi)} &= -\,2\pi N(v_1,\,\theta,\,\varphi)\,G^2m_1m_2(m_1+m_2)\,\frac{V_g}{V^2}\cos\Phi \\ &\times\log\left(1+\frac{D_0^2\,V^4}{G^2(m_1+m_2)^2}\right)dv_1d\theta d\varphi dt\,. \end{aligned}\right\} \tag{11}$$

If we adopt a spherical distribution of the velocities v_1 for purposes of averaging, equation (11) becomes (cf. I, egs. [40] and [41])

$$\left.\begin{aligned} \Sigma\Delta E_{(v_1,\theta,\varphi)} &= -\,\tfrac{1}{2}N(v_1)\,G^2m_1\,m_2\,(m_1+m_2)\,\frac{V_g}{V^2}\cos\Phi\sin\theta \\ &\times\log\left(1+\frac{D_0^2\,V^4}{G^2(m_1+m_2)^2}\right)dv_1d\theta d\varphi dt\,. \end{aligned}\right\} \tag{12}$$

The averaging of the foregoing equation over φ is immediate. To average over θ, we should first express to incorporate in a satisfactory manner the $[D_0 V^2/G(m_1 + m_2)]$ term within the framework of a theory which idealizes stellar encounters as two-body problems. The problems which we have to face in the present connection are several. First, if we retain the two-body approximation, the main contribution to $\Sigma\Delta E$ would arise, as we have already pointed out, from the random fluctuations which would give preference to certain values of Θ. Second, any error in estimating the "cut-off" distance D_0 will directly influence the results, and this, we may note, is contrary to our earlier experience in Parts I and II, where an uncertainty of a factor of 2 or 3 in D_0 introduced errors of less than 10 per cent. Third, the extreme importance of the dominant term in the present problem indicates that very great care should be exercised in including the distant encounters. And finally, if the distant encounters are as important as they appear to be, it would appear more profitable to abandon the two-body approximation of stellar encounters altogether and devise a more satisfactory statistical method. It is not the object of this note to go into these matters here. We wish only to draw attention to the fundamental difficulties which a proper discussion of stellar encounters must necessarily confront.

YERKES OBSERVATORY
September 18, 1940

V_g, V, and $\cos\Phi$ in terms of θ. Using the relations (30), (31), and (35), and after some further reductions, we find

$$\left.\begin{aligned}\Sigma\Delta E v_1 &= -N(v_1)\frac{\pi G^2 m_1 m_2(m_1+m_2)}{2v_1 v_2}dv_1 dt\\ &\times\int_{|v_1-v_2|}^{|v_1+v_2|}\left[\frac{v_2^2-v_1^2}{V^2}-\frac{m_1-m_2}{m_1+m_2}\right]\log(1+q^2V^4)dV\,,\end{aligned}\right\}\tag{13}$$

where

$$q=\frac{D_0}{G(m_1+m_2)}\,.\tag{14}$$

The integral occurring in equation (13) can be evaluated by the methods described in I and II (cf. pp. 294 and 295). We find

$$\Sigma\Delta E_{v_1} = -\,N(v_1)\,\frac{4\pi G^2 m_1 m_2}{v_1 v_2}\,dv_1 dt$$

$$\times\left\{\begin{array}{ll}\left[m_2 v_1\log q\,(v_2^2-v_1^2)-m_1 v_2\log\dfrac{v_2+v_1}{v_2-v_1}+2m_1 v_1\right] & (v_2>v_1)\,,\\ \tfrac{1}{2}(m_2-m_1)(\log 4qv^2-2)v & (v_2=v_1)\,,\\ \left[-\,m_1 v_2\log q(v_1^2-v_2^2)+m_2 v_1\log\dfrac{v_1+v_2}{v_1-v_2}-2m_2 v_2\right] & (v_2<v_1)\,.\end{array}\right\}\tag{15}$$