AN INTEGRAL THEOREM ON THE EQUILIBRIUM OF A STAR

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ABSTRACT

In this paper an integral theorem on the equilibrium of a star is proved which gives the lower limit to the value of $P_c/\rho_c^{(n+1)/n}$, assuming that both ρ and $P/\rho^{(n+1)/n}$ do not increase outward. As a special case of the theorem ($n = 3$) it is shown that for a gaseous star of a given mass in radiative equilibrium, in which ρ and $[\overline{\kappa\eta}]_R^r$ do not increase outward, the minimum value of $1 - \beta_c$ is the constant value of $(1 - \beta)$ ascribed to a standard model configuration of the same mass. For $n = \infty$ the theorem gives the minimum central temperature for a gaseous star with negligible radiation pressure.

In some recent papers[1] the author has proved a number of integral theorems on the equilibrium of a star. In particular it was shown that for any equilibrium configuration (of prescribed mass and radius) in which the mean density $\bar{\rho}(r)$ inside r decreases outward it is possible to set an *upper limit* to the value of $P_c/\rho_c^{(n+1)/n}$ for $1 < n \leqslant 3$. The inequality in question is (II, Eqs. [14] and [15])

$$\frac{P_c}{\rho_c^{(n+1)/n}} \leqslant S_n G M^{(n-1)/n} R^{(3-n)/n} , \qquad (1 < n \leqslant 3) , \quad (1)$$

where

$$S_n = (\tfrac{4}{3}\pi)^{1/n} \frac{n}{3(n-1)} . \qquad (2)$$

Furthermore, (1) is a *strict* inequality for $n < 3$, and for $n = 3$ is equivalent to setting an upper limit to $(1 - \beta_c)$ for gaseous stars (cf. Theorem 2, II).

The problem of finding a *lower limit* to the ratio $P_c/\rho_c^{(n+1)/n}$ has proved to be rather an elaborate one. In this paper we prove theorems in this direction.

The numbering of the theorems is continued from II and III.

I

THEOREM 10.—*In any equilibrium configuration of prescribed mass and radius in which both* ρ *and* $K = P/\rho^{(n+1)/n}$, $(n > 1)$ *do not increase*

[1] *M.N.*, **96**, 644, 1936; *Ap. J.*, **85**, 372, 1937, and **86**, 78, 1937. These papers will be referred to as "I," "II," and "III," respectively.

outward, the minimum value of K_c *is attained in the sequence of equilibrium configurations which consist of polytropic cores of index* n *and homogeneous envelopes.*

More explicitly, we consider a *composite* configuration in which the polytropic core extends to a fraction A of the radius R of the star. Inside the polytropic core, K is constant and equal to K_c. For such composite configurations, K_c will be a function $K_c(A)$ of A only. The theorem states that the minimum value of the function $K_c(A)$ is the *absolute* minimum of K_c for any equilibrium configuration in which ρ and K are restricted not to increase outward.

Proof: We shall first prove the following lemma:

Lemma: The configuration in which K_c *attains the minimum, either* $d\rho/dr = 0$ *or* $dK/dr = 0$ *for all* $0 \leqslant r \leqslant R$.

For, if not, in the configuration in which K attains its minimum there must exist a finite interval

$$0 < r_1 \leqslant r \leqslant r_2 < R\,, \tag{3}$$

in which

$$\frac{d\rho}{dr} < 0\,; \qquad \frac{dK}{dr} < 0\,. \tag{4}$$

Let P and ρ refer to the configuration we are considering, namely, the one in which K_c attains its minimum.

By means of the following transformation we construct the pressure and density distributions defined by

$$0 \leqslant r \leqslant r_1: \qquad P^* = (1-\epsilon)^2 P\,; \qquad \rho^* = (1-\epsilon)\rho\,, \tag{5}$$

$$r_1 < r \leqslant r_2: \qquad P^* = P + \epsilon P_1\,; \qquad \rho^* = \rho + \epsilon\rho_1\,, \tag{6}$$

$$r_2 < r \leqslant R: \qquad P^* = P\,; \qquad \rho^* = \rho\,, \tag{7}$$

where P^* and ρ^* refer to the new distributions of pressure and density, ϵ is a sufficiently small *positive* constant, and P_1 and ρ_1 (which are functions of r in the interval $r_1 \leqslant r \leqslant r_2$) are, for the present, unspecified.

If P^* and ρ^* should refer to an *equilibrium* configuration of the same mass M as the original configuration, then the following conditions would be fulfilled.

i) *Continuity of* P^* *and* ρ^*.—From (5), (6), and (7) we see that to insure continuity at $r = r_1$ and $r = r_2$ we should have

$$\rho_1 = -\rho \,, \qquad (r = r_1) \,; \qquad \rho_1 = 0 \,, \qquad (r = r_2) \,; \tag{8}$$

$$P_1 = -2P \,, \qquad (r = r_1) \,; \qquad P_1 = 0 \,, \qquad (r = r_2) \,. \tag{9}$$

ii) *Constancy of mass.*—This requires

$$4\pi \int_0^R \rho^* r^2 dr = 4\pi \int_0^R \rho r^2 dr \,. \tag{10}$$

By (5), (6), and (7) we find that (10) reduces to

$$M(r_1) = 4\pi \int_{r_1}^{r_2} \rho_1 r^2 dr \,. \tag{11}$$

The left-hand side of (11) is a known quantity. We can clearly choose a function ρ_1 in the interval $r_1 \leqslant r \leqslant r_2$ such that the equation (11) and the boundary conditions at r_1 and r_2 (Eq. [8]) are all satisfied. We assume that ρ_1 has been chosen to satisfy these conditions.

iii) *The distributions* P^* *and* ρ^* *satisfy the equation of hydrostatic equilibrium.*—The pressure-density distributions in any configuration of equilibrium must satisfy the equation

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho \,. \tag{12}$$

Given that P and ρ satisfy (12), we have to show that P^* and ρ^* distributions (with a suitable choice of P_1) satisfy (12).

It is immediately obvious that in the interval $0 \leqslant r \leqslant r_1$ and

$r_2 \leqslant r \leqslant R$, equation (12) is satisfied. For $r_1 \leqslant r \leqslant r_2$ we have, according to (5), (6), and (7),

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP_1}{dr} - \frac{r^2}{\rho^2}\rho_1\frac{dP}{dr}\right) = -4\pi G\rho_1 . \tag{13}$$

From (13) we derive

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP_1}{dr}\right) = F(r) , \tag{14}$$

where

$$F(r) = -4\pi G\rho_1 + \frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho^2}\rho_1\frac{dP}{dr}\right) . \tag{15}$$

Since P and ρ are assumed to be known functions of r, and ρ_1 has been chosen according to (ii) above, we can regard $F(r)$ as a known function of r. From (14) we easily derive

$$P_1 = \int_{r_1}^{r}\frac{\rho}{r^2}\left\{\int_{r_1}^{r}\xi^2 F(\xi)d\xi + c_1\right\}dr + c_2 , \tag{16}$$

where c_1 and c_2 are two integration constants. We now choose c_1 and c_2 such that P_1 defined by (16) satisfies the boundary conditions (9).

We thus see that with P_1 and ρ_1 chosen as specified in (ii) and (iii) above, P^* and ρ^* refer to an equilibrium configuration of the same mass and radius as the original configuration.

Finally, since in the interval $r_1 \leqslant r \leqslant r_2$, ρ and K are strictly decreasing (Eq. [4]), it is clear that we can choose a positive (*nonzero*) ϵ sufficiently small that ρ^* and K^* (defined with respect to P^* and ρ^*) are decreasing functions of r.

We have thus shown that from the given equilibrium configuration we can construct another satisfying the restrictions on ρ and K. But the configuration specified by the functions P^* and ρ^* defines

$$K_c^* = (1 - \epsilon)^{1-(1/n)} K_c . \tag{17}$$

Since we have assumed $n > 1$, we see that (17) implies

$$K_c^* < K_c \, , \tag{18}$$

which contradicts our hypothesis that in the configuration in which K_c attains its minimum there exists an interval (3) in which (4) holds. This proves the lemma.

The theorem now follows almost immediately. It is only necessary to exclude the types of density distributions shown by the full-line curves in Figures 1 and 2. In Figure 1, the regions *1* and *3* are regions of constant K, while *2* is a region of constant ρ; in Figure 2, *1* and *3* are regions of constant K, while *2* and *4* are regions of constant ρ.

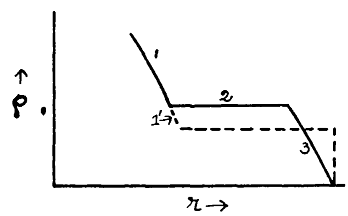

Fig. 1

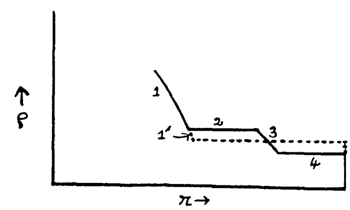

Fig. 2

But it is clear that by the constructions indicated by the dotted curves (in both cases *1'* corresponds to the analytic continuation of the density distribution specified by *1*) we are led to configurations with a smaller P_c, and hence a smaller K_c. This proves the theorem.

It has to be noticed that we have only proved that the minimum of K_c along the sequence of composite configurations (consisting of polytropic cores and homogeneous envelopes) is the absolute minimum of K_c under the restrictions $d\rho/dr \leqslant 0$, $dK/dr \leqslant 0$. But we have not yet specified the *particular* composite configuration in which the minimum of K_c is attained; to be able to do so, we shall have to study the function $K_c(A)$ where A is the fraction of the radius occupied by the polytropic core. We now proceed to study this function.

II

The composite configurations.—We consider a composite configuration in which the polytropic core extends to a fraction A of the radius R. Hence, if $r = r_1$ defines the place at which we have the *in-*

terface between the polytropic and the homogeneous regions, we have

$$r_1 = AR. \tag{19}$$

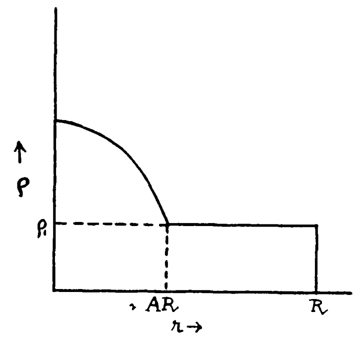

FIG. 3

Let $\bar{\rho}_1$ be the mean density of the polytropic core and ρ_1 the constant density in the homogeneous part. Let B denote the ratio

$$B = \frac{\bar{\rho}_1 - \rho_1}{\rho_1}. \tag{20}$$

Finally, let P_1 be the pressure at the interface.

Consider first the equilibrium of the homogeneous envelope: We have

$$M(r) = \tfrac{4}{3}\pi r_1^3 \bar{\rho}_1 + \tfrac{4}{3}\pi(r^3 - r_1^3)\rho_1 , \tag{21}$$

or, by (19) and (20),

$$M(r) = \tfrac{4}{3}\pi R^3 \rho_1 A^3 B + \tfrac{4}{3}\pi r^3 \rho_1 . \tag{22}$$

The mass M of the configuration is therefore given by

$$M = \tfrac{4}{3}\pi R^3 \rho_1 (1 + A^3 B) . \tag{23}$$

The equation of hydrostatic equilibrium is

$$\frac{dP}{dr} = -\frac{GM(r)}{r^2}\rho_1 , \tag{24}$$

or, by (22),

$$\frac{dP}{dr} = -G\,\tfrac{4}{3}\pi\rho_1^2\left(\frac{R^3A^3B}{r^2} + r\right), \tag{25}$$

or, integrating,

$$P = G\,\tfrac{4}{3}\pi\rho_1^2\left[\frac{R^3A^3B}{r} - \tfrac{1}{2}r^2\right]_R^r. \tag{26}$$

The pressure at the interface is obtained by putting $r = AR$ in (26). After some reductions we find that

$$P_1 = \tfrac{2}{3}\pi G\rho_1^2R^2[1 - A^2 + 2A^2B(1 - A)]\,. \tag{27}$$

Consider, now, the equilibrium of the polytropic core: in the core we can write

$$P = K\rho^{(n+1)/n}\,, \tag{28}$$

where K is a constant. The reduction to Emden's equation of index n is made by the substitutions

$$\rho = \lambda\theta^n\,; \qquad P = K\,\lambda^{(n+1)/n}\,\theta^{n+1}\,, \tag{29}$$

$$r = \left[\frac{(n+1)K}{4\pi G}\right]^{1/2}\lambda^{(1-n)/2n}\xi\,. \tag{30}$$

Let θ and ξ refer to the interface. Then P_1, ρ_1, and r_1 are given by the foregoing formulae. By (27), (29), and (30) we have

$$K\lambda^{(n+1)/n}\theta^{n+1} = \tfrac{2}{3}\pi G(\lambda\theta^n)^2\left[\frac{(n+1)K}{4\pi G}\right]\lambda^{(1-n)/n}\xi^2\,\frac{1 - A^2 + 2A^2B(1 - A)}{A^2}\,. \tag{31}$$

After some reductions the foregoing equation reduces to

$$1 = \xi^2\theta^{n-1}\cdot\frac{n+1}{6}\cdot\frac{1 - A^2 + 2A^2B(1 - A)}{A^2}\,. \tag{32}$$

Now introduce the homology invariant functions u and v defined by

$$u = -\frac{\xi\theta^n}{\theta'}\,; \qquad v = -\frac{\xi\theta'}{\theta}\,, \tag{33}$$

where θ' refers to the derivative of θ with respect to ξ. In the terms of u and v (32) can be re-written as

$$\frac{n+1}{6} uv = \frac{A^2}{1 - A^2 + 2A^2 B(1-A)} . \tag{34}$$

Furthermore,

$$B = \frac{\bar{\rho}_1 - \rho_1}{\rho_1} = \frac{\bar{\rho}_1}{\rho_1} - 1 , \tag{35}$$

or, using the well-known relation between the mean and the central densities for polytropic configurations, we have

$$B = -\frac{\lambda \frac{3}{\xi}\frac{d\theta}{d\xi}}{\lambda\theta^n} - 1 = -3\frac{\theta'}{\xi\theta^n} - 1 , \tag{36}$$

or, by (33),

$$B = \frac{3}{u} - 1 . \tag{37}$$

Equations (34) and (37) are our *equations of fit*. If the Emden function $\theta_n(\xi)$ is known, then for a given ξ, u and v are known and (34) and (37) determine A as the solution of a cubic equation. The configuration thus becomes determinate.

We have next to determine K in terms of A, R, and M. Using (34), we can re-write (27) as

$$P_1 = \tfrac{2}{3}\pi G\rho_1^2 R^2 A^2 \frac{6}{(n+1)uv} . \tag{38}$$

By (28) and (38) we now have

$$K = \tfrac{2}{3}\pi G\rho_1^{(n-1)/n} R^2 A^2 \frac{6}{(n+1)uv} . \tag{39}$$

We now eliminate ρ_1 between (39) and the mass relation (23). We thus have

$$K = \tfrac{2}{3}\pi G R^2 A^2 \frac{6}{(n+1)uv}\left[\frac{M}{\tfrac{4}{3}\pi R^3(1 + A^3 B)}\right]^{(n-1)/n} , \tag{40}$$

which, after some reductions, can be expressed as

$$K = \tfrac{1}{2}(\tfrac{4}{3}\pi)^{1/n} G M^{(n-1)/n} R^{(3-n)/n} \cdot Q_n \,, \tag{41}$$

where

$$Q_n = \frac{6}{(n+1)uv} \frac{A^2}{(1+A^3B)^{(n-1)/n}} \,. \tag{42}$$

We verify the following: Q_n *measures* K *in units of the value of* K_c *for the configuration of uniform density of mass* M *and radius* R. *The minimum of* $Q_n(A)$ *defines, according to our theorem, the minimum value of* K_c *in the specified units.*

Equation (34) is a cubic equation for A. Eliminating B between (34) and (37), we find that the equation for A can be written more conveniently as

$$2(n+1)v(3-u)A^3 + 3A^2[(n+1)v(u-2)+2] - (n+1)uv = 0 \,. \tag{43}$$

Furthermore,

$$B = \frac{3}{u} - 1 \,. \tag{44}$$

Equations (42), (43), and (44) define, then, the function $Q_n(A)$.

We notice that, as $u \rightarrow 0$,

$$A \rightarrow 1 \,; \qquad B \rightarrow \frac{3}{u} \,, \tag{45}$$

$$Q_n \rightarrow \frac{6}{n+1}\left(\frac{1}{3\omega_n}\right)^{(n-1)/n} , \tag{46}$$

where

$$\omega_n = -\left(\xi^{(n+1)/(n-1)} \frac{d\theta_n}{d\xi}\right)_{\xi=\xi_1} . \tag{47}$$

Inserting (46) in (41), we obtain equation (17), II.

III

The case n = 5.—The case $n = 5$ presents some interesting features. It is found that the Schuster-Emden integral for the case $n = 5$ reduces to

$$3v + u = 3 \tag{48}$$

in the (u, v)-plane. The cubic equation for A (Eq. [43]) now becomes

$$3(1+2A)(1-A)^2v^2 - 3(1-A^2)v + A^2 = 0\,, \tag{49}$$

or, solving for v,

$$v = \frac{3(1+A) - \sqrt{3(1-A)(3+9A+8A^2)}}{6(1+2A)(1-A)}\,. \tag{50}$$ [2]

Also,

$$B = \frac{v}{1-v}\,; \qquad Q_5 = \frac{1}{3v(1-v)}\,\frac{A^2}{(1+A^3B)^{4/5}}\,. \tag{51}$$

Equations (50) and (51) present the explicit solution for the problem.

Furthermore, we notice that if $v = 1$, (49) reduces to

$$6A^3 - 5A^2 = 0\,, \qquad \text{or} \qquad A = \tfrac{5}{6}\,. \tag{52}$$

Hence, for $A = \frac{5}{6}$, $v = 1$, $B = \infty$, $Q_5 = \infty$. Hence,

$$Q_5(A) \to \infty\,, \qquad A \to \tfrac{5}{6}\,. \tag{53}$$

The details of the solution are given in Table 1.

IV

The case $1 < n < 5$.—The solutions of the equations of fit have been effected for $n = 4.5$, 4, 3, and 2. The details of the solution are given in Table 1. The respective $Q_n(A)$ curves are shown in Figure 4. From the figure we infer the following theorems.

THEOREM 11.—*In any equilibrium configuration of prescribed mass and radius in which both* ρ *and* $K = P/\rho^{(n+1)/n}$, $(1 < n \leqslant 3)$ *do not increase outward, the minimum value of* K_c *is the constant value of* K *which must be ascribed to a complete polytrope of index* n *having the given mass and radius.*

For the case $n = 3$ the foregoing theorem can be stated in the following alternative form:

THEOREM 12.—*In a gaseous stellar configuration in which both* ρ *and* $(1 - \beta)$ *do not increase outward,* $(1 - \beta_c)$ *must be greater than the*

[2] The positive sign before the square root in (50) is easily seen to correspond to a physically impossible solution.

constant value of $(1 - \beta)$ *ascribed to a standard model configuration of the same mass.*

We shall comment on the implications of Theorem 12 in the studies of stellar structure in Section VIII.

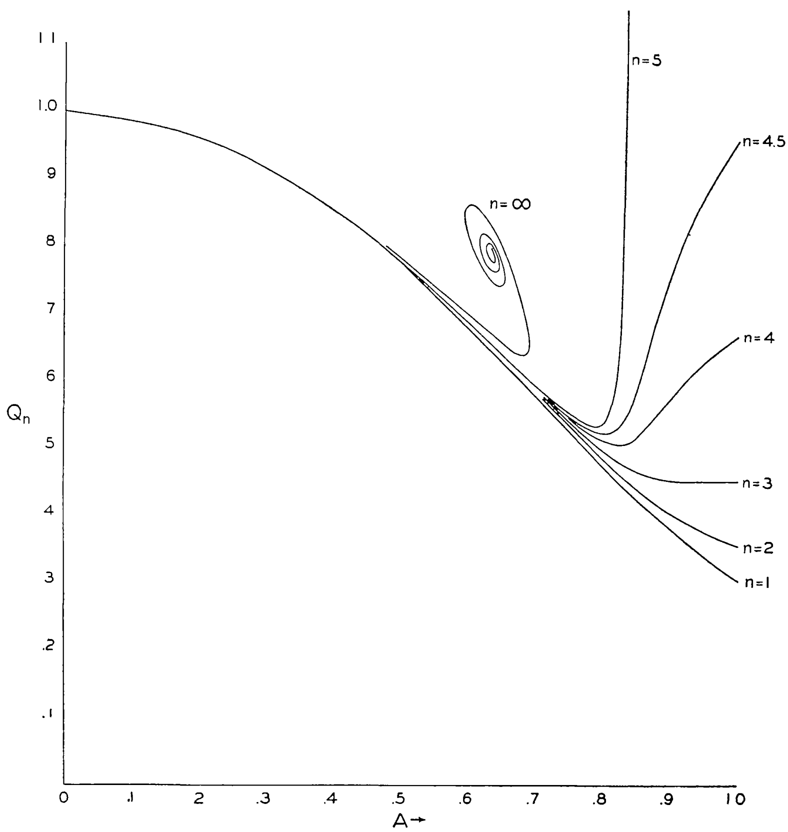

FIG. 4

V

The case n = 1.—The theorem has been proved only for the case $n > 1$. The arguments of Section I fail for this case (cf. Eqs. [17] and [18]). However, it seems likely that the theorem is also true for $n = 1$. If it is true, Theorem 11 is also valid for this case.

For the case $n = 1$ the Emden function is known:

$$\theta_1 = \frac{\sin \xi}{\xi} . \tag{54}$$

The appropriate equations are

$$\tfrac{1}{3}\xi^2 = \frac{A^2}{1 - A^2 + 2A^2B(1 - A)}\,, \tag{55}$$

$$B = 3\left[\frac{1}{\xi^2} - \frac{\cot\xi}{\xi}\right] - 1\,, \tag{56}$$

$$Q_1 = \frac{3A^2}{\xi^2}\,. \tag{57}$$

As $\xi \rightarrow \pi$, $A \rightarrow 1$ and $Q_1 = 3/\pi^2$. The details of the solution are given in Table 1.

VI

The case n $= \infty$.—For this case the conditions of the theorem are that both ρ and P/ρ do not increase outward. For gaseous stars with negligible radiation pressure $P/\rho = (k/\mu H)\,T$, and the theorem therefore sets a lower limit to the central temperature of such configurations.[3]

To determine the minimum value of Q_∞, we have to consider composite configurations consisting of *isothermal* cores and homogeneous envelopes. The analysis of these configurations is quite similar to that given in Section II.

If ψ is the Emden isothermal function, we define the functions u and v by

$$u = \frac{\xi e^{-\psi}}{\psi'}\,; \qquad v = \xi\,\frac{d\psi}{d\xi}\,. \tag{58}$$

The equations of fit are

$$\tfrac{1}{6}uv = \frac{A^2}{1 - A^2 + 2A^2B(1 - A)}\,, \tag{59}$$

$$B = \frac{3}{u} - 1\,, \tag{60}$$

[3] The problem of determining the minimum central temperature of a gaseous star with negligible radiation pressure has been considered earlier by Eddington (*Internal Constitution of the Stars* [Cambridge, 1924], pp. 91–93). However, his treatment of the problem is different from our approach to it, as it is a special case of a more general problem.

and

$$\left(\frac{P_c}{\rho_c}\right)_{\min} = \frac{1}{2}\frac{GM}{R} Q_\infty , \tag{61}$$

where

$$Q_\infty = \frac{6}{uv}\frac{A^2}{1 + A^3B} . \tag{62}$$

Equation (59) can be written alternatively as

$$2v(3 - u)A^3 + 3A^2[v(u - 2) + 2] - uv = 0 . \tag{63}$$

Now, it is well known that the isothermal function oscillates about the singular solution

$$e^{-\psi} = \frac{2}{\xi^2} \tag{64}$$

as $\xi \rightarrow \infty$. In the (u, v)-plane this corresponds to the appropriate curve spiraling around

$$u = 1 , \qquad v = 2 . \tag{65}$$

Introducing (65) into (60) and (64) we find that

$$B = 2 ; \qquad 4A^3 = 1 \qquad \text{or} \qquad A = \sqrt[3]{0.25} . \tag{66}$$

With these values of A and B, (62) gives

$$Q_\infty = \sqrt[3]{0.5} . \tag{67}$$

Hence, the $Q_\infty(A)$ curve spirals around the point

$$Q_\infty^{(s)} = \sqrt[3]{0.5} = 0.7937 ; \qquad A^{(s)} = 0.62996 . \tag{68}$$

The details of the solution are given in Table 1*a*. See also Figure 4.

VII

Numerical results.—In Tables 1 and 1*a* we give the details of the solution for the cases

$$n = 1,\ 2,\ 3,\ 4,\ 4.5,\ 5,\ \text{and}\ \infty\ . \qquad (69)$$

TABLE 1

VALUES OF Q_n

$n=1$		$n=2$		$n=3$		$n=4$		$n=4.5$		$n=5$	
A	Q_1	A	Q_2	A	Q_3	A	Q_4	A	$Q_{4.5}$	A	Q_5
0.	1.	0.	1.	0.	1.0	0.	1.	0.	1.0	0.	1.
				0.366	0.8710					0.15	0.9768
				0.423	0.8339			0.410	0.841	0.30	0.914
0.685	0.592	0.557	0.722	0.590	0.6946			.....	......	0.40	0.8505
0.737	0.537	.....	......	0.708	0.5787	0.662	0.626	0.716	0.576	0.50	0.7738
0.784	0.489	0.785	0.499	0.818	0.4833	0.775	0.526	0.764	0.538	0.60	0.6859
0.866	0.410	.....	......	0.872	0.4578	0.819	0.508	0.807	0.524	0.70	0.5931
0.919	0.363	0.893	0.408	0.912	0.4523	0.845	0.516	0.840	0.564	0.75	0.5521
......	....	.....	......	0.949	0.4519	0.879	0.551	0.875	0.681	0.78	0.5372
......	....	.....	......	0.964	0.4517	0.907	0.585	0.933	0.837	0.80	0.5410
......	....	.....	......	0.968	0.4516	0.931	0.613	0.961	0.893	0.832	1.0361
0.968	0.325	0.934	0.384	0.971	0.4516	0.974	0.648	.....	......	0.833	1.1500
0.984	0.314	0.996	0.358	0.986	0.4515	.....	......	0.990	0.940	0.8333	2.2054
1.000	0.30396	1.000	0.3564	1.000	0.45154	1.000	0.6671	1.000	0.9572	$\frac{5}{6}$	∞

TABLE 1*a*

VALUES OF Q_∞

ξ	A	Q_∞	ξ	A	Q_∞
0.........	0.	1.	70........	0.608	0.849
2.0.......	.542	0.743	100........	.594	.833
4.0.......	.663	0.646	150........	.616	.808
5.0.......	.680	0.640	200........	.615	.786
7.0.......	.689	0.655	300........	.629	.783
10.0.......	.683	0.695	400........	.633	.778
12.........	.675	0.721	500........	.635	.776
16.........	.661	0.766	700........	.636	.778
25.........	.636	0.826	1000........	.636	.784
30.........	.626	0.840	1500........	.627	.821*
35.........	.620	0.850	2000........	.632	.796
50.........	0.609	0.859	∞..........	0.62996	0.7937

* There seems to be an error in Emden's table of the isothermal function at this point.

In Table 2 the minimum values of $P_c/\rho_c^{(n+1)/n}$ under the conditions of Theorem 10 are given. From (41) and (42) it is clear that we can express the theorem as

$$\frac{P_c}{\rho_c^{(n+1)/n}} \geqslant \Sigma_n G M^{(n-1)/n} R^{(n-3)/n} , \tag{70}$$

where

$$\Sigma_n = \tfrac{1}{2}(\tfrac{4}{3}\pi)^{1/n} \text{ Minimum } Q_n(A) . \tag{71}$$

The values of Σ_n are also given in Table 2.

TABLE 2

n	A	Min Q_n	Σn
1.0	1.0	0.30396	0.63662
1.5	1.0	.3265	.42422
2.0	1.0	.3564	.36475
2.5	1.0	.3964	.35150
3.0	1.0	.45154	.36394
4.0	0.821	.508	.363
4.5	0.801	.524	.360
5.0	0.788	.535	.356
∞	0.678	0.640	0.320

VIII

Some remarks on Theorem 12.—We have shown that if ρ and $(1-\beta)$ decrease outward, then it is possible to set a lower limit to $(1-\beta_c)$ which depends on the mass only. We now examine the physical meaning of the assumption that "$(1-\beta)$ does not increase outward" for the case of radiative equilibrium.

We then have Strömgren's relation (cf. III, Eq. [9])

$$1-\beta = \frac{L}{4\pi c G M}\, \overline{\kappa\eta}(r) , \tag{72}$$

where

$$\overline{\kappa\eta}(r) = \frac{1}{P}\int_R^r \kappa\eta dP . \tag{73}$$

Hence, $(1 - \beta)$ *will decrease outward if* $\overline{\kappa\eta}(\mathrm{r})$ *decreases outward.* Analytically, the condition is

$$\frac{d}{dr}\,\overline{\kappa\eta}(r) \leqslant 0\,, \tag{74}$$

or, since dP/dr is negative,

$$\frac{d}{dP}\,\overline{\kappa\eta}(r) \geqslant 0\,. \tag{75}$$

From (73) and (75) we derive that (74) is equivalent to

$$\kappa\eta(r) \geqslant [\overline{\kappa\eta}]_R^r\,. \tag{76}$$

In words: *The necessary and sufficient condition for* $(1 - \beta)$ *decreasing outward is that* $\kappa\eta$ *at any point inside the star must be greater than the average value of* $\kappa\eta$ *for material exterior to* r. It should be noticed that the condition stated is less restrictive than the requirement that $\kappa\eta$ decreases outward. It is clear from (76) that we can actually allow a decrease of $\kappa\eta$ (within limits) as we approach the center. In actual stellar configurations η might be expected to decrease outward, but this will not generally be true of κ. For this reason it is important to realize that (76) does not require $\kappa\eta$ to decrease outward. We can now express Theorem 12 in the following alternative way. *In a wholly gaseous configuration in radiative equilibrium in which the density and* $\overline{\kappa\eta}(\mathrm{r})$ *as defined by* (73) *both decrease outward, the central value* $(1 - \beta_c)$ *of the ratio of the radiation pressure to the total pressure must satisfy the inequality.*

$$1 - \beta_c \geqslant 1 - \beta_s\,, \tag{77}$$

where $(1 - \beta_s)$ *satisfies the quartic equation*

$$M = -4\pi\,\frac{1}{(\pi G)^{3/2}}\left[\left(\frac{k}{\mu H}\right)^4 \frac{3}{a}\,\frac{1 - \beta_s}{\beta_s^4}\right]^{1/2}\left(\xi^2\,\frac{d\theta_3}{d\xi}\right)_{\xi=\xi_1}. \tag{78}$$

It might be recalled that under very much less restrictive circumstances than in the foregoing theorem we have shown that

$$1 - \beta_c \leqslant 1 - \beta^*\,, \tag{79}$$

where $(1-\beta^*)$ satisfies a similar quartic equation (cf. II, Theorem 2). It is thus seen that we can solve the problem of finding both the upper and the lower limits of $1-\beta_c$. We have (cf. III, Eq. [11])

$$L=\frac{4\pi cGM(1-\beta_c)}{\overline{\kappa\eta}}\,, \tag{80}$$

where $\overline{\kappa\eta}$ refers now to the average over the whole star. Hence, under the conditions of Theorem 12 by (77),

$$L\geqslant\frac{4\pi cGM(1-\beta_s)}{\overline{\kappa\eta}}\,. \tag{81}$$

Let

$$\overline{\kappa\eta}=\kappa_c\tilde{\eta}_c\,, \tag{82}$$

where

$$\tilde{\eta}_c=\frac{1}{P_c}\int_0^{P_c}\left(\frac{\kappa}{\kappa_c}\right)\eta dP\,. \tag{83}$$

If, further, we assume a law of opacity of the form

$$\kappa=\kappa_1\rho T^{-3-S}\,, \tag{84}$$

then

$$\kappa_c=\kappa_1\frac{\mu H}{k}\frac{a}{3}\frac{\beta_c}{1-\beta_c}T_c^{-S}\,, \tag{85}$$

or, again by (77),

$$\kappa_c\leqslant\kappa_1\frac{\mu H}{k}\frac{a}{3}\frac{\beta_s}{1-\beta_s}T_c^{-S}\,. \tag{86}$$

Combining (81), (82), and (86), we have

$$L\geqslant\frac{4\pi cGM}{\kappa_1\tilde{\eta}_c}\frac{k}{\mu H}\frac{3}{a}\frac{(1-\beta_s)^2}{\beta_s}T_c^S\,. \tag{87}$$

Comparing (77), (81), and (87) with the standard formulae in Eddington's theory, we see that the equations in that theory now become inequalities. This makes the conclusions drawn on the basis of the standard model have a "minimal" character which is of considerable physical importance.

If in addition to the conditions of Theorem 12 we assume that T decreases outward, then, according to Theorem 7 (II),

$$T_c > \frac{1}{5}\frac{\mu H}{k}\frac{GM}{R}\beta^* . \tag{88}$$

For the case of vanishing radiation pressure we can improve (88). For then (cf. the last row of Table 2)

$$T_c \geqslant 0.32\frac{\mu H}{k}\frac{GM}{R} . \tag{89}$$

We can now eliminate T_c between (87) and (88) or (89) and obtain an inequality of the same form as the luminosity formula used in current studies on gaseous stars.

The main Theorem 10 was conjectured by the writer over a year ago, but the fundamental idea in the proof as given in the text suggested itself only during a discussion with Professor J. von Neumann. It is a pleasure to record here my appreciation of the kind interest which Professor J. von Neumann has shown in this and other problems of the stellar interior. I am also indebted to Mr. E. Ebbighausen for his assistance in the numerical work connected with Tables 1 and 1*a*.

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