THE OPACITY IN THE INTERIOR OF A STAR

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ABSTRACT

In this paper two integral theorems on the radiative equilibrium of a gaseous star are proved.

I

In two recent papers[1] the author has proved some general theorems on the equilibrium of a star. These theorems are of some importance in the theory of stellar structures, in so far as they provide inequalities for the physical variables, e.g., central pressure, mean pressure, central radiation pressure $(1 - \beta_c)$, etc., which should be valid under very general circumstances. The method consists in obtaining inequalities for the physical variables which are direct consequences of the equation of hydrostatic equilibrium:

$$\frac{dP}{dr} = -\frac{GM(r)}{r^2}\rho . \tag{1}$$

In obtaining inequalities based on equation (1), one generally restricts one's self to such equilibrium configurations as are characterized by the mean density $\bar{\rho}(r)$, inside r decreasing outward. Further, in obtaining inequalities for the mean temperature a further restriction, namely, that $(1 - \beta)$ decreases outward, is introduced (cf. II, Theorem 7).

In this paper we shall obtain certain inequalities for equilibrium configurations in *radiative equilibrium*. We shall then have an additional differential equation for the radiation pressure $p_r(=\frac{1}{3}aT^4)$, namely,

$$\frac{dp_r}{dr} = -\frac{\kappa L(r)}{4\pi c r^2}\rho , \tag{2}$$

where κ is the opacity coefficient, c is the velocity of light, and $L(r)$ is the amount of energy crossing the spherical surface of radius r.

The numbering of the theorems is continued from II, and references to the equations of that paper are inclosed in square brackets.

[1] *M.N.*, **96**, 644, 1935; *Ap. J.*, **85**, 372, 1937. These papers will be referred to as "I" and "II," respectively.

II

In the following, L, M, and R refer to the luminosity, the mass, and the radius of a star; and the auxiliary variable η is defined by

$$\eta = \frac{L(r)}{M(r)} \Big/ \frac{L}{M} \, . \tag{3}$$

THEOREM 8.—*In a wholly gaseous configuration in radiative equilibrium, in which the mean density* $\bar{\rho}(r)$ *inside* r *decreases outward, we have*

$$L \leqslant \frac{4\pi c G M(1 - \beta^*)}{\overline{\kappa\eta}} \, , \tag{4}$$

where $(1 - \beta^*)$ *has the same meaning as in Theorem 2 and* $\overline{\kappa\eta}$ *is defined by*

$$P_c \overline{\kappa\eta} = \int_R^0 \kappa\eta dP \, . \tag{5}$$

Part of the analysis leading up to this theorem is originally due to B. Strömgren.[2]

Proof: The equation of radiative equilibrium (2) can be written as

$$dp_r = -\frac{\kappa}{4\pi^2 c} \frac{L(r) dM(r)}{r^4} \, . \tag{6}$$

Since

$$dP = -\frac{G}{4\pi} \frac{M(r) dM(r)}{r^4} \, , \tag{7}$$

we have, using equation (3), that

$$dp_r = \frac{L}{4\pi c G M} \kappa\eta dP \, . \tag{8}$$

Integrating equation (8) and using the boundary condition that $p_r = 0$ at $r = R$, we have

$$p_r = \frac{L}{4\pi c G M} \int_R^r \kappa\eta dP \, . \tag{9}$$

[2] *Handbuch der Astrophysik*, **8**, 159, 1936.

The foregoing equation has been given before by Strömgren. From equation (9) we can easily show that *if $\kappa\eta$ decreases outward,* $(1 - \beta)$ *would also decrease outward.* Extending the integral from 0 to R and using the definition (5) for the average value for $\kappa\eta$, we clearly have that

$$1 - \beta_c = \frac{L}{4\pi c G M}\,\overline{\kappa\eta}\,, \tag{10}$$

or

$$L = \frac{4\pi c G M(1 - \beta_c)}{\overline{\kappa\eta}}\,. \tag{11}$$

Since we have assumed that the mean density decreases outward, we can apply Theorem 2, which states that

$$1 - \beta_c \leq 1 - \beta^*\,, \tag{12}$$

where β^* satisfies a certain quartic equation (equation [3]) and is determined by the mass M, uniquely. Combining equations (11) and (12), we have

$$L \leqslant \frac{4\pi c G M(1 - \beta^*)}{\overline{\kappa\eta}}\,, \tag{13}$$

which proves the theorem.

III

THEOREM 9.—*In a wholly gaseous configuration in which the mean density* $\bar{\rho}(r)$ *inside* r *and the rate of generation of energy* ϵ *decrease outward, we have*

$$\bar{\kappa} \leqslant \frac{4\pi c G M(1 - \beta^*)}{L}\,, \tag{14}$$

where $\bar{\kappa}$ *is the mean opacity coefficient defined by*

$$P_c\bar{\kappa} = \int_R^0 \kappa dP\,, \tag{15}$$

and the equality sign in equation (14) is possible only when ϵ *is constant.*

Proof: This is an immediate consequence of Theorem 8. For, if ϵ decreases outward, η must also decrease outward, and consequently the *minimum* value of η is unity. Hence

$$\overline{\kappa\eta} \geqslant \bar{\kappa} , \tag{16}$$

the equality sign in equation (16) being possible only when $\eta =$ constant $= 1$, i.e., when ϵ is constant. By Theorem 8

$$\overline{\kappa\eta} \leqslant \frac{4\pi c G M(1 - \beta^*)}{L} . \tag{17}$$

Combining equations (16) and (17), we have the required result.

IV

We will apply equation (14) to certain practical cases of interest. Numerically, equation (14) reduces to

$$\bar{\kappa} \leqslant 1.318 \times 10^4 \frac{M}{\odot} \cdot \frac{L_\odot}{L} (1 - \beta^*) , \tag{18}$$

where $L_\odot$ refers to the luminosity of the sun.

For Capella we have $M = 4.18 \odot$ and $L = 126\ L_\odot$. Assuming $\mu = 1$, the solution of the quartic equation for β^* yields $1 - \beta^* = 0.22$. Hence

$$\bar{\kappa}_{\text{Capella}} < 96.1\ gm^{-1}\ cm^2 . \tag{19}$$

In the same way for the sun, we find ($\mu = 1$, $1 - \beta^* = 0.03$)

$$\bar{\kappa}_\odot < 395\ gm^{-1}\ cm^2 . \tag{20}$$

V

There is one interesting application of equation (14) to stellar models in which the opacity coefficient κ is assumed to be constant. For equation (14) can then be written as

$$L \leqslant L^* = \frac{4\pi c G M(1 - \beta^*)}{\kappa} . \tag{21}$$

Equation (21) has to be interpreted in the following sense: If $L > L^*$, then the configuration *must* be characterized by the occurrence of negative density gradients (i.e., $\bar{\rho}(r)$ increases outward in some finite regions of the interior), no matter what the law of energy generation is, provided only the rate of generation of energy ϵ decreases outward. On the other hand, if $L < L^*$, it does not *necessarily* follow that the configuration is characterized by positive density gradients throughout its interior. But if $L < L^*$ we can always find a "mild"-enough law for the rate of generation of energy such that the configuration is characterized by a positive density gradient throughout its interior. Further, it should be noticed that if

$$L > L_1 = \frac{4\pi cGM}{\kappa} , \tag{22}$$

then no equilibrium configuration is possible.[3] The inequality (22) is interpreted by the statement that if $L > L_1$ then the configuration would "blow up." We now see that this tendency to "blow up" must set in at lower values for the luminosity, in the event of negative density gradients in its interior. Negative density gradients must certainly exist for configurations with $L > L^*$. Depending on the concentration of the energy sources toward the center, the negative density gradients will set in for some $L < L^*$.

VI

Corollary to Theorem 9.—If, in addition to the conditions of Theorem 9, κ is assumed to increase outward, then

$$\kappa_c \leqslant \frac{4\pi cGM(1 - \beta^*)}{L} , \tag{23}$$

where κ_c is the opacity at the center. This is, of course, obvious.

If we assume any definite law for opacity, then equation (23) can be converted into an inequality for the central temperature, for a star of known mass and luminosity. Thus, if we assumed that

$$\kappa = \kappa_1 \frac{\rho}{T^{3+S}} , \qquad (S > 0) , \tag{24}$$

[3] E. A. Milne, *M.N.*, **91**, 4, 1930. See esp. pp. 12, 13, and 53 of this paper.

then κ would *increase* outward if $(1-\beta)$, and T would *decrease* outward; for we can write equation (24) as

$$\kappa = \kappa_1 \frac{\mu H}{k} \frac{a}{3} \frac{\beta}{1-\beta} T^{-S} . \tag{25}$$

Hence, by Theorem 2,

$$\kappa_c \geqslant \kappa_1 \frac{\mu H}{k} \frac{a}{3} \frac{\beta^*}{1-\beta^*} T_c^{-S} . \tag{26}$$

Combining this with (23), we have

$$T_c^S \geqslant \frac{L\kappa_1}{4\pi cGM} \frac{\mu H}{k} \frac{a}{3} \frac{\beta^*}{(1-\beta^*)^2} . \tag{27}$$

The foregoing inequality giving the minimum central temperature for a star of known M and L is generally not as good as the minimum central temperature set by Theorem 7 for a star of known M and R.

APPENDIX

We have seen that $(1-\beta^*)$, giving the maximum possible $(1-\beta_c)$ in a wholly gaseous configuration in which the mean density $\bar{\rho}(r)$, inside r, is assumed to decrease outward, plays an important role in Theorems 7, 8, and 9. It is therefore convenient to have a table giving M for different values of $(1-\beta^*)$. Table I should be sufficient for most purposes.

TABLE 1

SOLUTIONS OF THE EQUATION

$$M = \left(\frac{6}{\pi}\right)^{1/2} \left[\left(\frac{k}{\mu H}\right)^4 \frac{3}{a} \frac{1-\beta^*}{\beta^{*4}} \right]^{1/2} \frac{1}{G^{3/2}}$$

$1-\beta^*$	$\left(\frac{M}{\odot}\right)\mu^2$	$1-\beta^*$	$\left(\frac{M}{\odot}\right)\mu^2$
0.025	0.908	0.5	15.432
.05	1.352	0.6	26.41
.1	2.130	0.7	50.72
.2	3.812	0.8	122.0
.3	6.099	0.9	517.6
0.4	9.585	1.0	....

YERKES OBSERVATORY
June 8, 1937