ON THE STABILITY OF BINARY SYSTEMS

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ABSTRACT

In this paper the theory of spatial correlations developed in an earlier paper is applied to the discussion of the stability of binary stars, and a formula for estimating the time of dissolution, τ, of a binary is derived. It is found that

$$\tau \simeq \frac{(M_1 + M_2)^{1/2}}{4\pi G^{1/2} M N a^{3/2}},$$

where M_1 and M_2 denote the masses of the two components of the binary system, M an average mass of the field stars, N the number of stars per unit volume, and a the semimajor axis of the relative orbit. According to the foregoing formula, the time of dissolution of binaries with semimajor axes in the range of 10^3–10^4 astronomical units can be estimated to lie in the interval 7×10^{10} to 2×10^9 years. Since there is evidence that the distribution of energies of the binary systems does not conform to Boltzmann's law (Ambarzumian), the present theory would lead one to conclude that the statistics of binary stars lends further weight to the adoption of a time scale in the neighborhood of 3×10^9 years. Some additional remarks on the related problems of the common proper-motion stars and on cometary orbits which are very close to the parabolic limit are also made.

1. *Introduction.*—As Jeans[1] has emphasized on various occasions, the statistics of binary stars should provide an important basis for drawing conclusions concerning the time scale appropriate for the Galaxy. In Jeans's own considerations relating to this problem, stress is laid on a comparison of the observed frequencies of occurrence of the different periods and eccentricities with those predicted under conditions of thermal equilibrium. Thus, under conditions of thermal equilibrium[2] the number of binaries with eccentricity less than e will be proportional to e^2; and, according to Jeans, the agreement of the data of observations with this e^2-law is "far too good to be accidental." On the strength principally of this evidence[3] Jeans concludes that the time scale of the Galaxy must be long enough to allow for the establishment of conditions approaching those of equipartition among the parameters of a binary. However, an important lacuna in Jeans's argument, and one which appears to vitiate his principal conclusions, was first pointed out by Ambarzumian,[4] who showed that the e^2-law can be derived under conditions far more general than those contemplated under thermal equilibrium. Indeed, Ambarzumian proved that *for any arbitrary distribution in the phase space of the binary which is a function of the energy* E *of the binary system only, the number of systems with eccentricity less than* e *will be proportional to* e². In other words, the validity of the e^2-law is a necessary but not a sufficient condition for the existence of thermal equilibrium. Accordingly, the agreement of the known statistics of binary systems with the e^2-law can be regarded as evidence only for supposing that the distribution function in the phase space has the form $f(E)$; it cannot be regarded as evidence for the special form of the distribution function, namely, $\exp(-E/\theta)$, demanded when conditions of thermal equilibrium obtain, and neither does it provide evidence for supposing that conditions approximating equipartition among the parameters of the binary have been established.

[1] *M.N.*, **79**, 100 and 408, 1919; *Astronomy and Cosmogony*, pp. 323–326, Cambridge, England, 1929; *Nature*, **136**, 432, 1935.

[2] As we shall presently indicate, the result we are about to quote is actually valid under conditions far more general than those of thermal equilibrium.

[3] See, e.g., *Nature*, **136**, 432, 1935.

[4] *Russian Astr. J.*, **14**, 207, 1937. See also *Nature*, **137**, 537, 1936.

On the other hand, as Ambarzumian has shown, there exist other features of the known statistics (e.g., the distribution of energies as inferred from the distribution of the semi-major axis) which quite definitely contradict the assumption that conditions even remotely resembling those of thermal equilibrium have been attained.[5] We may therefore conclude with Ambarzumian that the time scale of the Galaxy *cannot* be so long that Boltzmann's law describes the distribution in the phase space of the binary. It is now clear that, in order that we may go beyond this point and draw specific conclusions concerning the time scale itself, it is necessary that we make some estimate of the effectiveness of the tidal forces, due to the neighboring stars, in modifying the orbital elements of a binary. It is in the details of the estimation of these latter effects that Ambarzumian's considerations appear to lose rigor. Thus, he uses the two-body approximation of stellar encounters to evaluate the *tidal* effects of the near-by stars. But it is apparent that the essentially characteristic features of the problem are ignored if an attempt is made to evaluate the *differential* effects of the neighboring stars on the components of a binary along the conventional lines of treating stellar encounters as a series of independent two-body problems. However, it appears that the theory of spatial correlations developed in an earlier paper[6] provides an adequate basis for incorporating in the dynamics of binary systems the fluctuating gravitational field always present on account of the continually changing complexion of the local stellar distribution. It is the object of this paper to take a first step in this direction.

2. *On the rate of dissolution of a binary.*—As we have already indicated in § 1, the orbital elements of a binary system will be subject to changes on account of the differential effect of the neighboring stars on the two components of the system. Thus, for any given separation between the two stars, there exists a definite distribution function, $W(\boldsymbol{F}_0, \boldsymbol{F}_1)$, governing the probability that forces of intensities $\boldsymbol{F}_0$ and $\boldsymbol{F}_1$, respectively, will act simultaneously on the two components of the system. In other words, a differential acceleration governed by a definite probability law will operate on the system, which will tend to accelerate one star relative to the other. Thus,

$$\Delta \boldsymbol{F} = \boldsymbol{F}_0 - \boldsymbol{F}_1 \tag{1}$$

represents the amount by which the star "0" will be accelerated relative to the star "1." More particularly, the component

$$\Delta F_{||} = (\boldsymbol{F}_0 - \boldsymbol{F}_1) \cdot \mathbf{1}_{\boldsymbol{F}_0}\,, \tag{2}$$

where $\mathbf{1}_{\boldsymbol{F}_0}$ is a unit vector parallel to the direction of $\boldsymbol{F}_0$, represents the amount by which the star "0" will be *systematically* accelerated relative to the star "1." It is clear that it is this systematic acceleration of one star relative to the other which will chiefly be responsible for such changes in the orbital elements as may be effected. For a component of $\Delta \boldsymbol{F}$ in a direction at right angles to the direction of $\boldsymbol{F}_0$ represents a differential acceleration of the star "0" relative to the star "1," which (in contrast to $\Delta F_{||}$) is of a *random* character. Accordingly, the average net increase in the velocity of the star "0" relative to the star "1" during a time Δt long, compared to the periods of the elementary fluctuations in $\boldsymbol{F}_0$ will be given by

$$\overline{\Delta v_{0,\,1}} = \overline{\Delta F_{||}}\Delta t\,. \tag{3}$$

Now, if the relative orbit is an eccentric one, we should allow for the fact that, on account of the changing separation between the two stars, the parameter in the distribution function $W(\boldsymbol{F}_0, \boldsymbol{F}_1)$ will also change. However, in this paper we shall restrict ourselves to

[5] In spite of the conclusiveness of Ambarzumian's arguments it is unfortunate that his very significant discussion of this problem has so far remained ignored. (See, e.g., H. N. Russell, *Science*, **92**, 19, 1940.)

[6] See p. 25 in this issue. This paper will be referred to as III.

the case of circular orbits, so that the separation remains constant. Under these conditions equation (3) can be written alternatively in the form

$$\overline{\Delta v_{0,\,1}} = \Delta t \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (\boldsymbol{F}_0 - \boldsymbol{F}_1) \cdot \mathbf{1}_{\boldsymbol{F}_0} W(\boldsymbol{F}_0, \boldsymbol{F}_1)\, d\boldsymbol{F}_0 d\boldsymbol{F}\ , \tag{4}$$

or, in the notation of III, equation (99),

$$\overline{\Delta v_{0,\,1}} = (\overline{|\boldsymbol{F}_0|} - \overline{\overline{F}}_{1,\,\boldsymbol{F}_0})\,\Delta t\,. \tag{5}$$

If we measure $|\boldsymbol{F}|$ in units of

$$Q_H = (\tfrac{4}{15})^{2/3} 2\pi G M N^{2/3} \tag{6}$$

(where M denotes a certain average mass for the field stars and N the number of stars per unit volume), equation (5) can be expressed as (cf. III, eqs. [100] and [126])

$$\overline{\Delta v_{0,\,1}} = Q_H \left[\frac{4}{\pi}\Gamma(\tfrac{1}{3}) - \overline{\overline{\beta}}_{\boldsymbol{\beta}}(s)\right]\Delta t\,. \tag{7}$$

In equation (7) s denotes the constant separation between the two stars expressed in units of (III, eq. [72])

$$l = \frac{15^{1/3}}{4^{1/3}(2\pi)^{1/2}} N^{-1/3}\,. \tag{8}$$

As we shall presently verify (in § 3, below), the case of greatest practical interest arises when the separation between the two components of the binary system is a small fraction of the average distance between the stars. We may then use the asymptotic expansion (III, eq. [123]):

$$\overline{\overline{\beta}}_{\boldsymbol{\beta}}(s) = \frac{4}{\pi}\Gamma(\tfrac{1}{3}) - \frac{15}{4}\left(\frac{2}{\pi}\right)^{1/2} s + O(s^4)\,. \tag{9}$$

For values of $s < \frac{1}{2}$ we can therefore replace equation (7) to a sufficient approximation by

$$\overline{\Delta v_{0,\,1}} = \frac{15}{4}\left(\frac{2}{\pi}\right)^{1/2} Q_H s \Delta t\,. \tag{10}$$

If a denotes the separation between the two stars,

$$a = sl\,. \tag{11}$$

Accordingly, equations (7), (8), (10), and (11) can be combined to give

$$\overline{\Delta v_{0,\,1}} = 4\pi G M N a \Delta t\,. \tag{12}$$

The foregoing equation for $\overline{\Delta v_{0,\,1}}$ may be regarded as the fundamental formula of the present theory.

Now the mean square velocity of a star in its relative binary orbit is given by

$$\overline{V^2} = \frac{G(M_1 + M_2)}{a}\,, \tag{13}$$

where M_1 and M_2 denote the masses of the two components and a the semimajor axis of the orbit. We may therefore *define* the time of dissolution τ of a binary by the equation

$$\overline{\Delta v_{0,\,1}} = \sqrt{\frac{G(M_1 + M_2)}{a}}\,. \tag{14}$$

Using equation (12) for $\overline{\Delta v_{0,1}}$, we have

$$\tau \simeq \frac{(M_1+M_2)^{1/2}}{4\pi G^{1/2} M N a^{3/2}} . \tag{15}$$

Comparing this formula with the one given by Ambarzumian,[7] we notice that the two are of altogether different forms. In particular, we may draw attention to the fact that, while Ambarzumian's formula contains the mean velocity of the stars as a factor, equation (14) contains no term in the velocities.

3. *Numerical applications.*—Expressing the mass in solar units, N as numbers per cubic parsec, and the separation a in astronomical units, the formula (15) for the time of dissolution of a binary reduces to

$$\tau = 1.11 \times 10^{14} \frac{(M_1+M_2)^{1/2}}{M N a^{3/2}} \text{ years} . \tag{16}$$

In the neighborhood of the sun

$$N = 0.1 \quad \text{and} \quad M = 0.5 ; \tag{17}$$

and, setting

$$M_1 + M_2 = 1 , \tag{18}$$

equation (16) becomes

$$\tau = 2.22 \times 10^{15} a^{-3/2} \text{ years} . \tag{19}$$

Table 1 is derived from this formula.

Since the average distance between the stars is of the order of 3 parsecs, a separation of 60,000 astronomical units still corresponds to only a tenth of the average distance between the stars. This fully justifies the use of the expansion (9) in equation (7).

TABLE 1

THE TIME OF DISSOLUTION OF BINARIES

(N = 0.1 Stars/(Parsec)^3; M = 0.5$\odot$; $M_1 + M_2$ = $\odot$)

a (Astronomical Units)	τ (Years)	a (Astronomical Units)	τ (Years)
1000	7.0×10^{10}	10,000	2.2×10^9
2000	2.5×10^{10}	20,000	7.8×10^8
4000	8.8×10^9	40,000	2.8×10^8
6000	4.8×10^9	60,000	1.5×10^8

From Table 1 it is apparent that the time of dissolution of binaries with semimajor axes between 10^3 and 10^4 astronomical units lies in the range of 7×10^{10} to 2×10^9 years. And, since, according to Ambarzumian, we can infer from the distribution of the semimajor axis of binaries that conditions of equipartition have not been attained in the Galaxy, we may conclude that the time scale cannot exceed 3×10^9 years by any very large factor. In other words, "the data of double star astronomy do not support the long time scale hypothesis" (Ambarzumian).

4. *On common proper-motion stars.*—The existence of common proper-motion stars with relatively wide separations has been known for some time. Thus, in a recent paper, Kuiper[8] lists three pairs of common proper-motion stars with separations exceeding 10^4 astronomical units. According to Table 1, the time of dissolution of binaries with such separations is appreciably less than 3×10^9 years, and the question now arises as to how justifiable it is to class such common proper-motion stars among the binaries.

[7] *Russian Astr. J.*, **14**, 207, 1937. The formula for the time of dissolution is given on p. 216 (eq. [14]).

[8] *Ap. J.*, **95**, 201, 1942 (see particularly Table 5 in this paper).

To take a specific case, consider the pair $-32°$ 16135 A and $-31°$ 17815 for which Kuiper estimates a separation of 3.3×10^4 astronomical units. Now a binary with this separation will be disrupted in a time of the order of 4×10^8 years. What this means is simply this: In a time of this duration one of the components will be accelerated relative to the other by an amount which would make the kinetic energy of motion exceed the binding energy. Accordingly, we shall not be justified in regarding them *dynamically* as binaries. But this does *not* imply that the two stars will have widely different space motions in a chosen local standard of rest; for this to happen, we shall have to wait for a time of the order of the time of relaxation of the system, which is of an altogether different order of magnitude. It appears, therefore, that we cannot strictly regard common proper-motion stars with separations exceeding 10^4 astronomical units as binaries: they are more appropriately regarded as the residue of disrupted binaries. In other words, they are what they are: common proper-motion stars!

5. *On cometary orbits which are very close to the parabolic limit.*—It has been known that cometary orbits with eccentricities extremely near unity exist. Thus, Van Biesbroeck[9] finds that the comet Delavan ($1913f = 1914V$) had an eccentricity of 0.9999781 when it *entered* the outermost parts of the solar system. This implies that the semimajor axis of the orbit is in the neighborhood of 10^5 astronomical units, at which distance the tidal disturbances due to the near-by stars must be appreciable (see Table 1). Indeed, the very existence of such orbits (if fully substantiated) presents some exceedingly delicate considerations. It should, however, be borne in mind that comets such as comet Delavan do not remain *all* the time at distances of the order of 10^5 astronomical units from the sun, though they would, for an appreciable fraction of their periods ($\sim 2.4 \times 10^7$ years). It will therefore be of very great interest to see if cometary orbits as close to the parabolic limit as the orbit of comet Delavan will *leave* the solar system with eccentricities appreciably different from those with which they entered the system; to settle this question, the orbits would have to be integrated *forward* instead of backward, as has been customary.

[9] *Yerkes Obs. Pub.*, 5, Part II, 34, 1927. The writer is greatly indebted to Professor Van Biesbroeck for discussions on these and related matters.