THE STATISTICS OF THE GRAVITATIONAL FIELD ARISING FROM A RANDOM DISTRIBUTION OF STARS

IV. THE STOCHASTIC VARIATION OF THE FORCE ACTING ON A STAR

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ABSTRACT

In this paper the theory of the stochastic variation of the force acting on a star is considered, and the solution to the formal problem is obtained in terms of the average force per unit mass $\boldsymbol{F}_t$ acting at time t, given that a force of intensity $\boldsymbol{F}_0$ acted at time $t = 0$. Various related quantities are also considered, and in particular an explicit formula for the correlation coefficient $R(\boldsymbol{F}_0, t)$ is derived.

1. *Introduction.*—A basic problem in statistical stellar dynamics is the characterization of the entire stochastic variation of the gravitational force acting on a star. And, as we have already explained in the introductory section to the preceding paper,[1] this requires the specification of the average force per unit mass $\boldsymbol{F}_t$ acting on a star at time t, given that a force of intensity $\boldsymbol{F}_0$ acted at time $t = 0$. In other words, the essential physical quantity which is needed concerns the correlation in the forces acting on a star at two different instants of time. In this paper we propose to present the formal parts of this theory. In a later paper we shall undertake a fuller discussion of the various formulae derived and also outline the applications of the theory developed here.

2. *The first moments of* $\mathbf{F}_t$.—As in III, § 2, the solution to the formal problem soon reduces to one of evaluating the characteristic function $C(\boldsymbol{\rho}, \boldsymbol{\sigma})$, associated with the distribution function $W(\boldsymbol{F}_0, \boldsymbol{F}_t)$ governing the probability that forces of intensities $\boldsymbol{F}_0$ and $\boldsymbol{F}_t$ will act, respectively, at times $t = 0$ and $t = t$. Similar to III, equation (7), we now have

$$C(\boldsymbol{\rho}, \boldsymbol{\sigma}) = N\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left[1 - e^{iGM\left\{\frac{\boldsymbol{r}\cdot\boldsymbol{\rho}}{|\boldsymbol{r}|^3} + \frac{(\boldsymbol{r}+\boldsymbol{V}t)\cdot\boldsymbol{\sigma}}{|\boldsymbol{r}+\boldsymbol{V}t|^3}\right\}}\right]\chi(\boldsymbol{V})\, d\boldsymbol{r}\, d\boldsymbol{V}\,, \qquad (1)$$

where $\chi(\boldsymbol{V})$ denotes the probability distribution of the *relative velocity* $\boldsymbol{V}$. In writing equation (1) we have assumed (as in III) that all the stars have the same mass M. (The generalization necessary to allow for a distribution over the masses is straightforward and will be indicated later.) The foregoing equation can be re-written in the form (cf. III, eq. [9])

$$\left.\begin{aligned} C(\boldsymbol{\rho}, \boldsymbol{\sigma}) &= \tfrac{4}{15}(2\pi GM)^{3/2}N|\boldsymbol{\rho}|^{1/2} \\ &\quad + N\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{iGM\boldsymbol{r}\cdot\boldsymbol{\rho}/|\boldsymbol{r}|^3}\left[1 - e^{iGM(\boldsymbol{r}+\boldsymbol{V}t)\cdot\boldsymbol{\sigma}/|\boldsymbol{r}+\boldsymbol{V}t|^3}\right]\chi(\boldsymbol{V})\, d\boldsymbol{r}\, d\boldsymbol{V}\,; \end{aligned}\right\} \qquad (2)$$

and, since we are interested only in the first moments of $\boldsymbol{F}_t$, it would be sufficient to examine the behavior of $\text{grad}_{\boldsymbol{\sigma}}\, C(\boldsymbol{\rho}, \boldsymbol{\sigma})$ for $|\boldsymbol{\sigma}| \to 0$. Accordingly, we express $C(\boldsymbol{\rho}, \boldsymbol{\sigma})$ in the form

$$C(\boldsymbol{\rho}, \boldsymbol{\sigma}) = \tfrac{4}{15}(2\pi GM)^{3/2}N|\boldsymbol{\rho}|^{1/2} + \mathfrak{D}(\boldsymbol{\rho}, \boldsymbol{\sigma}) + O(|\boldsymbol{\sigma}|^2) \qquad (3)$$

[1] P. 25. This paper will be referred to as III. The earlier papers, *Ap. J.*, **95**, 489, 1942, and **97**, 1, 1943, will be referred to as I and II, respectively.

and verify that

$$\mathfrak{D}(\boldsymbol{\rho}, \boldsymbol{\sigma}) = \int_{-\infty}^{+\infty} D(\boldsymbol{\rho}, \boldsymbol{\sigma}; Vt)\, \chi(V)\, dV, \tag{4}$$

where

$$D(\boldsymbol{\rho}, \boldsymbol{\sigma}; Vt) = -iGMN \int_{-\infty}^{+\infty} e^{iGMr\cdot\boldsymbol{\rho}/|r|^3} \frac{\boldsymbol{\sigma}\cdot(r+Vt)}{|r+Vt|^3}\, dr. \tag{5}$$

It is seen that our present definition of $D(\boldsymbol{\rho}, \boldsymbol{\sigma}; Vt)$ agrees with our earlier definition of $D(\boldsymbol{\rho}, \boldsymbol{\sigma})$ in III (eq. [12]) with $+Vt$ replacing $-r_1$. Hence (cf. III, eqs. [38] and [41]),

$$D(\boldsymbol{\rho}, \boldsymbol{\sigma}; Vt) = -\sigma_3 \sum_{l=0}^{\infty} A_l(|\boldsymbol{\rho}|; z_1) P_l(\mu) - \sigma_1 \sum_{l=1}^{\infty} B_l(|\boldsymbol{\rho}|; z_1) P_l^1(\mu), \tag{6}$$

where σ_3 and σ_1 are the components of $\boldsymbol{\sigma}$ in a Cartesian system of co-ordinates in which the z-axis is in the direction of $\boldsymbol{\rho}$ and the xz-plane contains the vector V and where

$$\mu = \cos\vartheta; \qquad \vartheta = \sphericalangle(\boldsymbol{\rho}, V), \tag{7}$$

$$\left.\begin{aligned} A_l = (-i)^l 2^{1/2}\pi^{3/2}(GM)^{3/2}N|\boldsymbol{\rho}|^{1/2}\int_0^\infty dz\, \xi^{l/2}\frac{\tilde{z}^{1/2}}{z^{3/2}}\Big[&-\frac{3l(l-1)}{(2l+1)(2l-1)}J_{l-3/2} \\ &+\frac{2l(l+1)}{(2l+3)(2l-1)}J_{l+1/2}-\frac{3(l+2)(l+1)}{(2l+3)(2l+1)}J_{l+5/2}\Big], \end{aligned}\right\} \tag{8}$$

and

$$\left.\begin{aligned} B_l = (-i)^l 2^{1/2}\pi^{3/2}(GM)^{3/2}N|\boldsymbol{\rho}|^{1/2}\int_0^\infty dz\, \xi^{l/2}\frac{\tilde{z}^{1/2}}{z^{3/2}}\Big[&-\frac{3(l-1)}{(2l+1)(2l-1)}J_{l-3/2} \\ &+\frac{3}{(2l+3)(2l-1)}J_{l+1/2}+\frac{3(l+2)}{(2l+1)(2l+3)}J_{l+5/2}\Big]. \end{aligned}\right\} \tag{9}$$ [2]

Further, in the foregoing equations (cf. III, eq. [29])

$$z_1 = \frac{GM|\boldsymbol{\rho}|}{|V|^2 t^2}. \tag{10}$$

And, finally, we may also recall that the integrals defining A_l and B_l should be broken at $z = z_1$ according to the scheme III, equations (31) and (32).

To evaluate $\mathfrak{D}(\boldsymbol{\rho}, \boldsymbol{\sigma})$, we have first to refer the solution (6) to a system of co-ordinates which is independent of the direction of V. Letting the z-axis still point in the direction of $\boldsymbol{\rho}$, we can write

$$D(\boldsymbol{\rho}, \boldsymbol{\sigma}; Vt) = -\sigma_3 \sum_{l=0}^{\infty} A_l P_l(\mu) - (\sigma_1\cos\varphi + \sigma_2\sin\varphi) \sum_{l=1}^{\infty} B_l P_l^1(\mu), \tag{11}$$

where φ denotes the azimuth of the meridian plane containing the vector V.

Now, let $\boldsymbol{u}$ and $\boldsymbol{v}$ denote, respectively, the velocities of a typical field star and the star under consideration in an appropriately chosen local standard of rest. Then

$$V = \boldsymbol{u} - \boldsymbol{v}. \tag{12}$$

[2] It will be noticed that our present definitions of A_l and B_l agree with those given in III, eqs. [39] and [40], except that $-i$ now replaces i. This difference can be traced to the circumstance that V and r_1 occur with opposite signs in the relevant equations.

It would be natural to suppose that the distribution of the "peculiar" velocities $\boldsymbol{u}$ is spherical (cf. II, eqs. [58] and [59]). While it would be feasible to work with a general spherical distribution of the velocities $\boldsymbol{u}$, in the present investigation we shall assume, for the sake of simplicity, that the distribution of the velocities $\boldsymbol{u}$ is actually Gaussian. And it is evident that a Gaussian distribution of the velocities $\boldsymbol{u}$ implies that $\chi(V)$ has the form

$$\chi(V) = \frac{j^3}{\pi^{3/2}} e^{-j^2|V+v|^2}, \tag{13}$$

where j is a certain parameter related to the mean-square residual velocities of the stars. Using the foregoing form for $\chi(V)$ in equation (4) and changing to polar co-ordinates, we have

$$\left.\begin{aligned}\mathfrak{D}(\boldsymbol{\rho}, \boldsymbol{\sigma}) = \frac{j^3}{\pi^{3/2}} \int_0^\infty \int_{-1}^{+1} \int_0^{2\pi} & e^{-j^2(|V|^2+|v|^2+2|V||v| \cos \Theta)} \\ & \times D(\boldsymbol{\rho}, \boldsymbol{\sigma}; Vt)\, |V|^2 d|V|\, d\mu\, d\varphi ,\end{aligned}\right\} \tag{14}$$

where Θ denotes the angle between the directions of $\boldsymbol{v}$ and $\boldsymbol{V}$. If the xz-plane of our co-ordinate system is now assumed to contain the vector $\boldsymbol{v}$, then

$$\cos \Theta = \cos \vartheta \cos \vartheta_1 + \sin \vartheta \sin \vartheta_1 \cos \varphi , \tag{15}$$

where

$$\vartheta = \sphericalangle (\boldsymbol{\rho}, V) ; \qquad \vartheta_1 = \sphericalangle (\boldsymbol{\rho}, \boldsymbol{v}) . \tag{16}$$

Writing

$$j|V| = \lambda \quad \text{and} \quad j|\boldsymbol{v}| = \nu , \tag{17}$$

we find equation (16) taking the form

$$\left.\begin{aligned}\mathfrak{D}(\boldsymbol{\rho}, \boldsymbol{\sigma}) = -\frac{1}{\pi^{3/2}} \int_0^\infty \int_{-1}^{+1} \int_0^{2\pi} & d\lambda\, d\mu\, d\varphi\, \lambda^2 e^{-(\lambda^2+\nu^2+2\lambda\nu \cos \Theta)} \\ & \times \left[\sigma_3 \sum_{l=0}^{\infty} A_l P_l(\mu) + (\sigma_1 \cos \varphi + \sigma_2 \sin \varphi) \sum_{l=1}^{\infty} B_l P_l^1(\mu) \right] ,\end{aligned}\right\} \tag{18}$$

where we have further substituted for $D(\boldsymbol{\rho}, \boldsymbol{\sigma}; Vt)$ from equation (11). The integrations over μ and φ in equation (18) can be performed by first expanding $\exp(-2\lambda\nu \cos \Theta)$ in tesseral harmonics. We have[3]

$$\left.\begin{aligned}e^{-2\lambda\nu \cos \Theta} = \left(\frac{\pi}{4\lambda\nu}\right)^{1/2} & \sum_{m=0}^{\infty} (-1)^m (2m+1) I_{m+1/2}(2\lambda\nu) \\ & \times \left\{ P_m(\mu) P_m(\mu_1) + 2 \sum_{n=1}^{m} \frac{(m-n)!}{(m+n)!} P_n^m(\mu) P_n^m(\mu_1) \cos m\varphi \right\} ,\end{aligned}\right\} \tag{19}$$

where the I's are the Bessel functions for purely imaginary arguments.[4] Introducing the foregoing expansion into equation (18) and using the orthogonality properties of the Legendre functions, we find that $\mathfrak{D}(\boldsymbol{\rho}, \boldsymbol{\sigma})$ reduces to

$$\left.\begin{aligned}& \mathfrak{D}(\boldsymbol{\rho}, \boldsymbol{\sigma}) = -\int_0^\infty d\lambda \frac{4}{\pi^{1/2}} e^{-(\lambda^2+\nu^2)} \lambda^2 \\ & \times \left(\frac{\pi}{4\lambda\nu}\right)^{1/2} \left\{ \sigma_3 \sum_{l=0}^{\infty} (-1)^l I_{l+1/2} A_l P_l(\mu_1) + \sigma_1 \sum_{l=1}^{\infty} (-1)^l I_{l+1/2} B_l P_l^1(\mu_1) \right\} .\end{aligned}\right\} \tag{20}$$

[3] See, e.g., G. N. Watson, *Theory of Bessel Functions*, p. 369, Cambridge, England, 1922.

[4] For the definition of these functions see *ibid.*, pp. 77–80.

We can thus express $\mathfrak{D}(\boldsymbol{\rho}, \boldsymbol{\sigma})$ in the form

$$\mathfrak{D}(\boldsymbol{\rho}, \boldsymbol{\sigma}) = -\sigma_3 \sum_{l=0}^{\infty} \mathfrak{A}_l(|\boldsymbol{\rho}|; \nu; \tau) P_l(\mu_1) - \sigma_1 \sum_{l=1}^{\infty} \mathfrak{B}_l(|\boldsymbol{\rho}|; \nu; \tau) P_l^1(\mu_1) , \quad (21)$$

where we have introduced the quantities $\mathfrak{A}_l$ and $\mathfrak{B}_l$, defined by

$$\mathfrak{A}_l = (-1)^l \frac{4}{\pi^{1/2}} \int_0^{\infty} e^{-(\lambda^2+\nu^2)} \lambda^2 \left(\frac{\pi}{4\lambda\nu}\right)^{1/2} I_{l+1/2}(2\lambda\nu) A_l(|\boldsymbol{\rho}|; \lambda\tau)\, d\lambda \quad (22)$$

and

$$\mathfrak{B}_l = (-1)^l \frac{4}{\pi^{1/2}} \int_0^{\infty} e^{-(\lambda^2+\nu^2)} \lambda^2 \left(\frac{\pi}{4\lambda\nu}\right)^{1/2} I_{l+1/2}(2\lambda\nu) B_l(|\boldsymbol{\rho}|; \lambda\tau)\, d\lambda , \quad (23)$$

where we have used τ to denote the time measured in the unit

$$t_0 = jl ; \qquad t = t_0 \tau , \quad (24)$$

l denoting the unit of length introduced in III, equation (72), so that (cf. III, eq. [73])

$$z_1 = \frac{x}{\lambda^2 \tau^2 \beta} . \quad (25)$$

It is now seen that $\mathfrak{D}(\boldsymbol{\rho}, \boldsymbol{\sigma})$ has exactly the same form as $D(\boldsymbol{\rho}, \boldsymbol{\sigma})$ in the theory of spatial correlations (cf. eq. [21] with III, eq. [38]). We can therefore write down at once the expressions for all the first moments of $\boldsymbol{F}_t$. Thus, choosing a co-ordinate system in which the ζ-axis is in the direction of $\boldsymbol{v}$ (i.e., the direction of motion) and the $\xi\zeta$-plane contains the vector $\boldsymbol{F}_0$ (see III, Fig. 1), we have (cf. III, eqs. [60], [65], and [66])

$$\overline{F}_{t,\xi} = -\frac{2i}{\pi\beta H(\beta)} \left(\frac{\pi}{2}\right)^{1/2} \sum_{n=1}^{\infty} (-i)^n P_n^1(\mu) \int_0^{\infty} e^{-(x/\beta)^{3/2}} x^{3/2} J_{n+1/2}(x) \times [\mathfrak{C}_{n-1} - \mathfrak{D}_{n+1}]\, dx , \quad (26)$$

$$\overline{F}_{t,\eta} = 0 , \quad (27)$$

and

$$\overline{F}_{t,\zeta} = -\frac{2i}{\pi\beta H(\beta)} \left(\frac{\pi}{2}\right)^{1/2} \sum_{n=0}^{\infty} (-i)^n P_n(\mu) \int_0^{\infty} e^{-(x/\beta)^{3/2}} x^{3/2} J_{n+1/2}(x) \times [n\mathfrak{C}_{n-1} + (n+1)\mathfrak{D}_{n+1}]\, dx , \quad (28)$$

where

$$\mathfrak{C}_l = \frac{1}{2l+1}\{\mathfrak{A}_l - l\mathfrak{B}_l\} \quad \text{and} \quad \mathfrak{D}_l = \frac{1}{2l+1}\{\mathfrak{A}_l + (l+1)\mathfrak{B}_l\} \quad (29)$$

and

$$\mu = \cos\vartheta ; \qquad \vartheta = \measuredangle(\boldsymbol{v}, \boldsymbol{F}_0) . \quad (30)$$

Combining equations (8), (9), (22), and (23), we find that the formulae defining $\mathfrak{C}_l$ and $\mathfrak{D}_l$ are explicitly (cf. III, eqs. [68] and [69])

$$\mathfrak{C}_l = i^l \frac{15 Q_H}{8(2l+3)(2l+1)} \left(\frac{x}{\beta}\right)^{1/2} \int_0^{\infty} d\lambda \frac{4}{\pi^{1/2}} e^{-(\lambda^2+\nu^2)} \lambda^2 \left(\frac{\pi}{4\lambda\nu}\right)^{1/2} I_{l+1/2}(2\lambda\nu) \times \int_0^{\infty} dz\, \xi^{l/2} \frac{\tilde{z}^{1/2}}{z^{3/2}} \{lJ_{l+1/} - 3(l+2) J_{l+5/2}\} \quad (31)$$

and

$$\left.\begin{aligned}\mathfrak{D}_l = i^l \frac{15 Q_H}{8(2l-1)(2l+1)} \left(\frac{x}{\beta}\right)^{1/2} \int_0^\infty d\lambda \frac{4}{\pi^{1/2}} e^{-(\lambda^2+\nu^2)} \lambda^2 \left(\frac{\pi}{4\lambda\nu}\right)^{1/2} I_{l+1/2}(2\lambda\nu) \\ \times \int_0^\infty dz\, \xi^{l/2} \frac{z^{1/2}}{z^{3/2}} \{-3(l-1) J_{l-3/2} + (l+1) J_{l+1/2}\} .\end{aligned}\right\} \quad (32)$$

With this we have formally solved the problem.

It is of interest to note that, since

$$I_{l+1/2}(2\lambda\nu) \to \frac{1}{\Gamma(l+\frac{3}{2})} (\lambda\nu)^{l+1/2} \qquad (\nu \to 0) , \quad (33)$$

it follows that

$$\left(\frac{\pi}{4\lambda\nu}\right)^{1/2} I_{l+1/2}(2\lambda\nu) \to \frac{\pi^{1/2}}{2\Gamma(l+\frac{3}{2})} (\lambda\nu)^l \qquad (\nu \to 0) . \quad (34)$$

Hence, for $\nu = 0$ all the functions $\mathfrak{C}_l$ and $\mathfrak{D}_l$ except $\mathfrak{C}_0$ vanish identically. This implies (cf. § 3 below) that for $|\boldsymbol{v}| = 0$ the only nonvanishing moment of $\boldsymbol{F}_t$ is in the direction of $\boldsymbol{F}_0$. In other words, the problem of the stochastic variation of the force acting on a "fixed" point in a system containing stars in random motion is exceptionally simple.

3. *The first moment of* $\mathbf{F}_t$ *in the direction of* $\mathbf{F}_0$ *and its averages*.—A quantity of considerable interest is the average value of $\boldsymbol{F}_t$ in the direction of $\boldsymbol{F}_0$. Analogous to III, equation (76), we now have

$$\left.\begin{aligned}\overline{F}_{t,\,\boldsymbol{F}_0} = -\frac{2i}{\pi\beta H(\beta)} \left(\frac{\pi}{2}\right)^{1/2} \Bigg\{ \sum_{n=1}^{\infty} (-i)^n P_1^1(\mu) P_n^1(\mu) \int_0^\infty e^{-(x/\beta)^{3/2}} x^{3/2} J_{n+1/2}(x) \\ \times [\mathfrak{C}_{n-1} - \mathfrak{D}_{n+1}]\, dx + \sum_{n=0}^{\infty} (-i)^n P_1(\mu) P_n(\mu) \int_0^\infty e^{-(x/\beta)^{3/2}} x^{3/2} J_{n+1/2}(x) \\ \times [n\mathfrak{C}_{n-1} + (n+1)\mathfrak{D}_{n+1}]\, dx \Bigg\} .\end{aligned}\right\} \quad (35)$$

Again, as in the theory of spatial correlations, greatest interest is attached to $\overline{F}_{t,\,\boldsymbol{F}_0}$ only after it has been further averaged over all mutual orientations of the vectors $\boldsymbol{F}_0$ and $\boldsymbol{v}$. When this additional averaging process is carried out, we are left with

$$\overline{\overline{F}}_{t,\,\boldsymbol{F}_0} = -\frac{2}{\pi\beta H(\beta)} \left(\frac{\pi}{2}\right)^{1/2} \int_0^\infty e^{-(x/\beta)^{3/2}} x^{3/2} J_{3/2}(x)\, \mathfrak{C}_0 dx , \quad (36)$$

where, according to equation (31),

$$\mathfrak{C}_0 = -Q_H \frac{15}{4} \left(\frac{x}{\beta}\right)^{1/2} \int_0^\infty d\lambda \frac{4}{\pi^{1/2}} e^{-(\lambda^2+\nu^2)} \lambda^2 \left(\frac{\pi}{4\lambda\nu}\right)^{1/2} I_{1/2}(2\lambda\nu) \int_0^\infty dz \frac{z^{1/2}}{z^{3/2}} J_{5/2}(z) . \quad (37)$$

Combining equations (36) and (37) and expressing $\overline{\overline{F}}_{t,\,\boldsymbol{F}_0}$ in units of Q_H and t in units of t_0 (eq. [24]), we can write

$$\overline{\overline{\beta}}_{\tau,\,\mathbf{B}} = \frac{15}{2\pi\beta^{3/2} H(\beta)} \int_0^\infty e^{-(x/\beta)^{3/2}} x^{3/2} \left(\frac{\sin x}{x} - \cos x\right) \mathfrak{Q}\left(\frac{x}{\tau^2\beta};\, \nu\right) dx , \quad (38)$$

where

$$\overline{\mathfrak{Q}}(y;\nu)=\int_0^\infty d\lambda\,\frac{4}{\pi^{1/2}}\,e^{-(\lambda^2+\nu^2)}\lambda^2\left(\frac{\pi}{4\lambda\nu}\right)^{1/2}I_{1/2}(2\lambda\nu)\,\mathfrak{Q}\left(\frac{y}{\lambda^2}\right), \tag{39}$$

the function $\mathfrak{Q}(x)$ being defined as in III, equation (80). Since

$$I_{1/2}(2\lambda\nu)=\frac{1}{(4\pi\lambda\nu)^{1/2}}\,(e^{2\lambda\nu}-e^{-2\lambda\nu})\,, \tag{40}$$

we can re-write equation (39) more explicitly in the form

$$\overline{\mathfrak{Q}}(y;\nu)=\int_0^\infty d\lambda\,\frac{4}{\pi^{1/2}}\,e^{-(\lambda^2+\nu^2)}\lambda^2\left[\frac{1}{2\lambda\nu}\sinh\,2\lambda\nu\right]\frac{1}{2}\int_0^{y/\lambda^2}\frac{dz}{z^2}J_{3/2}(z)\,, \tag{41}$$

where we have further substituted for $\mathfrak{Q}(y/\lambda^2)$ according to III, equation (85).

Now, the correlation coefficient $R(\boldsymbol{\beta},\tau)$ is related to $\overline{\overline{\beta}}_{\tau,\,\boldsymbol{\beta}}$ very simply. Thus,

$$R(\boldsymbol{\beta},\tau)=\frac{\overline{\boldsymbol{\beta}\cdot\boldsymbol{\beta}_\tau}}{|\boldsymbol{\beta}^2|}=\frac{1}{\beta}\overline{\overline{\beta}}_{\tau,\,\boldsymbol{\beta}}\,. \tag{42}$$

Hence,

$$R(\boldsymbol{\beta},\tau)=\frac{15}{2\pi\beta^{5/2}H(\beta)}\int_0^\infty e^{-(x/\beta)^{3/2}}x^{3/2}\left(\frac{\sin x}{x}-\cos x\right)\overline{\mathfrak{Q}}\left(\frac{x}{\tau^2\beta};\nu\right)dx\,. \tag{43}$$

It is evident that (cf. III, eqs. [95] and [97])

$$R(\boldsymbol{\beta},\tau)\rightarrow 1\qquad\text{as}\quad\tau\rightarrow 0\,, \tag{44}$$

while

$$R(\boldsymbol{\beta},\tau)\propto\tau^{-1}\qquad\text{as}\quad\tau\rightarrow\infty\,. \tag{45}$$

According to equation (45), the correlation coefficient $R(\boldsymbol{\beta},\tau)$ decreases relatively rather slowly for $\tau\rightarrow\infty$. In this respect the stochastic process we are considering is strikingly in contrast with processes of the usual Markoff type, in which the correlations are expected to decrease exponentially with the time.

As in III, § 7, we can also consider the result of further averaging $\overline{\overline{\beta}}_{\tau,\,\boldsymbol{\beta}}$ over all initial values of β weighted according to the Holtsmark function, $H(\beta)$. Analogous to III, equation (111), we now have

$$\overline{\overline{\beta}}_{\boldsymbol{\beta}}(\tau,\nu)=\frac{15}{\pi}\int_0^\infty e^{-y^{3/2}}\overline{\mathfrak{Q}}\left(\frac{z}{\tau^2};\nu\right)\frac{dy}{y^{1/2}}\,. \tag{46}$$

The appropriate asymptotic expansions for $\overline{\overline{\beta}}_{\boldsymbol{\beta}}(\tau,\nu)$ can be readily written down from the corresponding expansions for $\overline{\overline{\beta}}_{\boldsymbol{\beta}}(s)$ given in III, § 7.

4. *The average value of* $\mathbf{F}_t$ *in the direction of* $\mathbf{v}$.—The average value, $\overline{F}_{t,\,v}$ of $\boldsymbol{F}_t$ in the direction of $\boldsymbol{v}$ is, according to our choice of the co-ordinate system, the same as $\overline{F}_{t,\,\zeta}$, and this is given by equation (28). And, as in the case of $\overline{F}_{t,\,F_0}$, the quantity of greatest physical interest is that which results from further averaging $\overline{F}_{t,\,v}$ over all mutual orientations of the vectors $\boldsymbol{F}_0$ and $\boldsymbol{v}$. Analogous to III, equation (138), we now have

$$\overline{\overline{\beta}}_{\tau,\,v}=\frac{5}{2\pi\beta^{3/2}H(\beta)}\int_0^\infty e^{-(x/\beta)^{3/2}}x^{3/2}\sin x\,\overline{\mathfrak{R}}\left(\frac{x}{\tau^2\beta};\nu\right)dx\,, \tag{47}$$

where

$$\overline{\mathfrak{R}}(y;\nu)=\int_0^\infty d\lambda\,\frac{4}{\pi^{1/2}}\,e^{-(\lambda^2+\nu^2)}\lambda^2\left(\frac{\pi}{4\lambda\nu}\right)^{1/2}I_{3/2}(2\lambda\nu)\,\mathfrak{R}\left(\frac{y}{\lambda^2}\right),\tag{48}$$

the function $\mathfrak{R}(x)$ being defined as in III, equation (139). Since

$$I_{3/2}(2\lambda\nu)=\frac{1}{(4\pi\lambda\nu)^{1/2}}\left[e^{2\lambda\nu}\left(1-\frac{1}{2\lambda\nu}\right)+e^{-2\lambda\nu}\left(1+\frac{1}{2\lambda\nu}\right)\right],\tag{49}$$

we can re-write equation (48) more conveniently in the form

$$\overline{\mathfrak{R}}(y;\nu)=\int_0^\infty d\lambda\,\frac{4}{\pi^{1/2}}\,e^{-(\lambda^2+\nu^2)}\lambda^2\,\frac{1}{(2\lambda\nu)^2}(2\lambda\nu\cosh 2\lambda\nu-\sinh 2\lambda\nu)\,\mathfrak{R}\left(\frac{y}{\lambda^2}\right).\tag{50}$$

Again, the relevant asymptotic expansions for $\overline{\overline{\beta}}_{T,\,v}$ can be derived from those given for $\overline{\overline{\beta}}_{1,\,s}$ in III, § 8.

This completes the formal parts of the theory of the stochastic variation of the force acting on a star. In later papers we shall return to fuller discussions of the various implications of this theory to the problems of stellar dynamics.