THE STATISTICS OF THE GRAVITATIONAL FIELD ARISING FROM A RANDOM DISTRIBUTION OF STARS

III. THE CORRELATIONS IN THE FORCES ACTING AT TWO POINTS SEPARATED BY A FINITE DISTANCE

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ABSTRACT

In this paper we consider the theory of the correlations in the forces acting at two points separated by a finite distance in a system containing a random distribution of stars. A problem central in this theory is the evaluation of the average force acting at a point and in the same direction as the force acting at another specified point. It is shown how this and other similar problems in the theory of *spatial correlations* can be solved. An application of this theory to the problem of the stability of wide binaries is contained in a later paper.

1. *Introduction.*—In the two earlier papers[1] of this series we have analyzed certain statistical features of the fluctuating gravitational field acting on a star. More specifically, the particular problems considered in the earlier papers arose out of an attempt to answer certain questions relating to the speed of fluctuations in the force per unit mass $\boldsymbol{F}$ acting on a star and required the evaluation of all the first and the second moments of $\dot{\boldsymbol{F}}$, the rate of change of $\boldsymbol{F}$ for a given $\boldsymbol{F}$. While the specification of these moments of $\dot{\boldsymbol{F}}$ are sufficient for the purposes of determining the instantaneous rates of change of $\boldsymbol{F}$ that are to be expected, they are very far from providing all the information that is necessary for a complete statistical description of the fluctuating force acting on a star.[2] For the entire stochastic variation of $\boldsymbol{F}$ with time can be described fully only in terms of the average force $\bar{\boldsymbol{F}}_t$ acting at any later time t, given that a force of some prescribed intensity acted at time $t = 0$. In other words, we need a complete "integration" of the stochastic equations of $\boldsymbol{F}$.

Now the problem of specifying the average force $\bar{\boldsymbol{F}}_t$ acting at time t (for a given $\boldsymbol{F}_0$ at time $t = 0$) is essentially equivalent to determining the correlations in the force acting at a given point but at two instants separated by a finite interval of time. Accordingly, it would appear that the solution to the problem of the stochastic variation of $\boldsymbol{F}$ acting on a star can be derived from that of the somewhat simpler one of the correlations in the force acting simultaneously at two points separated by a finite distance. For the correlations between $\boldsymbol{F}_0$ and $\boldsymbol{F}_t$ will be determined in terms of the distribution function $W(\boldsymbol{F}_0, \boldsymbol{F}_t)$, where

$$\boldsymbol{F}_0 = G\sum_i M_i \frac{\boldsymbol{r}_i}{|\boldsymbol{r}_i|^3} \tag{1}$$

and

$$\boldsymbol{F}_t = G\sum_i M_i \frac{\boldsymbol{r}_i + \boldsymbol{V}_i t}{|\boldsymbol{r}_i + \boldsymbol{V}_i t|^3}\,. \tag{2}$$

[1] *Ap. J.*, **95**, 489, 1942, and **97**, 1, 1943: these two papers will be referred to as I and II, respectively. For a more general account of the basic ideas see S. Chandrasekhar, *Rev. Mod. Phys.*, **15**, 1, 1943, and also an essay entitled "New Methods in Stellar Dynamics," *Annals of the New York Academy of Sciences*, **45**, 131, 1943.

[2] See, e.g., the remarks in S. Chandrasekhar, *Ap. J.*, **97**, 255, 1943 (§ 3 of this paper).

In equations (1) and (2) $\boldsymbol{r}_i$ and $\boldsymbol{V}_i$ denote respectively the position and the velocity of a typical field star *relative* to the one under consideration. On the other hand, the problem of the spatial correlations requires the consideration of the distribution function $W(\boldsymbol{F}_0, \boldsymbol{F}_1)$, where $\boldsymbol{F}_0$ is again defined as in equation (1), while

$$\boldsymbol{F}_1 = G \sum_i \frac{\boldsymbol{r}_i - \boldsymbol{r}_1}{|\boldsymbol{r}_i - \boldsymbol{r}_1|^3}, \tag{3}$$

where $\boldsymbol{r}_1$ now denotes the position of the second point considered relative to the first. Comparing equations (2) and (3), it is apparent that the two theories associated respectively with the quantities $\boldsymbol{F}_t$ and $\boldsymbol{F}_1$ differ formally from each other only in one respect, namely, that, while when considering $\boldsymbol{F}_t$ we have to allow for an appropriate asymmetric distribution of the relative velocities $\boldsymbol{V}_i$, we have no such problem of averaging when considering $\boldsymbol{F}_1$, since $\boldsymbol{r}_1$ is a certain prescribed constant vector. It is therefore seen that the consideration of spatial correlations provides us with a problem basic to the whole general theory. But, even apart from this, the theory of spatial correlations has its own independent interest for stellar dynamics. Thus, questions relating to the stability of wide binaries can be answered satisfactorily only in terms of such a theory.[3] We shall accordingly devote this paper entirely to the development of the theory of spatial correlations. The more difficult problem of the stochastic variation of $\boldsymbol{F}$ acting on a star is taken up in the paper following this one.[4]

2. *The general formula for* W($\mathbf{F}_0$, $\mathbf{F}_1$).—As we have already indicated, the problem of spatial correlations requires us to consider the probability that a force $\boldsymbol{F}_0$ will act at a point and that simultaneously a force $\boldsymbol{F}_1$ will act at a second point distant $|\boldsymbol{r}_1|$ from the first. The general expression for the corresponding probability distribution $W(\boldsymbol{F}_0, \boldsymbol{F}_1)$ can be readily written down by an application of Markoff's method.[5] We have

$$W(\boldsymbol{F}_0, \boldsymbol{F}_1) = \frac{1}{64\pi^6} \int_{|\boldsymbol{\rho}|=0}^{\infty} \int_{|\boldsymbol{\sigma}|=0}^{\infty} e^{-i(\boldsymbol{\rho}\cdot\boldsymbol{F}_0 + \boldsymbol{\sigma}\cdot\boldsymbol{F}_1)} A(\boldsymbol{\rho}, \boldsymbol{\sigma})\, d\boldsymbol{\rho}\, d\boldsymbol{\sigma}, \tag{4}$$

where

$$A(\boldsymbol{\rho}, \boldsymbol{\sigma}) = \lim_{R\to\infty} \left[\frac{3}{4\pi R^3} \int_{|\boldsymbol{r}|<R} e^{iGM\left\{\frac{\boldsymbol{r}\cdot\boldsymbol{\rho}}{|\boldsymbol{r}|^3} + \frac{(\boldsymbol{r}-\boldsymbol{r}_1)\cdot\boldsymbol{\sigma}}{|\boldsymbol{r}-\boldsymbol{r}_1|^3}\right\}} d\boldsymbol{r}\right]^{4\pi R^3 N/3}. \tag{5}$$

In equations (4) and (5) $\boldsymbol{\rho}$ and $\boldsymbol{\sigma}$ are two auxiliary vectors and N denotes the average number of stars per unit volume. (It should be noted here that in writing equation [5] we have assumed that all the stars have the same mass M. However, the generalizations needed to allow for a distribution over the masses is fairly straightforward and will be indicated later.) By a series of transformations customary in this theory we can express $A(\boldsymbol{\rho}, \boldsymbol{\sigma})$ in the form[6]

$$A(\boldsymbol{\rho}, \boldsymbol{\sigma}) = e^{-C(\boldsymbol{\rho}, \boldsymbol{\sigma})}, \tag{6}$$

where the characteristic function $C(\boldsymbol{\rho}, \boldsymbol{\sigma})$ is given by

$$C(\boldsymbol{\rho}, \boldsymbol{\sigma} = N \int_{-\infty}^{+\infty} \left[1 - e^{iGM\left\{\frac{\boldsymbol{r}\cdot\boldsymbol{\rho}}{|\boldsymbol{r}|^3} + \frac{(\boldsymbol{r}-\boldsymbol{r}_1)\cdot\boldsymbol{\sigma}}{|\boldsymbol{r}-\boldsymbol{r}_1|^3}\right\}}\right] d\boldsymbol{r}. \tag{7}$$

[3] See a later paper appearing in this same issue (p. 54).

[4] See p. 47; this paper will be referred to as IV.

[5] Cf. S. Chandrasekhar, *Rev. Mod. Phys.*, **15**, 1, 1943. (See particularly chap. i, § 3, and chap. iv, §§ 2 and 3.)

[6] Cf., e.g., II, § 1, particularly the transformations leading from eq. (6) to eq. (12).

An alternative form for $C(\boldsymbol{\rho}, \boldsymbol{\sigma})$ is

$$\left.\begin{aligned} C(\boldsymbol{\rho}, \boldsymbol{\sigma}) &= N\int_{-\infty}^{+\infty}[1 - e^{iGM\boldsymbol{r}\cdot\boldsymbol{\rho}/|\boldsymbol{r}|^3}]\,d\boldsymbol{r} \\ &\quad + N\int_{-\infty}^{+\infty} e^{iGM\boldsymbol{r}\cdot\boldsymbol{\rho}/|\boldsymbol{r}|^3}[1 - e^{iGM(\boldsymbol{r}-\boldsymbol{r}_1)\cdot\boldsymbol{\sigma}/|\boldsymbol{r}-\boldsymbol{r}_1|^3}]\,d\boldsymbol{r}. \end{aligned}\right\} \quad (8)$$

The first of the two integrals which occur in the foregoing equation is equivalent to the one we have already evaluated in I (eqs. [55]–[58]). We thus have

$$C(\boldsymbol{\rho}, \boldsymbol{\sigma}) = \tfrac{4}{15}(2\pi GM)^{3/2}N|\boldsymbol{\rho}|^{3/2} + N\int_{-\infty}^{+\infty} e^{iGM\boldsymbol{r}\cdot\boldsymbol{\rho}/|\boldsymbol{r}|^3}[1 - e^{iGM(\boldsymbol{r}-\boldsymbol{r}_1)\cdot\boldsymbol{\sigma}/|\boldsymbol{r}-\boldsymbol{r}_1|^3}]\,d\boldsymbol{r}. \quad (9)$$

Equations (4), (6), and (9) formally solve the problem. However, an explicit evaluation of the entire probability distribution $W(\boldsymbol{F}_0, \boldsymbol{F}_1)$ will require a complete knowledge of the characteristic function $C(\boldsymbol{\rho}, \boldsymbol{\sigma})$. But, if we are interested only in the moments of $\boldsymbol{F}_1$, we need only the behavior of $C(\boldsymbol{\rho}, \boldsymbol{\sigma})$ for $|\boldsymbol{\sigma}| \to 0$. For these purposes we can therefore expand

$$1 - e^{iGM(\boldsymbol{r}-\boldsymbol{r}_1)\cdot\boldsymbol{\sigma}/|\boldsymbol{r}-\boldsymbol{r}_1|^3}, \quad (10)$$

which occurs under the integral sign in equation (9) in a power series in $\boldsymbol{\sigma}$. Retaining only the first term in this expansion, we have

$$C(\boldsymbol{\rho}, \boldsymbol{\sigma}) = \tfrac{4}{15}(2\pi GM)^{3/2}N|\boldsymbol{\rho}|^{3/2} + D(\boldsymbol{\rho}, \boldsymbol{\sigma}) + O(|\boldsymbol{\sigma}|^2), \quad (11)$$

where

$$D(\boldsymbol{\rho}, \boldsymbol{\sigma}) = -iGMN\int_{-\infty}^{+\infty} e^{iGM\boldsymbol{r}\cdot\boldsymbol{\rho}/|\boldsymbol{r}|^3}\frac{(\boldsymbol{r}-\boldsymbol{r}_1)\cdot\boldsymbol{\sigma}}{|\boldsymbol{r}-\boldsymbol{r}_1|^3}\,d\boldsymbol{r}. \quad (12)$$

In equation (12) it is convenient to introduce two new variables, $\boldsymbol{R}$ and $\boldsymbol{S}$, in place of $\boldsymbol{\rho}$ and $\boldsymbol{\sigma}$. Let

$$\boldsymbol{R} = GM\boldsymbol{\rho} \quad \text{and} \quad \boldsymbol{S} = GM\boldsymbol{\sigma}. \quad (13)$$

Equation (12) becomes

$$D(\boldsymbol{R}\ \boldsymbol{S}) = -iN\int_{-\infty}^{+\infty} e^{i\boldsymbol{r}.\boldsymbol{R}/|\boldsymbol{r}|^3}\boldsymbol{S}\cdot\frac{(\boldsymbol{r}-\boldsymbol{r}_1)}{|\boldsymbol{r}-\boldsymbol{r}_1|^3}\,d\boldsymbol{r}, \quad (14)$$

or, alternatively,

$$D(\boldsymbol{R}, \boldsymbol{S}) = iN\int_{-\infty}^{+\infty} e^{i\boldsymbol{r}.\boldsymbol{R}/|\boldsymbol{r}|^3}\left(\boldsymbol{S}\cdot\operatorname{grad}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}_1|}\right)d\boldsymbol{r}. \quad (15)$$

Integrating by parts, we obtain

$$D(\boldsymbol{R}, \boldsymbol{S}) = -iN\int_{-\infty}^{+\infty}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}_1|}\boldsymbol{S}\cdot\operatorname{grad}(e^{i\boldsymbol{r}.\boldsymbol{R}/|\boldsymbol{r}|^3})\,d\boldsymbol{r}. \quad (16)$$

Explicitly, equation (16) has the form

$$D(\boldsymbol{R}, \boldsymbol{S}) = N\int_{-\infty}^{+\infty}\frac{e^{i\boldsymbol{r}.\boldsymbol{R}/|\boldsymbol{r}|^3}}{|\boldsymbol{r}-\boldsymbol{r}_1|}\left\{\frac{\boldsymbol{S}\cdot\boldsymbol{R}}{|\boldsymbol{r}|^3} - 3\,\frac{(\boldsymbol{r}\cdot\boldsymbol{R})\,(\boldsymbol{r}\cdot\boldsymbol{S})}{|\boldsymbol{r}|^5}\right\}d\boldsymbol{r}. \quad (17)$$

It is of interest to note that, according to equation (17), $D(\boldsymbol{R}, \boldsymbol{S})$ regarded as a function of r_1 satisfies the differential equation

$$\operatorname{div}\operatorname{grad} D = 4\pi N e^{i\boldsymbol{r}\cdot\boldsymbol{R}/|\boldsymbol{r}|^3}\left\{\frac{\boldsymbol{S}\cdot\boldsymbol{R}}{|\boldsymbol{r}|^3} - 3\,\frac{(\boldsymbol{r}\cdot\boldsymbol{R})\,(\boldsymbol{r}\cdot\boldsymbol{S})}{|\boldsymbol{r}|^5}\right\}. \tag{18}$$

It is possible that the existence of this differential equation of the Poisson type has a deeper significance than is apparent at first sight.

3. *The evaluation* D($\boldsymbol{\rho}$, $\boldsymbol{\sigma}$).—To evaluate the integral defining $D(\boldsymbol{R}, \boldsymbol{S})$, we shall chose a Cartesian system of co-ordinates in which the z-axis is in the direction of $\boldsymbol{R}$ (i.e., $\boldsymbol{\rho}$) and the xz-plane contains the vector $\boldsymbol{r}_1$. Let the components of $\boldsymbol{S}$ in this system of co-ordinates be S_1, S_2, and S_3. With this choice of the orientation of the co-ordinate system and transforming to polar co-ordinates, equation (17) reduces to

$$\left.\begin{aligned} D(\boldsymbol{R},\boldsymbol{S}) = -N|\boldsymbol{R}|\int_0^\infty\int_{-1}^{+1}\int_0^{2\pi}\frac{e^{i|\boldsymbol{R}|\mu/r^2}}{|\boldsymbol{r}-\boldsymbol{r}_1|\,r}&[2S_3P_2(\mu) \\ &+3\mu(1-\mu^2)^{1/2}(S_1\cos\varphi+S_2\sin\varphi)]\,dr\,d\mu\,d\varphi\,, \end{aligned}\right\} \tag{19}$$

where we have used μ to denote $\cos\vartheta$. Further, the P_n's denote, as usual, the Legendre functions.

We now expand $|\boldsymbol{r}-\boldsymbol{r}_1|^{-1}$, which occurs under the integral sign in equation (19) in spherical harmonics. We have

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{r}_1|} = \frac{1}{\tilde{r}}\sum_{l=0}^{\infty} x^l P_l(\cos\Theta)\,, \tag{20}$$

where

$$\tilde{r} \text{ is the larger of } r \text{ and } r_1 \tag{21}$$

and

$$x = \begin{cases} r/r_1 & \text{if } r < r_1 \\ r_1/r & \text{if } r > r_1 \end{cases}. \tag{22}$$

Further, in equation (20) Θ denotes the angle between the directions of $\boldsymbol{r}$ and $\boldsymbol{r}_1$. Accordingly,

$$\cos\Theta = \cos\vartheta\cos\vartheta_1 + \sin\vartheta\sin\vartheta_1\cos\varphi\,, \tag{23}$$

where

$$\vartheta_1 = \sphericalangle(\boldsymbol{R},\boldsymbol{r}_1)\,. \tag{24}$$

Moreover, by the addition theorem of spherical harmonics

$$P_l(\cos\Theta) = P_l(\mu)P_l(\mu_1) + 2\sum_{m=1}^{l}\frac{(l-m)!}{(l+m)!}P_l^m(\mu)P_l^m(\mu_1)\cos m\varphi\,, \tag{25}$$

where we have written μ_1 for $\cos\vartheta_1$. Combining equations (19), (20), and (25), we have

$$\left.\begin{aligned} D(\boldsymbol{R},\boldsymbol{S}) = -N|\boldsymbol{R}|\int_0^\infty\int_{-1}^{+1}\int_0^{2\pi}\frac{e^{i|\boldsymbol{R}|\mu/r^2}}{r\tilde{r}}&[2S_3P_2(\mu)+3\mu(1-\mu^2)^{1/2} \\ \times(S_1\cos\varphi+S_2\sin\varphi)]&\left[\sum_{l=0}^{\infty}x^l\left\{P_l(\mu)P_l(\mu_1)+2\sum_{m=1}^{l}\frac{(l-m)!}{(l+m)!}\right.\right. \\ &\left.\left.\times P_l^m(\mu)P_l^m(\mu_1)\cos m\varphi\right\}\right]dr\,d\mu\,d\varphi\,. \end{aligned}\right\} \tag{26}$$

Integration over φ is now readily performed, and we get

$$\left.\begin{aligned} D(\boldsymbol{R},\boldsymbol{S}) = -2\pi N|\boldsymbol{R}|\int_0^\infty\int_{-1}^{+1}\frac{e^{i|\boldsymbol{R}|\mu/r^2}}{r\tilde{r}}\Bigg[&2S_3P_2(\mu)\sum_{l=0}^{\infty}x^lP_l(\mu)P_l(\mu_1) \\ &+3S_1\mu(1-\mu^2)^{1/2}\sum_{l=1}^{\infty}\frac{(l-1)!}{(l+1)!}x^lP_l^1(\mu)P_l^1(\mu_1)\Bigg]dr\,d\mu\,. \end{aligned}\right\} \tag{27}$$

Now, introduce the new variable

$$z=|\boldsymbol{R}|\,r^{-2} \tag{28}$$

and let

$$z_1=|\boldsymbol{R}|\,r_1^{-2}=GM|\boldsymbol{\rho}|\,r_1^{-2}\,. \tag{29}$$

With this change of the variables, equation (27) takes the more convenient form

$$\left.\begin{aligned} D(\boldsymbol{R},\boldsymbol{S}) = -\pi N|\boldsymbol{R}|^{1/2}\int_0^\infty\int_{-1}^{+1}e^{iz\mu}\frac{\tilde{z}^{1/2}}{z}\Bigg[&2S_3P_2(\mu)\sum_{l=0}^{\infty}\xi^{l/2}P_l(\mu)P_l(\mu_1) \\ &+3S_1\mu(1-\mu^2)^{1/2}\sum_{l=1}^{\infty}\xi^{l/2}\frac{(l-1)!}{(l+1)!}P_l^1(\mu)P_l^1(\mu_1)\Bigg]dz\,d\mu\,, \end{aligned}\right\} \tag{30}$$

where

$$\tilde{z}\text{ is the smaller of } z \text{ and } z_1 \tag{31}$$

and

$$\xi=\begin{Bmatrix} z/z_1 & \text{if} & z<z_1 \\ z_1/z & \text{if} & z>z_1 \end{Bmatrix}. \tag{32}$$

To perform the integration over μ, we expand exp $(iz\mu)$ in spherical harmonics. We have the well-known expansion

$$e^{iz\mu}=\left(\frac{\pi}{2z}\right)^{1/2}\sum_{n=0}^{\infty}(2n+1)\,i^nJ_{n+1/2}(z)P_n(\mu)\,, \tag{33}$$

in which the $J_{n+1/2}$'s are the Bessel functions of half odd-integral orders. Hence,

$$\left.\begin{aligned} D(\boldsymbol{R},\boldsymbol{S}) = -\frac{\pi^{3/2}N|\boldsymbol{R}|^{1/2}}{2^{1/2}}\int_0^\infty dz\frac{\tilde{z}^{1/2}}{z^{3/2}}\Bigg[&2S_3\sum_{l=0}^{\infty}\xi^{l/2}P_l(\mu_1)\int_{-1}^{+1}d\mu P_2(\mu)P_l(\mu) \\ &\times\sum_{n=0}^{\infty}(2n+1)\,i^nJ_{n+1/2}(z)P_n(\mu)+3S_1\sum_{l=1}^{\infty}\xi^{l/2}P_l^1(\mu_1) \\ &\times\frac{(l-1)!}{(l+1)!}\int_{-1}^{+1}d\mu\,\mu(1-\mu^2)^{1/2}P_l^1(\mu)\sum_{n=0}^{\infty}(2n+1)\,i^nJ_{n+1/2}(z)P_n(\mu)\Bigg]. \end{aligned}\right\} \tag{34}$$

It is seen that in the foregoing equation we have to perform integrations over the product of three tesseral harmonics, the upper suffix of one being the sum of the upper suffixes of the other two. Such integrals have been studied by J. A. Gaunt.[7] However, the par-

[7] *Phil. Trans. Roy. Soc.* (London), Ser. A, **228**, 151, 1929. (See particularly the appendix to this paper, pp. 192–196.)

ticular results needed for our purposes can be derived more directly. Thus, using the recursion formulae

$$\left.\begin{aligned}\mu P_n^m &= \frac{1}{2n+1}\left[(n-m+1)P_{n+1}^m + (n+m)P_{n-1}^m\right],\\ (1-\mu^2)^{1/2}P_n^m &= \frac{1}{2n+1}\left(P_{n+1}^{m+1} - P_{n-1}^{m+1}\right),\end{aligned}\right\} \tag{35}$$

we readily obtain the equations

$$\left.\begin{aligned}P_2P_n &= \frac{3(n+1)(n+2)}{2(2n+1)(2n+3)}P_{n+2} + \frac{n(n+1)}{(2n+3)(2n-1)}P_n\\ &\qquad + \frac{3n(n-1)}{2(2n+1)(2n-1)}P_{n-2},\\ \mu(1-\mu^2)^{1/2}P_n &= \frac{n+1}{(2n+1)(2n+3)}P_{n+2}^1 + \frac{1}{(2n+3)(2n-1)}P_n^1\\ &\qquad - \frac{n}{(2n+1)(2n-1)}P_{n-2}^1 .\end{aligned}\right\} \tag{36}$$

Substituting these formulae in equation (34) and using the orthogonality properties of the Legendre functions, we find

$$\left.\begin{aligned}D(R,S) = -\frac{\pi^{3/2}|R|^{1/2}N}{2^{1/2}}\int_0^\infty dz\,\frac{z^{1/2}}{z^{3/2}}\Bigg[&2S_3\sum_{l=0}^{\infty} i^l \xi^{l/2}P_l(\mu_1)\\ \times\Big\{&-\frac{3l(l-1)}{(2l+1)(2l-1)}J_{l-3/2} + \frac{2l(l+1)}{(2l+3)(2l-1)}J_{l+1/2}\\ &-\frac{3(l+2)(l+1)}{(2l+1)(2l+3)}J_{l+5/2}\Big\} + 2S_1\sum_{l=1}^{\infty} i^l\xi^{l/2}P_l^1(\mu_1)\\ \times\Big\{&-\frac{3(l-1)}{(2l+1)(2l-1)}J_{l-3/2} + \frac{3}{(2l+3)(2l-1)}J_{l+1/2}\\ &+\frac{3(l+2)}{(2l+1)(2l+3)}J_{l+5/2}\Big\}\Bigg].\end{aligned}\right\} \tag{37}$$

Returning to our original variables $\boldsymbol{\rho}$ and $\boldsymbol{\sigma}$ (cf. eq. [13]), it is apparent that we can express $D(\boldsymbol{\rho}, \boldsymbol{\sigma})$ in the form

$$D(\boldsymbol{\rho},\boldsymbol{\sigma}) = -\sigma_3\sum_{l=0}^{\infty} A_l(|\boldsymbol{\rho}|;\, z_1)P_l(\mu) - \sigma_1\sum_{l=1}^{\infty} B_l(|\boldsymbol{\rho}|;\, z_1)P_l^1(\mu)\,, \tag{38}$$

where

$$\left.\begin{aligned}A_l = i^l 2^{1/2}\pi^{3/2}(GM)^{3/2}N|\boldsymbol{\rho}|^{1/2}\int_0^\infty dz\,\xi^{l/2}\frac{z^{1/2}}{z^{3/2}}\Bigg[&-\frac{3l(l-1)}{(2l+1)(2l-1)}J_{l-3/2}\\ &+\frac{2l(l+1)}{(2l+3)(2l-1)}J_{l+1/2} - \frac{3(l+2)(l+1)}{(2l+1)(2l+3)}J_{l+5/2}\Bigg]\end{aligned}\right\} \tag{39}$$

and

$$\left.\begin{aligned}B_l = i^l 2^{1/2}\pi^{3/2}(GM)^{3/2}N\,|\boldsymbol{\rho}|^{1/2}\int_0^\infty dz\,\xi^{l/2}\frac{\tilde{z}^{1/2}}{z^{3/2}}\Big[&-\frac{3(l-1)}{(2l+1)(2l-1)}J_{l-3/2}\\ &+\frac{3}{(2l+3)(2l-1)}J_{l+1/2}+\frac{3(l+2)}{(2l+1)(2l+3)}J_{l+5/2}\Big]\,.\end{aligned}\right\}\quad(40)$$

Further, in equation (38) we have suppressed the suffix "1" in μ_1, thus now letting μ denote the cosine of the angle between the directions of $\boldsymbol{\rho}$ and r_1.

4. *The expression for* $A(\boldsymbol{\rho}, \boldsymbol{\sigma})$ *for* $|\boldsymbol{\sigma}| \to 0$.—According to equation (4), $A(\boldsymbol{\rho}, \boldsymbol{\sigma})$ is the six-dimensional Fourier transform of the distribution function $W(\boldsymbol{F}_0, \boldsymbol{F}_1)$. Consequently, for the purposes of this equation the vectors $\boldsymbol{\rho}$ and $\boldsymbol{\sigma}$ must be referred to a fixed

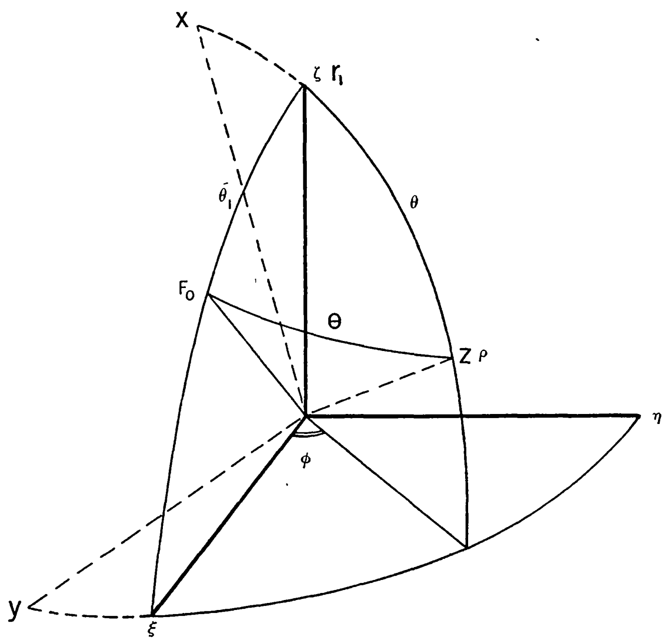

FIG. 1

system of co-ordinates. But in equation (38) for $D(\boldsymbol{\rho}, \boldsymbol{\sigma})$ we have expressed $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ in a variable system of co-ordinates depending on the direction of $\boldsymbol{\rho}$. We shall now pass from this variable xyz-system to a fixed $\xi\eta\zeta$-system (see Fig. 1). This fixed $\xi\eta\zeta$-system is so chosen that the ζ-axis is in the direction of r_1 and the $\xi\zeta$-plane contains the vector $\boldsymbol{F}_0$. Let $\boldsymbol{\rho}$ be along an arbitrary direction in this system of co-ordinates. The linear transformation which allows us to pass from the xyz- to the $\xi\eta\zeta$-system is clearly

$$\left.\begin{aligned}\sigma_1 &= -\sigma_\xi\cos\vartheta\cos\varphi - \sigma_\eta\cos\vartheta\sin\varphi + \sigma_\zeta\sin\vartheta\,,\\ \sigma_2 &= +\sigma_\xi\sin\varphi - \sigma_\eta\cos\varphi\,,\\ \sigma_3 &= +\sigma_\xi\sin\vartheta\cos\varphi + \sigma_\eta\sin\vartheta\sin\varphi + \sigma_\zeta\cos\vartheta\,.\end{aligned}\right\}\quad(41)$$

Thus, in this fixed system of co-ordinates $D(\boldsymbol{\rho}, \boldsymbol{\sigma})$ has the form

$$\left.\begin{aligned}D(\boldsymbol{\rho}, \boldsymbol{\sigma}) = &-(\sigma_\xi\sin\vartheta\cos\varphi + \sigma_\eta\sin\vartheta\sin\varphi + \sigma_\zeta\cos\vartheta)\sum_{l=0}^{\infty}A_lP_l(\mu)\\ &+(\sigma_\xi\cos\vartheta\cos\varphi + \sigma_\eta\cos\vartheta\sin\varphi - \sigma_\zeta\sin\vartheta)\sum_{l=1}^{\infty}B_lP_l^1(\mu)\,.\end{aligned}\right\}\quad(42)$$

Finally, combining equations (6), (11), and (42), we obtain

$$
\left.
\begin{aligned}
A(\boldsymbol{\rho},\boldsymbol{\sigma}) = e^{-a|\boldsymbol{\rho}|^{3/2}}\Big[1+(\sigma_\xi\sin\vartheta\cos\varphi+\sigma_\eta\sin\vartheta\sin\varphi+\sigma_\zeta\cos\vartheta)\sum_{l=0}^{\infty}A_l P_l(\mu) \\
-(\sigma_\xi\cos\vartheta\cos\varphi+\sigma_\eta\cos\vartheta\sin\varphi-\sigma_\zeta\sin\vartheta)\sum_{l=1}^{\infty}B_l P_l^1(\mu)+O(|\boldsymbol{\sigma}|^2)\Big],
\end{aligned}
\right\}
\tag{43}
$$

where

$$
a = \tfrac{4}{15}(2\pi GM)^{3/2}N\,. \tag{44}
$$

We can express (43) more simply in the form

$$
A(\boldsymbol{\rho},\boldsymbol{\sigma}) = e^{-a|\boldsymbol{\rho}|^{3/2}}[1-D(\boldsymbol{\rho},\boldsymbol{\sigma})+O(|\boldsymbol{\sigma}|^2)]\,, \tag{45}
$$

where $D(\boldsymbol{\rho},\boldsymbol{\sigma})$ is defined as in equation (42).

5. *The first moments of* $\boldsymbol{F}_1$.—To determine the average values of $\boldsymbol{F}_1$ (for a given $\boldsymbol{F}_0$) in any specified direction, it would clearly be sufficient to evaluate the first moments of the components of $\boldsymbol{F}_1$ (namely, $F_{1,\xi}$, $F_{1,\eta}$, and $F_{1,\zeta}$) along the three principal directions of the $\xi\eta\zeta$-system as defined in § 4.

According to equation (4),

$$
\int_{-\infty}^{+\infty} W(\boldsymbol{F}_0,\boldsymbol{F}_1)F_{1,\tau}d\boldsymbol{F}_1 = \frac{1}{64\pi^6}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-i(\boldsymbol{\rho}\cdot\boldsymbol{F}_0+\boldsymbol{\sigma}\cdot\boldsymbol{F}_1)}A(\boldsymbol{\rho},\boldsymbol{\sigma})F_{1,\tau}d\boldsymbol{\rho}\,d\boldsymbol{\sigma}\,d\boldsymbol{F}_1\,, \tag{46}
$$

where we have used τ to denote either of ξ, η, or ζ. From this equation we readily derive (cf. II, eqs. [73]–[76])

$$
\int_{-\infty}^{+\infty} W(\boldsymbol{F}_0,\boldsymbol{F}_1)F_{1,\tau}d\boldsymbol{F}_1 = -\frac{i}{8\pi^3}\int_{-\infty}^{+\infty} e^{-i\boldsymbol{\rho}\cdot\boldsymbol{F}_0}\left[\frac{\partial}{\partial\sigma_\tau}A(\boldsymbol{\rho},\boldsymbol{\sigma})\right]_{|\boldsymbol{\sigma}|=0}d\boldsymbol{\rho}\,; \tag{47}
$$

or, using equation (45) for $A(\boldsymbol{\rho},\boldsymbol{\sigma})$, we have

$$
\int_{-\infty}^{+\infty} W(\boldsymbol{F}_0,\boldsymbol{F}_1)F_{1,\tau}d\boldsymbol{F}_1 = \frac{i}{8\pi^3}\int_{-\infty}^{+\infty} e^{-i\boldsymbol{\rho}\cdot\boldsymbol{F}_0-a|\boldsymbol{\rho}|^{3/2}}\frac{\partial D}{\partial\sigma_\tau}d\boldsymbol{\rho}\,, \tag{48}
$$

since $D(\boldsymbol{\rho},\boldsymbol{\sigma})$ is linear in $\boldsymbol{\sigma}$. On the other hand, the left-hand side of the foregoing equation defines $\overline{F}_{1,\tau}$, for

$$
W(\boldsymbol{F}_0)\overline{F}_{1,\tau} = \int_{-\infty}^{+\infty} W(\boldsymbol{F}_0,\boldsymbol{F}_1)F_{1,\tau}d\boldsymbol{F}_1\,. \tag{49}
$$

Hence,

$$
W(\boldsymbol{F}_0)\overline{F}_{1,\tau} = \frac{i}{8\pi^3}\int_{-\infty}^{+\infty} e^{-i\boldsymbol{\rho}\cdot\boldsymbol{F}_0-a|\boldsymbol{\rho}|^{3/2}}\frac{\partial D}{\partial\sigma_\tau}d\boldsymbol{\rho}\,; \tag{50}
$$

or, using polar co-ordinates, we have

$$
W(\boldsymbol{F}_0)\overline{F}_{1,\tau} = \frac{i}{8\pi^3}\int_0^{\infty}\int_{-1}^{+1}\int_0^{2\pi} e^{-i|\boldsymbol{\rho}||\boldsymbol{F}_0|\cos\Theta-a|\boldsymbol{\rho}|^{3/2}}\frac{\partial D}{\partial\sigma_\tau}|\boldsymbol{\rho}|^2d|\boldsymbol{\rho}|\,d\mu\,d\varphi\,, \tag{51}
$$

where Θ denotes the angle between the directions of $\boldsymbol{F}_0$ and $\boldsymbol{\rho}$. Putting (cf. I, eq. [134], and II, eq. [81])

$$|\boldsymbol{\rho}|\,|\boldsymbol{F}_0| = x\,; \qquad |\boldsymbol{F}_0| = a^{2/3}\beta = Q_H\beta \tag{52}$$

in equation (51), we obtain

$$W(\boldsymbol{F}_0)\,\overline{F}_{1,\tau} = \frac{i}{8\pi^3 a^2\beta^3}\int_0^\infty\int_{-1}^{+1}\int_0^{2\pi} e^{-ix\cos\Theta-(x/\beta)^{3/2}}\,\frac{\partial D}{\partial\sigma_\tau}\,x^2\,dx\,d\mu\,d\varphi\,. \tag{53}$$

Substituting for $W(\boldsymbol{F}_0)$ in the foregoing equation from I, equation (117), we obtain the general formula

$$\overline{F}_{1,\tau} = \frac{i}{2\pi^2\beta H(\beta)}\int_0^\infty\int_{-1}^{+1}\int_0^{2\pi} e^{-ix\cos\Theta-(x/\beta)^{3/2}}\,\frac{\partial D}{\partial\sigma_\tau}\,x^2\,dx\,d\mu\,d\varphi\,, \tag{54}$$

where it might be recalled that $H(\beta)$ is the "Holtsmark function,"

$$H(\beta) = \frac{2}{\pi\beta}\int_0^\infty e^{-(x/\beta)^{3/2}}x\sin x\,dx\,. \tag{55}$$

To carry out the integrations over μ and φ in equation (54), we first expand exp $(-ix\cos\Theta)$ in spherical harmonics. We have (cf. eqs. [25] and [33])

$$\left.\begin{aligned} e^{-ix\cos\Theta} = \left(\frac{\pi}{2x}\right)^{1/2}\sum_{n=0}^\infty (2n+1)(-i)^n J_{n+1/2}(x)\Big\{P_n(\mu)P_n(\mu_1) \\ +2\sum_{m=1}^n \frac{(n-m)!}{(n+m)!}P_n^m(\mu)P_n^m(\mu_1)\cos m\varphi\Big\}\,, \end{aligned}\right\} \tag{56}$$

where

$$\mu_1 = \cos\vartheta_1\,; \qquad \vartheta_1 = \sphericalangle(\boldsymbol{F}_0, \boldsymbol{r}_1)\,. \tag{57}$$

Combining equations (54) and (56) and substituting for D from equation (42), we obtain

$$\left.\begin{aligned} \overline{F}_{1,\tau} = -\frac{i}{2\pi^2\beta H(\beta)}\left(\frac{\pi}{2}\right)^{1/2}\int_0^\infty\int_{-1}^{+1}\int_0^{2\pi} dx\,d\mu\,d\varphi\, e^{-(x/\beta)^{3/2}}x^{3/2} \\ \times\Bigg[\sum_{n=0}^\infty (-i)^n(2n+1)J_{n+1/2}(x)\Big\{P_n(\mu)P_n(\mu_1) \\ +2\sum_{m=1}^n\frac{(n-m)!}{(n+m)!}P_n^m(\mu)P_n^m(\mu_1)\cos m\varphi\Big\}\Bigg] \\ \times\left\{\begin{aligned} &\left[\sin\vartheta\cos\varphi\sum_{l=0}^\infty A_lP_l(\mu)-\cos\vartheta\cos\varphi\sum_{l=1}^\infty B_lP_l^1(\mu)\right] \quad (\tau=\xi) \\ &\left[\sin\vartheta\sin\varphi\sum_{l=0}^\infty A_lP_l(\mu)-\cos\vartheta\sin\varphi\sum_{l=1}^\infty B_lP_l^1(\mu)\right] \quad (\tau=\eta) \\ &\left[\cos\vartheta\sum_{l=0}^\infty A_lP_l(\mu)+\sin\vartheta\sum_{l=1}^\infty B_lP_l^1(\mu)\right] \quad (\tau=\zeta) \end{aligned}\right\}. \end{aligned}\right\} \tag{58}$$

The integration over φ can be carried out directly, and we are left with

$$\left.\begin{aligned}\overline{F}_{1,\xi}=&-\frac{i}{\pi\beta H(\beta)}\left(\frac{\pi}{2}\right)^{1/2}\sum_{n=1}^{\infty}(-i)^{n}P^{1}(\mu_1)\int_0^{\infty}dx\,e^{-(x/\beta)^{3/2}}x^{3/2}J_{n+1/2}(x)\\&\times(2n+1)\frac{(n-1)!}{(n+1)!}\int_{-1}^{+1}d\mu P_n^1(\mu)\left\{\sum_{l=0}^{\infty}A_l(1-\mu^2)^{1/2}P_l(\mu)\right.\\&\left.\qquad-\sum_{l=1}^{\infty}B_l\mu P_l^1(\mu)\right\},\end{aligned}\right\}\quad(59)$$

$$\overline{F}_{1,\eta}=0\,,\quad(60)$$

and

$$\left.\begin{aligned}\overline{F}_{1,\zeta}=&-\frac{i}{\pi\beta H(\beta)}\left(\frac{\pi}{2}\right)^{1/2}\sum_{n=0}^{\infty}(-i)^{n}P_n(\mu_1)\int_0^{\infty}dx\,e^{-(x/\beta)^{3/2}}x^{3/2}J_{n+1/2}(x)\\&\times(2n+1)\int_{-1}^{+1}d\mu P_n(\mu)\left\{\sum_{l=0}^{\infty}A_l\mu P_l(\mu)+\sum_{l=1}^{\infty}B_l(1-\mu^2)^{1/2}P_l^1(\mu)\right\}.\end{aligned}\right\}\quad(61)$$

The result (60) is, of course, to be expected. Using the recursion formulae (35) and the orthogonality relations among the Legendre functions, we can readily effect the integrations over μ in equations (59) and (61). We find

$$\left.\begin{aligned}\overline{F}_{1,\xi}=&-\frac{2i}{\pi\beta H(\beta)}\left(\frac{\pi}{2}\right)^{1/2}\sum_{n=1}^{\infty}(-i)^{n}P_n^1(\mu_1)\int_0^{\infty}e^{-(x/\beta)^{3/2}}x^{3/2}J_{n+1/2}(x)\\&\times\left[\frac{1}{2n-1}\{A_{n-1}-(n-1)B_{n-1}\}-\frac{1}{2n+3}\{A_{n+1}+(n+2)B_{n+1}\}\right]dx\end{aligned}\right\}\quad(62)$$

and

$$\left.\begin{aligned}\overline{F}_{1,\zeta}=&-\frac{2i}{\pi\beta H(\beta)}\left(\frac{\pi}{2}\right)^{1/2}\sum_{n=0}^{\infty}(-i)^{n}P_n(\mu_1)\int_0^{\infty}e^{-(x/\beta)^{3/2}}x^{3/2}J_{n+1/2}(x)\\&\times\left[\frac{n}{2n-1}\{A_{n-1}-(n-1)B_{n-1}\}+\frac{n+1}{2n+3}\{A_{n+1}+(n+2)B_{n+1}\}\right]dx\,.\end{aligned}\right\}\quad(63)$$

It is seen that the foregoing expressions for $\overline{F}_{1,\xi}$ and $\overline{F}_{1,\zeta}$ can be written somewhat more compactly if we introduce the quantities

$$C_l=\frac{1}{2l+1}\{A_l-lB_l\}\quad\text{and}\quad D_l=\frac{1}{2l+1}\{A_l+(l+1)B_l\}\,.\quad(64)$$

Then

$$\left.\begin{aligned}\overline{F}_{1,\xi}=&-\frac{2i}{\pi\beta H(\beta)}\left(\frac{\pi}{2}\right)^{1/2}\sum_{n=1}^{\infty}(-i)^{n}P_n^1(\mu)\int_0^{\infty}e^{-(x/\beta)^{3/2}}x^{3/2}J_{n+1/2}(x)\\&\qquad\times[C_{n-1}-D_{n+1}]\,dx\end{aligned}\right\}\quad(65)$$

and

$$\left.\begin{aligned}\overline{F}_{1,\zeta}=&-\frac{2i}{\pi\beta H(\beta)}\left(\frac{\pi}{2}\right)^{1/2}\sum_{n=0}^{\infty}(-i)^{n}P_n(\mu)\int_0^{\infty}e^{-(x/\beta)^{3/2}}x^{3/2}J_{n+1/2}(x)\\&\qquad\times[nC_{n-1}+(n+1)D_{n+1}]\,dx\,,\end{aligned}\right\}\quad(66)$$

where we have further suppressed the suffix "1" in μ_1, thus letting μ denote the cosine of the angle between the directions of $\boldsymbol{r}_1$ and $\boldsymbol{F}_0$.

Using equations (39) and (40), we shall now obtain explicit formulae for C_l and D_l. First, we notice that the factor which occurs in front of the integrals on the right-hand sides of the equations (39) and (40) can be re-written as (cf. eqs. [44] and [52])

$$i^l \frac{15}{8}\left(\frac{x}{\beta}\right)^{1/2} Q_H . \tag{67}$$

Next, evaluating C_l and D_l as defined in equation (64), we find that

$$C_l = i^l \frac{15 Q_H}{8(2l+3)(2l+1)}\left(\frac{x}{\beta}\right)^{1/2} \int_0^\infty dz\, \xi^{l/2} \frac{\tilde{z}^{1/2}}{z^{3/2}} \left[l J_{l+1/2} - 3(l+2) J_{l+5/2} \right] \tag{68}$$

and

$$D_l = i^l \frac{15 Q_H}{8(2l-1)(2l+1)}\left(\frac{x}{\beta}\right)^{1/2} \int_0^\infty dz\, \xi^{l/2} \frac{\tilde{z}^{1/2}}{z^{3/2}} \left[-3(l-1) J_{l-3/2} + (l+1) J_{l+1/2} \right], \tag{69}$$

where it might be recalled that the range of integration over z has to be broken at z_1, according to the scheme (31) and (32). Thus, the "infinite" integrals occurring on the right-hand sides of the foregoing equations are really functions of z_1 and therefore also of $|\boldsymbol{\rho}|$ and $|\boldsymbol{r}_1|$, according to equation (29). In terms of our new variables x and β (eq. [52]), we can express z_1 as follows:

$$z_1 = \frac{GM|\boldsymbol{\rho}||\boldsymbol{F}_0|}{|\boldsymbol{r}_1|^2 |\boldsymbol{F}_0|} = \frac{GM}{|\boldsymbol{r}_1|^2 Q_H} \frac{x}{\beta}; \tag{70}$$

or, alternatively,

$$z_1 = \frac{15^{2/3}}{4^{2/3} 2\pi N^{2/3} |\boldsymbol{r}_1|^2} \frac{x}{\beta}. \tag{71}$$

This suggests that we measure $|\boldsymbol{r}_1|$ in units of the distance

$$l = \frac{15^{1/3}}{4^{1/3}(2\pi)^{1/2}} N^{-1/3} = 0.619804 N^{-1/3}. \tag{72}$$ [8]

If s denotes $|\boldsymbol{r}_1|$ measured in this unit, we clearly have

$$z_1 = \frac{x}{s^2 \beta}. \tag{73}$$

Thus, the integrals occurring in the formulae for C_l and D_l are functions of x, s, and β only through the combination $x/s^2\beta$.

Equations (60), (65), (66), (68), and (69) together provide the complete formal solution to the problem of spatial correlations.

6. *The first moment of* $\mathbf{F}_1$ *in the direction of* $\mathbf{F}_0$ *and its average.*—A quantity of considerable interest is the average value of $\boldsymbol{F}_1$ in the direction of $\boldsymbol{F}_0$. Since (see Fig. 1)

$$F_{1,\boldsymbol{F}_0} = \cos\vartheta F_{1,\zeta} + \sin\vartheta F_{1,\xi}, \tag{74}$$

we have

$$\overline{F}_{1,\boldsymbol{F}_0} = \mu \overline{F}_{1,\zeta} + (1-\mu^2)^{1/2} \overline{F}_{1,\xi}. \tag{75}$$

[8] Since the average distance $\overline{D}$ between the stars is $0.55396 N^{-1/3}$ (cf. S. Chandrasekhar, *Rev. Mod. Phys.*, **15**, 1, 1943, eq. [676]), it follows that the unit of distance adopted is of the same order as $\overline{D}$.

Hence, according to equations (65) and (66),

$$\left.\begin{aligned}\overline{F}_{1,\,F_0} = -\frac{2i}{\pi\beta H(\beta)}\left(\frac{\pi}{2}\right)^{1/2}\Bigg\{\sum_{n=1}^{\infty}(-i)^n P_1^1(\mu)P_n^1(\mu)\int_0^\infty e^{-(x/\beta)^{3/2}}x^{3/2}J_{n+1/2}(x)\\ \times[C_{n-1}-D_{n+1}]\,dx+\sum_{n=0}^{\infty}(-1)^n P_1(\mu)P_n(\mu)\int_0^\infty e^{-(x/\beta)^{3/2}}x^{3/2}J_{n+1/2}(x)\\ \times[nC_{n-1}+(n+1)D_{n+1}]\,dx\Bigg\}.\end{aligned}\right\} \tag{76}$$

Now, from the point of view of the applications of the theory, greatest interest is attached to $\overline{F}_{1,\,F_0}$ only after it has been further averaged over all mutual orientations of the vectors F_0 and r_1.[9] When this additional averaging is performed, it is seen that the only two terms in the infinite series in equation (76) which survive are those with $n = 1$. We are thus left with

$$\overline{\overline{F}}_{1,\,F_0} = -\frac{2}{\pi\beta H(\beta)}\left(\frac{\pi}{2}\right)^{1/2}\int_0^\infty e^{-(x/\beta)^{3/2}}x^{3/2}J_{3/2}(x)\,C_0\,dx\,. \tag{77}$$

On the other hand (cf. eq. [68]),

$$C_0 = -Q_H\,\frac{15}{4}\left(\frac{x}{\beta}\right)^{1/2}\int_0^\infty dz\,\frac{z^{1/2}}{z^{3/2}}J_{5/2}(z)\,. \tag{78}$$

Hence, combining equations (77) and (78), we obtain the relatively simple formula

$$\overline{\overline{F}}_{1,\,F_0} = \frac{15Q_H}{2\pi\beta^{3/2}H(\beta)}\left(\frac{\pi}{2}\right)^{1/2}\int_0^\infty e^{-(x/\beta)^{3/2}}x^2J_{3/2}(x)\,\mathcal{Q}\left(\frac{x}{s^2\beta}\right)dx\,, \tag{79}$$

where we have used $\mathcal{Q}(y)$ to denote the function

$$\mathcal{Q}(y) = \int_0^\infty \frac{z^{1/2}}{z^{3/2}}J_{5/2}(z)\,dz\,. \tag{80}$$

Letting

$$F_1 = Q_H\boldsymbol{\beta}_1 \quad\text{and}\quad F_0 = Q_H\boldsymbol{\beta}\,, \tag{81}$$

we can re-write equation (79) more conveniently in the form

$$\overline{\overline{\beta}}_{1,\,\boldsymbol{\beta}} = \frac{15}{2\pi\beta^{3/2}H(\beta)}\int_0^\infty e^{-(x/\beta)^{3/2}}x^{3/2}\left(\frac{\sin x}{x}-\cos x\right)\mathcal{Q}\left(\frac{x}{s^2\beta}\right)dx\,, \tag{82}$$

where we have further substituted explicitly for $J_{3/2}(x)$.

We now proceed to a closer discussion of the equations (80) and (82).

i) $\mathcal{Q}(y)$ *and its asymptotic behavior.*—According to the scheme (31) and (32), the equation defining $\mathcal{Q}(y)$ has explicitly the form

$$\mathcal{Q}(y) = \int_0^y \frac{1}{z}J_{5/2}(z)\,dz + y^{1/2}\int_y^\infty \frac{1}{z^{3/2}}J_{5/2}(z)\,dz\,. \tag{83}$$

[9] E.g., see the paper "On The Stability of Binary Systems" appearing later in this same issue of the *Astrophysical Journal* (p. 54).

Using the formula

$$\frac{d}{dz}(z^{-n}J_n) = -z^{-n}J_{n+1}, \tag{84}$$

well known in the theory of Bessel functions, we can directly evaluate the second of the two integrals occurring on the right-hand side of equation (83), while the first can be simplified by an integration by parts. The two integrated parts cancel each other, and we are left with

$$\mathcal{Q}(y) = \tfrac{1}{2}\int_0^y \frac{1}{z^2} J_{3/2}(z)\, dz\,; \tag{85}$$

or, using the formula for $J_{3/2}$, we have

$$\mathcal{Q}(y) = \frac{1}{(2\pi)^{1/2}}\int_0^y \frac{1}{z^{7/2}}\,(\sin z - z\cos z)\, dz\,. \tag{86}$$

After two further integrations by parts the foregoing equation can be reduced to the form

$$\mathcal{Q}(y) = \frac{4}{5(2\pi)^{1/2}}\int_0^y \frac{\cos z}{z^{1/2}}\, dz - \frac{1}{5}\left(\frac{2}{\pi}\right)^{1/2}\frac{1}{y^2}[(1+2y^2)\sin y - y\cos y]\,, \tag{87}$$

or, somewhat differently,

$$\mathcal{Q}(y) = \frac{1}{5}\left[4\mathcal{F}(y) - \frac{1}{y}J_{3/2}(y) - 2J_{1/2}(y)\right], \tag{88}$$

where we have used $\mathcal{F}(y)$ to denote the *Fresnel integral,*

$$\mathcal{F}(y) = \frac{1}{(2\pi)^{1/2}}\int_0^y \frac{\cos z}{z^{1/2}}\, dz = \frac{1}{2}\int_0^y J_{-1/2}(z)\, dz\,. \tag{89}$$

The form (88) for $\mathcal{Q}(y)$ is particularly convenient for the purposes of numerically evaluating the function.

From equation (88) and the known asymptotic expansions for the Fresnel integral[10] it can be readily shown that

$$\mathcal{Q}(y) = \frac{2}{5} - \frac{1}{(2\pi)^{1/2}}\frac{\sin y}{y^{5/2}} + O(y^{-7/2}) \qquad (y \to \infty) \tag{90}$$

and

$$\mathcal{Q}(y) = \frac{1}{3}\left(\frac{2}{\pi}\right)^{1/2} y^{1/2} + O(y^{5/2}) \qquad (y \to 0)\,. \tag{91}$$

ii) *The asymptotic behavior of* $\overline{\overline{\beta}}_{1,\,\beta}$ *for* $s \to 0$ *and* $s \to \infty$.—It is of interest to consider the asymptotic behavior of $\overline{\overline{\beta}}_{1,\,\beta}$. First, considering the behavior for $s \to 0$, we have, according to equations (82) and (90),

$$\overline{\overline{\beta}}_{1,\,\beta} \to \frac{3}{\pi\beta^{3/2}H(\beta)}\int_0^\infty e^{-(x/\beta)^{3/2}} x^{3/2}\left(\frac{\sin x}{x} - \cos x\right) dx\,. \tag{92}$$

[10] Cf. G. N. Watson, *Theory of Bessel Functions*, p. 545, Cambridge, England, 1922.

The integral occurring on the right-hand side of this equation can be expressed simply in terms of the Holtsmark function. For, writing it in the form

$$-\tfrac{2}{3}\beta^{3/2}\int_0^\infty \frac{d}{dx}\left[e^{-(x/\beta)^{3/2}}\right](\sin x - x\cos x)\,dx \tag{93}$$

and integrating by parts, we have

$$\left.\begin{aligned}\int_0^\infty e^{-(x/\beta)^{3/2}}x^{1/2}(\sin x - x\cos x)\,dx &= \tfrac{2}{3}\beta^{3/2}\int_0^\infty e^{-(x/\beta)^{3/2}}x\sin x\,dx\\ &= \frac{\pi}{3}\beta^{5/2}H(\beta)\,.\end{aligned}\right\} \tag{94}$$

Hence,

$$\bar{\bar{\beta}}_{1,\,\mathbf{B}}\to\beta\quad\text{as}\quad s\to 0\,, \tag{95}$$

a result which is to be expected. On the other hand, according to equations (82) and (91), we have for $s\to\infty$

$$\bar{\bar{\beta}}_{1,\,\mathbf{B}}\to\frac{5}{2\pi\beta^2 H(\beta)}\left(\frac{2}{\pi}\right)^{1/2}\frac{1}{s}\int_0^\infty e^{-(x/\beta)^{3/2}}x(\sin x - x\cos x)\,dx\qquad(s\to\infty)\,. \tag{96}$$

In other words,

$$\bar{\bar{\beta}}_{1,\,\mathbf{B}}\propto s^{-1}\quad\text{for}\quad s\to\infty\,, \tag{97}$$

the constant of proportionality depending, however, on β; this behavior for $S\to\infty$ has important consequences for the applications of the theory.[11]

7. *The average value of* $\mathbf{F}_{1,\,\mathbf{F}_0}$ *for all* $\mathbf{F}_0$ *and the correlation in the forces acting simultaneously at two different points.*—In the preceding section we have evaluated the average value of $\bar{F}_{1,\,\boldsymbol{F}_0}$ for all mutual directions between $\boldsymbol{F}_0$ and $\boldsymbol{r}_1$. The result of this averaging was to yield a function $\bar{\bar{F}}_{1,\,\boldsymbol{F}_0}$ of the two variables $|\boldsymbol{F}_0|$ and $|\boldsymbol{r}_1|$. If we now average $\bar{\bar{F}}_{1,\,\boldsymbol{F}_0}$ still further over all initial values of $|\boldsymbol{F}_0|$ (with the appropriate weight function $W(|\boldsymbol{F}_0|)$), we shall obtain a function of $|\boldsymbol{r}_1|$ only which will describe the correlation in the forces acting simultaneously at points separated by a distance $|\boldsymbol{r}_1|$. For,

$$\bar{\bar{\bar{F}}}_{1,\,\boldsymbol{F}_0}=\int_0^\infty \bar{\bar{F}}_{1,\,\boldsymbol{F}_0}W(|\boldsymbol{F}_0|)\,d|\boldsymbol{F}_0|\,; \tag{98}$$

and this is clearly the same as

$$\bar{\bar{\bar{F}}}_{1,\,\boldsymbol{F}_0}=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\boldsymbol{F}_1\cdot\mathbf{1}_{\boldsymbol{F}_0}W(\boldsymbol{F}_0,\boldsymbol{F}_1)\,d\boldsymbol{F}_0 d\boldsymbol{F}_1\,, \tag{99}$$

where we have used $\mathbf{1}_{\boldsymbol{F}_0}$ to denote a unit vector in the direction of $\boldsymbol{F}_0$.

Since the distribution of $|\boldsymbol{F}_0|$ is directly governed by the Holtsmark function $H(\beta)$ (cf. I, eq. [115]), we have

$$\bar{\bar{\bar{\beta}}}_{\mathbf{B}}(s)=\frac{\bar{\bar{\bar{F}}}_1}{Q_H}=\int_0^\infty \bar{\bar{\beta}}_{1,\,\mathbf{B}}H(\beta)\,d\beta\,, \tag{100}$$

[11] See IV, p. 52.

where it might be recalled that s measures $|r_1|$ in units of the distance l introduced in equation (72). Substituting for $\overline{\overline{\beta}}_{1,\,\mathbf{B}}$ from equation (82), we can write

$$\overline{\overline{\beta}}_{\mathbf{B}}(s)=\int_0^\infty I(\beta)\,d\beta\,, \tag{101}$$

where

$$I(\beta)=\frac{15}{2\pi\beta^{3/2}}\int_0^\infty e^{-(x/\beta)^{3/2}}x^{3/2}\left(\frac{\sin x}{x}-\cos x\right)\mathfrak{Q}\left(\frac{x}{s^2\beta}\right)dx\,. \tag{102}$$

Putting $x=\beta y$ in equation (102), we can express $I(\beta)$ alternatively in the form

$$I(\beta)=\int_0^\infty \Phi(y)\left(\frac{\sin\beta y}{y}-\beta\cos\beta y\right)dy\,, \tag{103}$$

where, for the sake of brevity, we have written

$$\Phi(y)=\frac{15}{2\pi}\,e^{-y^{3/2}}y^{3/2}\mathfrak{Q}\left(\frac{y}{s^2}\right). \tag{104}$$

Integrating by parts the term in $\cos\beta y$ in equation (103), we find

$$I(\beta)=\int_0^\infty \Phi(y)\,\frac{\sin\beta y}{y}\,dy+\int_0^\infty \Phi'(y)\sin\beta y\,dy\,, \tag{105}$$

where Φ' denotes the derivative of Φ.

Now, multiplying both sides of equation (105) by $\sin\beta z/\beta z$ (where z is some positive constant) and integrating over β from 0 to ∞, we obtain

$$\left.\begin{aligned}\int_0^\infty I(\beta)\frac{\sin\beta z}{\beta z}\,d\beta=\int_0^\infty\!\!\int_0^\infty \Phi(y)\frac{\sin\beta y\sin\beta z}{\beta y z}\,d\beta dy&\\ +\int_0^\infty\!\!\int_0^\infty \Phi'(y)\frac{\sin\beta y\sin\beta z}{\beta z}\,d\beta dy\,.&\end{aligned}\right\} \tag{106}$$

Since

$$\int_0^\infty\frac{\sin\beta y\sin\beta z}{\beta}\,d\beta=\tfrac{1}{4}\log\left(\frac{y+z}{y-z}\right)^2, \tag{107}$$

equation (106) reduces to

$$\left.\begin{aligned}\int_0^\infty I(\beta)\frac{\sin\beta z}{\beta z}\,d\beta=\int_0^\infty \Phi(y)\frac{1}{4yz}\log\left(\frac{y+z}{y-z}\right)^2dy&\\ +\int_0^\infty \Phi'(y)\frac{1}{4z}\log\left(\frac{y+z}{y-z}\right)^2dy\,,&\end{aligned}\right\} \tag{108}$$

a formula which is valid for all positive z. Passing now to the limit $z=0$, we obtain

$$\int_0^\infty I(\beta)\,d\beta=\int_0^\infty \Phi(y)\frac{dy}{y^2}+\int_0^\infty \Phi'(y)\frac{dy}{y}\,. \tag{109}$$

Integrating by parts the second of the two integrals occurring on the right-hand side, we find that

$$\int_0^\infty I(\beta)\,d\beta = 2\int_0^\infty \Phi(y)\frac{dy}{y^2}\,. \tag{110}$$

According to equations (101), (104), and (110), we therefore have

$$\bar{\bar{\beta}}_{\mathbf{B}}(s) = \frac{15}{\pi}\int_0^\infty e^{-y^{3}/{}^{2}}\mathfrak{Q}\left(\frac{y}{s^2}\right)\frac{dy}{y^{1/2}}\,. \tag{111}$$

We shall find that the function $\bar{\bar{\beta}}_{\mathbf{B}}(s)$ plays an important role in the applications of the theory. We shall therefore discuss the integral defining this function somewhat closely.

First, we shall derive a useful alternative form of equation (111). Writing $y = xs^2$ in this equation and substituting for $\mathfrak{Q}(x)$ according to equation (85), we find that $\bar{\bar{\beta}}_{\mathbf{B}}(s)$ can be expressed as a double integral in the form

$$\bar{\bar{\beta}}_{\mathbf{B}}(s) = \frac{15}{2\pi}\,s\int_0^\infty \frac{dx}{x^{1/2}}\,e^{-x^{3/2}s^3}\int_0^x \frac{dz}{z^2}\,J_{3/2}(z)\,, \tag{112}$$

or, inverting the order of the integration,

$$\bar{\bar{\beta}}_{\mathbf{B}}(s) = \frac{15}{2\pi}\,s\int_0^\infty \frac{dz}{z^2}\,J_{3/2}(z)\int_z^\infty \frac{dx}{x^{1/2}}\,e^{-x^{3/2}s^3}\,. \tag{113}$$

Putting

$$t = x^{3/2}s^3\,, \tag{114}$$

equation (113) becomes

$$\bar{\bar{\beta}}_{\mathbf{B}}(s) = \frac{5}{\pi}\int_0^\infty \frac{dz}{z^2}\,J_{3/2}(z)\int_{s^3z^{3/2}}^\infty dt\,t^{-2/3}e^{-t}\,, \tag{115}$$

which is the form required. Since the incomplete Γ-function is defined by

$$\Gamma_x(p+1) = \int_0^x e^{-t}t^p\,dt\,, \tag{116}$$

we can re-write equation (115) alternatively in the form

$$\bar{\bar{\beta}}_{\mathbf{B}}(s) = \frac{5}{\pi}\int_0^\infty \frac{dz}{z^2}\,J_{3/2}(z)\,\left[\Gamma\left(\tfrac{1}{3}\right) - \Gamma_{s^3z^{3/2}}\left(\tfrac{1}{3}\right)\right]\,. \tag{117}$$

i) *The asymptotic expansion for* $\bar{\bar{\beta}}_{\mathbf{B}}(s)$ *for* $s \to 0$.—The behavior of $\bar{\bar{\beta}}_{\mathbf{B}}(s)$ for $s \to 0$ can be derived from equation (117) by using an appropriate expansion for the incomplete Γ-function which occurs under the integral sign in this equation. Thus, since

$$\left.\begin{aligned}\Gamma_{s^3z^{3/2}}\left(\tfrac{1}{3}\right) &= \int_0^{s^3z^{3/2}} e^{-t}t^{-2/3}\,dt\\ &= \int_0^{s^3z^{3/2}}\left(1 - t + \frac{t^2}{2!} - \dots\right)t^{-2/3}\,dt\\ &= 3\,s\,z^{1/2} - \tfrac{3}{4}\,s^4z^2 + O(s^7)\,,\end{aligned}\right\} \tag{118}$$

we have

$$\overline{\overline{\beta}}_{\mathbf{B}}(s)=\frac{5}{\pi}\int_0^\infty\frac{dz}{z^2}J_{3/2}(z)\,[\Gamma(\tfrac{1}{3})-3sz^{1/2}+\tfrac{3}{4}s^4z^2+\dots]\quad(s\to0)\,. \tag{119}$$

Substituting explicitly for $J_{3/2}$ in the foregoing equation, we obtain

$$\overline{\overline{\beta}}_{\mathbf{B}}(s)=\frac{5}{\pi}\left(\frac{2}{\pi}\right)^{1/2}\int_0^\infty\frac{dz}{z^{7/2}}(\sin z-z\cos z)\,[\Gamma(\tfrac{1}{3})-3sz^{1/2}+\tfrac{3}{4}s^4z^2+\dots]\,. \tag{120}$$

Now the integrals

$$\int_0^\infty\frac{dz}{z^j}(\sin z-z\cos z) \tag{121}$$

for $j=7/2$, 3, and 3/2 are readily expressible in terms of known integrals and have respectively the values

$$\frac{4}{5}\left(\frac{\pi}{2}\right)^{1/2},\quad\frac{\pi}{4},\quad\text{and}\quad\left(\frac{\pi}{2}\right)^{1/2}\qquad(j=\tfrac{7}{2},\,3,\,\tfrac{3}{2})\,. \tag{122}$$

Accordingly, equation (120) leads to the following asymptotic expansion for $\overline{\overline{\beta}}_{\mathbf{B}}(s)$ valid for $s\to0$:

$$\overline{\overline{\beta}}_{\mathbf{B}}(s)=\frac{4}{\pi}\Gamma(\tfrac{1}{3})-\frac{15}{4}\left(\frac{2}{\pi}\right)^{1/2}s+\frac{15}{4\pi}s^4+\dots\,. \tag{123}$$

Numerically, this series has the form

$$\overline{\overline{\beta}}_{\mathbf{B}}(s)=3.41093-2.992067\,s+1.193662\,s^4+\dots\,. \tag{124}$$

According to equation (123),

$$\overline{\overline{\beta}}_{\mathbf{B}}(0)=\frac{4}{\pi}\Gamma(\tfrac{1}{3})\,. \tag{125}$$

On the other hand, it is clear that, as $s\to0$, $\overline{\overline{\beta}}_{\mathbf{B}}(s)$ must simply tend to the first moment of the Holtsmark function (cf. eqs. [95] and [100]). We have thus incidentally proved that

$$\int_0^\infty\beta H(\beta)\,d\beta=\frac{4}{\pi}\Gamma(\tfrac{1}{3})\,. \tag{126}$$ [12]

[12] It is perhaps of interest to establish equation (126) directly. We have (cf. eq. [55])

$$\beta H(\beta)=\frac{2}{\pi}\int_0^\infty e^{-(x/\beta)^{3/2}}x\sin x\,dx\,. \tag{1'}$$

Putting $x=\beta y$ in the right-hand side of this equation, we can write

$$\beta H(\beta)=\beta^2\int_0^\infty\Phi(y)\sin\beta y\,dy\,, \tag{2'}$$

where

$$\Phi(y)=\frac{2}{\pi}e^{-y^{3/2}}y\,. \tag{3'}$$

Integrating equation (2′) twice successively by parts, we obtain

$$\beta H(\beta)=-\int_0^\infty\Phi''(y)\sin\beta y\,dy\,. \tag{4'}$$

[Footnote continued on following page]

ii) *A series expansion for* $\bar{\bar{\beta}}_{\mathbf{B}}(s)$ *for* $s \rightarrow \infty$.—A rapidly converging series for $\bar{\bar{\beta}}_{\mathbf{B}}(s)$ for $s \rightarrow \infty$ can be obtained in the following manner:

Differentiating equation (115) with respect to s, we find that

$$-\frac{d\bar{\bar{\beta}}_{\mathbf{B}}(s)}{ds} = \frac{15}{\pi}\int_0^\infty \frac{dz}{z^{3/2}} J_{3/2}(z)\, e^{-s^3 z^{3/2}}; \tag{127}$$

or, substituting for $J_{3/2}$, we have

$$-\frac{d\bar{\bar{\beta}}_{\mathbf{B}}(s)}{ds} = \frac{15}{\pi}\left(\frac{2}{\pi}\right)^{1/2}\int_0^\infty e^{-s^3 z^{3/2}} \frac{1}{z^3}(\sin z - z\cos z)\, dz. \tag{128}$$

Replacing sin z and cos z in the foregoing equation by their respective series expansions and inverting the order of the integration and the summation, we obtain

$$-\frac{d\bar{\bar{\beta}}_{\mathbf{B}}(s)}{ds} = \frac{15}{\pi}\left(\frac{2}{\pi}\right)^{1/2}\sum_{n=1}^{\infty}(-)^{n+1}\frac{2n}{(2n+1)!}\int_0^\infty e^{-s^3 z^{3/2}} z^{2n-2}\, dz. \tag{129}$$

If we now introduce the variable $t = s^3 z^{3/2}$, equation (129) becomes

$$-\frac{d\bar{\bar{\beta}}_{\mathbf{B}}(s)}{ds} = \frac{10}{\pi}\left(\frac{2}{\pi}\right)^{1/2}\sum_{n=1}^{\infty}(-)^{n+1}\frac{2n}{(2n+1)!}\frac{1}{s^{4n-2}}\int_0^\infty e^{-t} t^{(4n-5)/3}\, dt. \tag{130}$$

Hence,

$$-\frac{d\bar{\bar{\beta}}_{\mathbf{B}}(s)}{ds} = \frac{10}{\pi}\left(\frac{2}{\pi}\right)^{1/2}\sum_{n=1}^{\infty}(-)^{n+1}\frac{2n}{(2n+1)!}\Gamma\left(\frac{4n-2}{3}\right)\frac{1}{s^{4n-2}}. \tag{131}$$

Now multiplying both sides of equation (4′) by sin $\beta z/\beta z$ (where z is some positive constant) and integrating over β from 0 to ∞, we find that

$$\int_0^\infty \beta H(\beta)\frac{\sin \beta z}{\beta z}\, d\beta = -\int_0^\infty \Phi'' \frac{1}{4z}\log\left(\frac{y+z}{y-z}\right)^2 dy. \tag{5′}$$

Passing now to the limit $z = 0$, we obtain

$$\int_0^\infty \beta H(\beta) d\beta = -\int_0^\infty \Phi''(y)\frac{dy}{y}. \tag{6′}$$

But, according to our definition of $\Phi(y)$ (eq. [3′]),

$$\Phi''(y) = -\frac{3}{2\pi}(5y^{1/2} - 3y^2)e^{-y^{3/2}}. \tag{7′}$$

Thus,

$$\int_0^\infty \beta H(\beta) d\beta = \frac{3}{2\pi}\int_0^\infty (5y^{1/2} - 3y^2)e^{-y^{3/2}}\frac{dy}{y}. \tag{8′}$$

The integrals which occur on the right-hand side of this equation can be reduced to Γ-function integrals by the substitution $t = y^{3/2}$. In this manner we find that

$$\left.\begin{aligned}\int_0^\infty \beta H(\beta) d\beta &= \frac{1}{\pi}\int_0^\infty (5t^{1/3} - 3t^{4/3})e^{-t}\frac{dt}{t} \\ &= \frac{1}{\pi}[5\Gamma(1/3) - 3\Gamma(4/3)] \\ &= \frac{4}{\pi}\Gamma(1/3),\end{aligned}\right\} \tag{9′}$$

which is the required result.

Integrating this series, term by term, and remembering that $\bar{\bar{\beta}}_{B}(s)$ must tend to 0 as $s \to \infty$, we obtain the following expansion for $\bar{\bar{\beta}}_{B}(s)$:

$$\bar{\bar{\beta}}_{B}(s) = \frac{10}{\pi}\left(\frac{2}{\pi}\right)^{1/2} \sum_{n=1}^{\infty} (-)^{n+1} \frac{2n}{(4n-3)(2n+1)!} \Gamma\left(\frac{4n-2}{3}\right) \frac{1}{s^{4n-3}} . \quad (132)$$

The dominant term of this series is

$$\bar{\bar{\beta}}_{B}(s) = \frac{10}{3\pi}\left(\frac{2}{\pi}\right)^{1/2} \Gamma\left(\frac{2}{3}\right) \frac{1}{s} + O(s^{-5}) \qquad (s \to \infty) , \quad (133)$$

or, numerically,

$$\bar{\bar{\beta}}_{B}(s) \to \frac{1.14637}{s} \qquad (s \to \infty) . \quad (134)$$

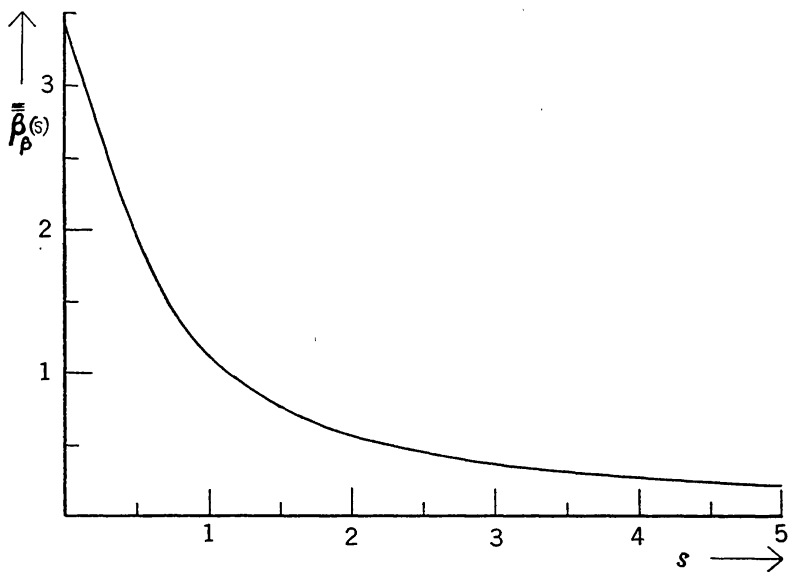

FIG. 2

This behavior of $\bar{\bar{\beta}}_{B}(s)$ for $s \to \infty$ is, of course, to be expected on the basis of our earlier result (97).

In a later paper we shall undertake a full numerical discussion of the various formulae obtained in this paper: but in the meantime we may refer to Figure 2, in which the run of the function $\bar{\bar{\beta}}_{B}(s)$ is illustrated.

8. *The average value of* $\mathbf{F}_1$ *in the direction of* $\mathbf{r}_1$.—In §§ 6 and 7 we have considered in some detail the various functions which arise from a discussion of the first moment of $\boldsymbol{F}_1$ in the direction of $\boldsymbol{F}_0$. We shall now consider certain other functions of comparable interest which result from a similar discussion of the first moment of $\boldsymbol{F}_1$ in the direction of $\boldsymbol{r}_1$. Since the ζ-axis of our co-ordinate system is in the direction of $\boldsymbol{r}_1$ (see Fig. 1), we have (cf. eq. [66])

$$\left.\begin{aligned} \overline{F}_{1,\, r_1} = -\frac{2i}{\pi\beta H(\beta)}\left(\frac{\pi}{2}\right)^{1/2} \sum_{n=0}^{\infty} (-i)^n P_n(\mu) \int_0^{\infty} e^{-(x/\beta)^{3/2}} x^{3/2} J_{n+1/2}(x) & \\ \times [n C_{n-1} + (n+1) D_{n+1}]\, dx \, . & \end{aligned}\right\} \quad (135)$$

As in the case of $\overline{F}_{1,\, F_0}$, greatest interest is attached to the foregoing equation only after it has been further averaged over all directions of $\boldsymbol{F}_0$ (keeping $\boldsymbol{r}_1$ and $|\boldsymbol{F}_0|$, however, fixed). When this further averaging process is carried out, it is seen that the only term in equation (135) which survives is that with $n = 0$, and we are left with

$$\overline{\overline{F}}_{1,\,r_1} = -\frac{2i}{\pi\beta H(\beta)}\left(\frac{\pi}{2}\right)^{1/2}\int_0^\infty e^{-(x/\beta)^{3/2}} x^{3/2} J_{1/2}(x)\, D_1 dx\,, \tag{136}$$

where, according to equation (69),

$$D_1 = iQ_H \frac{5}{4}\left(\frac{x}{\beta}\right)^{1/2}\int_0^\infty dz\, \xi^{1/2}\frac{\tilde{z}^{1/2}}{z^{3/2}} J_{3/2}(z)\,. \tag{137}$$

Combining equations (136) and (137) and expressing $\overline{\overline{F}}_{1,\,r_1}$ in units of Q_H, we have

$$\overline{\overline{\beta}}_{1,\,s} = \frac{5}{2\pi\beta^{3/2}H(\beta)}\int_0^\infty e^{-(x/\beta)^{3/2}} x^{3/2}\sin x\, \mathcal{R}\left(\frac{x}{s^2\beta}\right) dx\,, \tag{138}$$

where, for the sake of brevity, we have written

$$\mathcal{R}(y) = \int_0^\infty dz\, \xi^{1/2}\frac{\tilde{z}^{1/2}}{z^{3/2}} J_{3/2}(z)\,. \tag{139}$$

Now, according to the scheme (31) and (32), equation (139) has explicitly the form

$$\mathcal{R}(y) = \frac{1}{y^{1/2}}\int_0^y \frac{1}{z^{1/2}} J_{3/2}(z)\, dz + y\int_y^\infty \frac{1}{z^2} J_{3/2}(z)\, dz\,. \tag{140}$$

It is seen that, of the two integrals which occur in the foregoing equation, the first can be directly evaluated (cf. eq. [84]), while the second can be related quite simply to the function $\mathcal{Q}(y)$ introduced in § 6 (cf. eqs. [85] and [90]). Thus,

$$\mathcal{R}(y) = \left(\frac{2}{\pi y}\right)^{1/2} - \frac{1}{y} J_{1/2}(y) + 2y\left[\tfrac{2}{5} - \mathcal{Q}(y)\right]; \tag{141}$$

or, substituting for $\mathcal{Q}(y)$ from equation (88), we have

$$\mathcal{R}(y) = \left(\frac{2}{\pi y}\right)^{1/2} + \tfrac{2}{5} J_{3/2}(y) + \left(\tfrac{4}{5}y - \frac{1}{y}\right) J_{1/2}(y) + \tfrac{8}{5}y\left[\tfrac{1}{2} - \mathcal{F}(y)\right], \tag{142}$$

or, more explicitly,

$$\mathcal{R}(y) = \left(\frac{2}{\pi y}\right)^{1/2}\left[1 - \frac{3}{5}\frac{\sin y}{y} - \tfrac{2}{5}\cos y + \tfrac{4}{5}y\sin y\right] + \tfrac{8}{5}y\left[\tfrac{1}{2} - \mathcal{F}(y)\right]. \tag{143}$$

From equation (143) the behavior of $\mathcal{R}(y)$ for $y \to 0$ and $y \to \infty$ can be readily deduced. We find that

$$\mathcal{R}(y) = \tfrac{4}{5}y + O(y^{3/2}) \qquad (y \to 0) \tag{144}$$

and

$$\mathcal{R}(y) = \left(\frac{2}{\pi}\right)^{1/2} y^{-1/2} + O(y^{-5/2}) \qquad (y \to \infty)\,. \tag{145}$$

The asymptotic relations (144) and (145) for $\mathcal{R}(y)$ enables us to derive the behavior of $\overline{\overline{\beta}}_{1,\,s}$ for $s \to 0$ and $s \to \infty$. Thus, according to equations (138) and (145),

$$\overline{\overline{\beta}}_{1,\,s} \to \frac{5}{2\pi\beta H(\beta)} \left(\frac{2}{\pi}\right)^{1/2} s \int_0^\infty e^{-(x/\beta)^{3/2}} x \sin x dx \quad (s \to 0)\,, \tag{146}$$

or, remembering our definition of $H(\beta)$,

$$\overline{\overline{\beta}}_{1,\,s} \to \frac{5}{4}\left(\frac{2}{\pi}\right)^{1/2} s \quad (s \to 0)\,. \tag{147}$$

In other words, for short separations between two points, there is, on the average, a repulsive force (proportional to the distance) operating on one relative to the other. On the other hand, for $s \to \infty$, we have (cf. eqs. [138] and [144])

$$\overline{\overline{\beta}}_{1,\,s} \to \frac{2}{\pi\beta^{5/2} H(\beta)} \frac{1}{s^2} \int_0^\infty e^{-(x/\beta)^{3/2}} x^{5/2} \sin x dx \quad (s \to \infty)\,. \tag{148}$$

Since, however,

$$\frac{d}{d\beta}[\beta H(\beta)] = \frac{3}{\pi\beta^{5/2}} \int_0^\infty e^{-(x/\beta)^{3/2}} x^{5/2} \sin x dx\,, \tag{149}$$

we can re-write equation (148) in the form

$$\overline{\overline{\beta}}_{1,\,s} \to \frac{2}{3s^2} \frac{1}{H(\beta)} \frac{d}{d\beta}[\beta H(\beta)] \quad (s \to \infty)\,, \tag{150}$$

or, somewhat differently,

$$\overline{\overline{\beta}}_{1,\,s} \to \frac{2}{3s^2}\left(1 + \beta \frac{d \log H}{d\beta}\right) \quad (s \to \infty)\,. \tag{151}$$

According to this equation,

$$\overline{\overline{\beta}}_{1,\,s} \to \frac{2}{s^2} \quad (\beta \to 0) \quad \text{and} \quad \overline{\overline{\beta}}_{1,\,s} \to -\frac{1}{s^2} \quad (\beta \to \infty)\,. \tag{152}$$

In other words, for large s, the sign of $\overline{\overline{\beta}}_{1,\,s}$ depends on the magnitude of β. In this respect $\overline{\overline{\beta}}_{1,\,s}$ differs from $\overline{\overline{\beta}}_{1,\,\boldsymbol{\beta}}$, which is always positive and is, moreover, a monotonic decreasing function of β. An even more fundamental difference in the character of the functions $\overline{\overline{\beta}}_{1,\,s}$ and $\overline{\overline{\beta}}_{1,\,\boldsymbol{\beta}}$ is revealed when we average $\overline{\overline{\beta}}_{1,\,s}$ over all β to obtain a quantity similar to $\overline{\overline{\beta}}_{\boldsymbol{\beta}}(s)$ considered in § 7; for, as we shall now show,

$$\int_0^\infty \overline{\overline{\beta}}_{1,\,s} H(\beta)\, d\beta \equiv 0\,. \tag{153}$$

To prove this, write $x = \beta y$ in equation (138) and express $\overline{\overline{\beta}}_{1,\,s} H(\beta)$ in the form

$$\overline{\overline{\beta}}_{1,\,s} H(\beta) = \beta \int_0^\infty \Phi(y) \sin \beta y dy\,, \tag{154}$$

where

$$\Phi(y) = \frac{5}{2\pi} e^{-y^{3/2}} y^{3/2} \mathcal{R}\left(\frac{y}{s^2}\right). \tag{155}$$

Integrating equation (154) by parts, we have

$$\bar{\bar{\beta}}_{1,\,s}H(\beta)=\int_0^\infty \Phi'(y)\cos\beta y\,dy\,. \tag{156}$$

Multiplying both sides of this equation by sin $\beta z/\beta z$ (where z is some positive constant) and integrating over β from 0 to ∞, we obtain

$$\int_0^\infty \bar{\bar{\beta}}_{1,\,s}H(\beta)\frac{\sin\beta z}{\beta z}\,d\beta=\int_0^\infty\int_0^\infty \Phi'(y)\frac{\cos\beta y\sin\beta z}{\beta z}\,d\beta\,dy\,. \tag{157}$$

On the other hand, since

$$\left.\begin{aligned}\int_0^\infty \frac{\cos\beta y\sin\beta z}{\beta}\,d\beta&=\frac{\pi}{2}\qquad \text{if}\quad z>y\\ &=0\qquad \text{if}\quad z<y\,,\end{aligned}\right\} \tag{158}$$

equation (157) becomes

$$\left.\begin{aligned}\int_0^\infty \bar{\bar{\beta}}_{1,\,s}H(\beta)\frac{\sin\beta z}{\beta z}\,d\beta&=\frac{\pi}{2z}\int_0^z \Phi'(y)\,dy\\ &=\frac{\pi}{2}\frac{\Phi(z)}{z}\,,\end{aligned}\right\} \tag{159}$$

a formula which is valid for any positive finite z. Passing now to the limit $z = 0$, we find

$$\int_0^\infty \bar{\bar{\beta}}_{1,\,s}H(\beta)\,d\beta=\frac{\pi}{2}\lim_{z\to 0}\left[\frac{\Phi(z)}{z}\right]. \tag{160}$$

According to our definition of Φ (eq. [155]), the quantity on the right-hand side vanishes. Hence,

$$\bar{\bar{\beta}}_{1,\,s}\equiv 0\,, \tag{161}$$

which was the result to be proved.