Stellar Configurations with degenerate Cores.

In an article published in the March number of *The Observatory* the new orientation towards the general problem of Stellar Structure resulting from the use of degenerate statistics was discussed. A result obtained in that article and which is of importance in our present discussion can be recalled in the following terms:—For stellar material at a specified temperature T and density ρ we can define abstractly a quantity β denoting the ratio between the gas pressure p and the total pressure P (which is the sum of the gas kinetic and radiation pressure). Then if

$$\frac{960}{\pi^4}\frac{1-\beta}{\beta}>1, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

the material at density ρ and temperature T will be a perfect gas in the classical sense. If β_ω be such that relation (1) is an *equality* then in stellar configurations in which $(1-\beta)$ is always greater than $(1-\beta_\omega)$ the stellar material continues to be a perfect gas however high the density may become. On the standard model this means that if the mass be greater than a certain critical mass (say $\mathfrak{M}$) then finiteness of central density restricts us to consider only the non-singular solutions of Emden's equation with index 3. This means that if we plot $(1-\beta)$ against the radius R then the curves of constant mass $(M>\mathfrak{M})$ are lines parallel to the R-axis. For $(1-\beta)>(1-\beta_\omega)$ these lines (which I shall refer to as "Eddington lines") are not distorted by the introduction of degenerate states. The question of using degenerate states for these configurations does not arise at all. If, however $(1-\beta)<(1-\beta_\omega)$, the curves of constant mass are no longer fully represented by the Eddington lines, and in this region we have a non-trivial solution to Milne's fundamental problem, which for our purposes can be formulated as follows:—"For a star to be wholly gaseous the mass is a function of β only. Call the appropriate β, β_M. Has the star equilibrium configurations for $\beta\neq\beta_M$?" This problem is important, for it is precisely by formulating the problem of stellar structure in this way that we can fully analyse the structure of stars of mass less than $\mathfrak{M}$ $(=6{\cdot}623\,\mu^{-2}\,\odot)$ *.

* *Cf.* S. Chandrasekhar, *Zs. f. Astrophysik*, **5**, 321 (1932), equation (15).

To answer Milne's problem for $(1-\beta)<(1-\beta_\omega)$ it is essential to take the equation of state for the degenerate matter in the exact form. We cannot neglect relativistic degeneracy since we have seen already that precisely because of the relativistic effects the Eddington lines are undistorted in the greater part of the $(1-\beta, \mathrm{R})$ diagram. I shall refer to such a plot as a *Milne diagram.*

Now the equation of state for the degenerate state can be written parametrically as follows :—

$$p=\frac{\pi m^4c^5}{3h^3}f(x)\;;\;\rho=\frac{8\pi m^3c^3\mu m_{\mathrm{H}}}{3h^3}x^3=\mathrm{B}x^3\text{ (say)}, \quad . \quad (2)$$

where

$$f(x)=[x(x^2+1)^{1/2}(2x^2-3)+3\sinh^{-1}x], \quad . \quad . \quad (3)$$

the other symbols having their usual meaning. It may be noticed here that with the same definition for ρ as in (2) the pressure for a classical gas can be written as

$$p=\frac{\pi m^4c^5}{3h^3}\left(\frac{960}{\pi^4}\frac{1-\beta_1}{\beta_1}\right)^{1/3}.\,2x^4. \quad . \quad . \quad . \quad (4)$$

As a preliminary to the study of composite configurations with degenerate cores we firstly consider the structure of *completely* collapsed configurations with $\beta=1$. In this case the radiation pressure p' is zero and the total pressure p is given by (2). If one introduces the function ϕ defined as

$$\rho=\frac{\rho_c}{\left(1-\dfrac{1}{y_0^2}\right)^{3/2}}\left(\phi^2-\frac{1}{y_0^2}\right)^{3/2}, \quad . \quad . \quad . \quad (5)$$

where

$$y_0^2=x_0^2+1\;;\;\rho_c=\rho_{\text{central}}=\mathrm{B}x_0^3, \quad . \quad . \quad . \quad (5')$$

then one can prove that the structure of the configurations is completely specified by the solution of the differential equation

$$\frac{1}{\eta^2}\frac{d}{d\eta}\left(\eta^2\frac{d\phi}{d\eta}\right)=-\left(\phi^2-\frac{1}{y_0^2}\right)^{3/2}, \quad . \quad . \quad (6)$$

with

$$\phi=1 \text{ at } \eta=0\;;\;\phi(\eta_1)=\frac{1}{y_0},\ \eta_1 \text{ referring to the boundary}, \quad . \quad . \quad . \quad (7)$$

where η measures the radius vector in a suitable scale. (6) is an *exact* equation, and it is surprising that it has not been isolated before. The derivation of this exact equation has led to a considerable simplification in the analysis of the problem of stellar structure. For a specified y_0, *i.e.* for a specified central density, the structure is completely determined and in particular its mass. We see from (6) that as $y_0 \to \infty$, $\phi \to$ the Emden-function with index 3. The mass of these configurations therefore tends to a unique limit as $y_0 \to \infty$. This mass is naturally M_3 (which was first obtained by the writer (*M.N.* **91.** p. 456, 1931)). Configurations with mass less than M_3 have finite radii. On the Milne diagram we can therefore plot on the radius-axis a series of points corresponding to the radii of different masses of these configurations. M_3 in particular is at the origin of the two axes $(1-\beta, R)$.

With this necessary preliminary analysis of these configurations with $\beta = 1$ we can now see how the Eddington lines should be distorted in the region of the Milne diagram $(1-\beta) < (1-\beta_\omega)$. If we consider a star of mass M less than $\mathfrak{M}$ and contract it from infinite extension the star continues to be wholly gaseous till the central density is such that (*cf.* equations (2) and (4))

$$\left(\frac{960}{\pi^4}\frac{1-\beta_M}{\beta_M}\right)^{1/3} = \frac{f(x_0)}{2x_0^4}\ ;\ \rho_c = Bx_0^3. \quad . \quad . \quad . \quad (8)$$

The radius R_0 of this configuration (with $\rho_c = Bx_0^3$) can now be determined. We can therefore draw the curve $(R_0, 1-\beta_M)$ in the Milne diagram. This curve naturally intersects the $(1-\beta)$ axis, where $\beta_M = \beta_\omega$ corresponding to $M = \mathfrak{M}$. Hence the curves of constant mass are vertical lines parallel to the R-axis until they intersect the $(R_0, 1-\beta_M)$ curve. Below this curve the Eddington lines are distorted, and to study the curves of constant mass inside this region we have to consider composite configurations where the structure of the degenerate core is governed by the differential equation (6) and the outer envelope by Emden's equation with index 3. In considering these configurations it would be natural to work the *generalized*-standard-model in which "$\kappa\eta$" takes different values in the envelope and in the core. We will, however, first consider the usual standard model where "$\kappa\eta$" has the same constant value throughout the star.

The composite configurations can now be studied by writing down the "equations of fit" and solving them. It may be stated that for solving the equations of fit one can with some modifications adopt here the methods developed by Milne in a rather different connection. One can first prove that when β_1 has the same value in the envelope as in the core then *only collapsed configurations are possible*, i. e., a composite configuration has a "$1-\beta$" which is always less than the value $(1-\beta_M)$ which it has in the wholly gaseous state. The nature of the curves of constant mass can at once be predicted. If the mass is less than M_3 then in the completely collapsed state $(\beta_1=1)$ it has a unique radius already determined from our analysis of these configurations. For each mass M′ we can calculate $\beta_{M'}$. The vertical line through $(1-\beta_{M'})$ cuts the $(1-\beta_M, R_0)$ curve at the point $(1-\beta_{M'}, R_0(M'))$. When the star contracts further it goes along some smooth curve joining the point $(1-\beta_{M'}, R_0(M'))$ with a point on the R-axis corresponding to the radius which this M′ has in the completely collapsed state. In particular, the curve of constant mass for M_3 passes through the origin. One finds that β_{M_3} is specified by

$$\frac{960}{\pi^4}\frac{1-\beta_{M_3}}{\beta^4_{M_3}}=1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Let $\beta_{M_3}=\beta_0$. Clearly $(1-\beta_\omega)>(1-\beta_0)$.

The question arises what happens for stars with $M_3<M\leq\mathfrak{M}$. Now when $(1-\beta)<(1-\beta_\omega)$ then the configuration has a mass $M_3\beta^{-3/2}$ as $y_0\to\infty$. (This result was obtained in my paper in *M. N.* already referred to.) Hence when $M_3<M\leq\mathfrak{M}$ the curves of constant mass intersect the $(1-\beta)$ axis at a point β^* such that

$$M=M_3\beta^{*-3/2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

One can show that β^* is related to $\beta\dagger$—the value it has in the wholly gaseous state—by the relation

$$\beta^*=\left(\frac{\pi^4}{960}\frac{\beta\dagger^4}{1-\beta\dagger}\right)^{1/3} \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

We notice that $\beta^*=\beta\dagger=\beta_\omega$ is a solution. Also $\beta^*=1$ when $\beta\dagger=\beta_0$. These results are of course necessary for consistency. We should further have

$$\mathfrak{M}=M_3\beta_\omega^{-3/2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Relation (12) can in fact be shown to be true. Hence Milne's problem admits of a solution (consistent with our present knowledge of the equations of state for ionized material) for $(1-\beta)<(1-\beta_\omega)$, and in this region only collapsed configurations are possible on the standard model. We have also seen how the curves of constant mass run in this region.

The treatment of the generalized standard model ("β_2" of the core different from "β_1" of the envelope) can be carried out in a similar way, though the analysis is very much more complicated. If we consider $\beta_2=1$ as an extreme case, then one can prove for instance that *composite configurations with* $M>M_3$ *are necessarily centrally condensed*. When $M\leq M_3$, but greater than another critical mass, "quasi-diffuse" and centrally-condensed configurations make their appearance in addition to the usual collapsed configurations. It is clearly impossible to describe these results in this short communication, which is intended primarily as a preliminary statement of some of the results of the author's recent studies. The detailed investigations with full tables of solutions will be published elsewhere, but the purpose of writing this article was to show how the setting up of an exact differential equation to describe the degenerate state has led to an almost complete solution of the general problem of stellar structure along the lines indicated above.

Finally, it is necessary to emphasize one major result of the whole investigation, namely, that it must be taken as well established that the life-history of a star of small mass must be essentially different from the life-history of a star of large mass. For a star of small mass the natural white-dwarf stage is an initial step towards complete extinction. A star of large mass ($>\mathfrak{M}$) cannot pass into the white-dwarf stage, and one is left speculating on other possibilities.

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