## *The Physical State of Matter in the Interior of Stars.*

In 1924 Sir Arthur Eddington established his mass-luminosity relation on the basis of the perfect gas hypothesis. During the ten years that have elapsed since, our knowledge of the possible states of an ionized gas has advanced considerably, and the purpose of this review is to show how this new knowledge has gone far towards a clarification of ideas regarding the physical conditions prevailing in the interior of stars. It is necessary, however, to state explicitly that lack of complete information regarding the internal distribution of energy sources is not very serious when we are primarily concerned with the hydrostatic equilibrium of the star. The various numerical integrations that have been carried out with widely different laws for opacity and source distribution have shown that the qualitative nature of the information that can in principle be derived from steady state considerations alone is, within limits, independent of the particular type of laws we choose to discuss ; when it is therefore found necessary to particularize the situation, we shall work with the *Standard Model.*

Now the possible equations of state of a more or less completely ionized gas are :—

$$p=\left(\frac{k}{m_{\mathrm{H}}\mu}\right)\rho \mathrm{T}, \qquad (1)$$

$$p=\frac{1}{20}\left(\frac{3}{\pi}\right)^{2/3}\frac{h^2}{m(m_{\mathrm{H}}\mu)^{5/3}}\rho^{5/3}=\frac{9\cdot 890\times 10^{12}}{\mu^{5/3}}\rho^{5/3}$$
$$=\mathrm{K}_1\rho^{5/3}\ \text{(say)}, \qquad (2)$$

$$p=\left(\frac{3}{\pi}\right)^{1/3}\frac{hc}{8(m_{\mathrm{H}}\mu)^{4/3}}\rho^{4/3}=\frac{1\cdot 228\times 10^{15}}{\mu^{4/3}}\rho^{4/3}$$
$$=\mathrm{K}_2\rho^{4/3}\ \text{(say)}, \qquad (3)$$

where $p$=pressure, $\rho$=density, T=temperature, $k$= Boltzmann's constant, $m_{\mathrm{H}}$=mass of the hydrogen atom, $\mu$=mean molecular weight, $h$=Planck's constant, $c$= velocity of light, $m$=mass of the electron. Equations (1), (2), and (3) correspond to the perfect *, degenerate and relativistically degenerate gases respectively †. Equation (2) will be valid if the pressure given by this formula is, firstly, much greater than that given by (1), and, secondly, much less than that given by (3). These two conditions give the criteria for ordinary degeneracy. If the first criterion is satisfied and the second violated, then the material will be relativistically degenerate. On the other hand, if the pressure given by (1) is greater than that given by either (2) or (3) then the gas remains an ideal gas in the classical sense. Bearing these considerations in mind we shall examine the state of matter in the interior of stars.

(A) *Massive Stars.*—We shall first attempt to give precision as to what is meant by "massive" in this connection. To do that we assume that in the perfect gas regions of the star the equations of the standard

* We shall use the term "perfect gas" to denote an ideal gas in the classical sense (*i. e.* (1) will be valid).

† It may be remarked here that equation (1) is true, independent of relativity.

model hold, *i. e.*, the gas pressure $p$ is a constant fraction * $\beta$ of the total pressure (=the sum of the gas pressure and the radiation pressure which is equal to $a\mathrm{T}^4/3$). Then one easily finds by eliminating the temperature in (1) that in the perfect gas regions

$$p=\mathrm{C}\rho^{4/3}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where

$$\mathrm{C}=\left[\left(\frac{k}{m_{\mathrm{H}}\mu}\right)^4 \frac{3}{a}\frac{1-\beta}{\beta}\right]^{1/3}=\frac{2\cdot632\times10^{15}}{\mu^{4/3}}\left[\frac{1-\beta}{\beta}\right]^{1/3}, \quad . \quad . \quad . \quad (5)$$

where, in a well-known notation, $\beta=1-\kappa\eta\mathrm{L}/4\pi c\mathrm{GM}$ (G is the constant of gravitation).

It is now immediately obvious that if C as defined by (5) be greater than the relativistic-degenerate constant $\mathrm{K}_2$ then the perfect gas equation (1) cannot break down in any part of the internal regions of the star.

One finds by (3) and (4) that if $\mathrm{K}_2<\mathrm{C}$ then

$$\frac{1-\beta}{\beta}>\frac{\pi^4}{960}=0\cdot1015 \quad \text{or} \quad \beta<0\cdot9079. \quad . \quad . \quad . \quad (6)$$

The meaning of (6) is simply this. There is normally a certain temperature gradient in the star, and if the radiation pressure is greater than a tenth of the total pressure then the temperature increases sufficiently rapidly to prevent the matter from becoming degenerate †. We can in fact say much more than this. If for a moment we now consider that the configuration is a perfect gas sphere, then, on the standard model, $\beta$ is a function of the mass M and $\mu$ only, and is given by Eddington's quartic equation :

$$1-\beta=0\cdot00309\,(\mathrm{M}/\odot)^2\mu^4\beta^4, \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where $\odot$ refers to the mass of the Sun. For perfect gas configurations, the inequality (6) is formally equivalent to the inequality

$$\mathrm{M}>6\cdot623\mu^{-2}\times\odot. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Hence for all stars for which $\beta$ is less than or equal to

* That $\beta$ should be an absolute constant is not essential for the argument (see the following footnote).

† This result is actually much more general than the derivation would suggest. Defining $\beta$ abstractly as the ratio, gas pressure/total pressure, then it is clear that even if $\beta$ were not a constant, the stellar material would continue to be a perfect gas provided only $\beta<0\cdot91$ throughout the perfect gas parts of the star.

that given by Eddington's quartic equation * and mass M greater than $6{\cdot}623\mu^{-2}\times\odot$ the perfect gas equation of state cannot pass over into the degenerate equations of state. The conclusion is that *if equations* (1), (2), *and* (3) *represent the only possible states of stellar material then all stars with* $M>6{\cdot}623\mu^{-2}\times\odot$ *and which are not white dwarfs are necessarily wholly perfect gas configurations.* But the perfect gas law (1) can conceivably deviate in ways yet unknown to us. We shall come back to this point towards the end.

(B) *Stars of Small Mass.*—Some general conclusions regarding the physical conditions in the interior of stars of small mass can be deduced by an application of a theorem due to Eddington which states that the total pressure cannot anywhere exceed

$$P_{\text{max.}}=\frac{1}{2}\left(\frac{4}{3}\pi\right)^{1/3}GM^{2/3}\rho_0^{\,4/3}=B\rho_0^{\,4/3}\ (\text{say}), \quad . \quad . \quad (9)$$

where $\rho_0$ is the greatest density inside the star. $P_{\text{max}}$ given by (9) is just equal to the central pressure in a configuration of mass M with a uniform density $\rho_0$.

From (9) we can at once set a lower limit to the mass of stars for which zones of relativistic degeneracy can possibly occur. For if there are regions in the star which are relativistically degenerate then clearly by the above theorem

$$K_2<B \quad \text{or} \quad M>\sqrt{\left(\frac{K_2}{G}\right)^3\frac{6}{\pi}}, \quad . \quad . \quad (10)$$

or by (3)

$$M>1{\cdot}743\mu^{-2}\times\odot \; . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Hence *for stars of mass less than* $1{\cdot}743\mu^{-2}\times\odot$ *there can be no regions in which matter is relativistically degenerate.* For stars of mass less than this limit matter could be incipiently relativistically degenerate, but if equation (2) describes the state of affairs sufficiently well then by (9) we can now set an upper limit to the density (as was first shown by Eddington). Thus one finds that

$$\rho<\left(\frac{B}{K_1}\right)^3=6{\cdot}301\times10^5\,.\,\mu^5(M/\odot)^2\ \text{gms. cm.}^{-3}. \qquad (12)$$

We can also formally set an upper limit to the tempera-

* The former case corresponds to "centrally condensed" configurations in the sense defined by Milne.

ture using equation (9), but, since the matter is assumed to be degenerate, it is clear that physical considerations alone require T to be such that the pressure given by (1) is much less than that given by (2). This yields an inequality for T.

$$T << \left(\frac{\mu m_H}{k}\right)\frac{B^2}{K_1} = 8{\cdot}808 \times 10^8 \, . \, \mu^{8/3}(M/\odot)^{4/3}. \qquad (13)$$

We see therefore that for stars of small mass (*i. e.*, $< 1{\cdot}743\mu^{-2} \times \odot$) the physical conditions cannot be more extreme than the limits set above.

(C) *White Dwarfs.*—The degenerate equations of state have essentially clarified our views regarding the constitution of white dwarfs. Since these stars are of very small luminosity, radiation pressure must play a minor role, and the equilibrium can be studied more or less thoroughly *. The analysis yields in addition a confirmation of the inequalities obtained in (B). One finds that when $M < M_{3/2} = 3{\cdot}822\mu^{-2}\beta^{-3/2} \times \odot$ ($\beta$ has the usual meaning, but by hypothesis $\sim 1$) relativistic degeneracy does not appreciably set in, and the central density is given by

$$\rho_c = 1{\cdot}310 \times 10^5 \, . \, \mu^5 (M/\odot)^2 \beta^3 \text{ gms. cm.}^{-3}. \quad . \quad (14)$$

We see that $M_{3/2} > 1{\cdot}743\mu^{-2} \times \odot$ and that $\rho_c < \rho_{max.}$ given by (12). But when M becomes greater than $M_{3/2}$, relativistic degeneracy sets in very rapidly, and in fact when $M \rightarrow 5{\cdot}736\mu^{-2} \times \odot\beta^{-3/2}$ the star tends to contract to a point. Hence, by taking the mass sufficiently near this limit we can obtain arbitrarily high values for the central density; but it is very doubtful if this result has any particular significance.

(D) $1{\cdot}743\mu^{-2} \times \odot < M < 6{\cdot}623\mu^{-2} \times \odot$.—We now come to discuss these stars of intermediate mass. The situation is rather complicated, because the star can have relativistically degenerate zones †. The problem now is: Can these stars have cores of high density consistent with the equations of state (1), (2), and (3)? Now Milne has shown (*M. N.* **92.** 610, 1932) that if $\beta < 4/5$, then it is not possible to have centrally condensed stars. (This result was obtained on the assumption that the degenerate

* The results quoted in this section are taken from the author's paper (*M. N.* **91.** 456, 1931).

† If the star has no regions where the relativistic degeneracy has appreciably set in then the upper limits (12) and (13) continue to hold.

parts of the star are characterized by opacity which is negligible in comparison with that of the outer gaseous envelope. Actually Milne considered only two phase configurations, but the result quoted appears to be true even if we consider three phase configurations.) The stars in the range above specified necessarily satisfy this condition, and hence, when $M < 6{\cdot}623\mu^{-2} \times \odot$, "centrally-condensed" stars are not possible. By this statement one merely means that it is not possible to reach the centre with finite density provided only with the equations of state (1), (2), and (3), if the perfect gas part of the star is described by centrally-condensed solutions of the Emden's equation of index 3.

One can therefore summarize the present situation as follows :—

*Given that* (1), (2), *and* (3) *are the only possible equations of state for an ionized gas, then for a star* (*which is not a white dwarf*) *it is possible to have finite physical conditions at the centre if, and only if, the star is wholly a perfect gas configuration in the sense of equation* (1).

In this connection it is necessary to draw attention to another point. Strömgren's investigations on the (minimum) hydrogen content of stars indicate that the massive B-type stars should be practically wholly composed of hydrogen if there is to be no opacity-discrepancy for these stars on Eddington's model. If now these stars had dense central regions (governed by equations of state yet unknown to us) then to predict the correct luminosity we should have to increase the hydrogen content over the minimum value, and this would not be possible as the limit has already been reached.

The general evidence then is in favour of Eddington's perfect gas hypothesis for ordinary stars, and it would follow that the physical conditions in the interior of stars derived by him * should be near the truth. But it is well to emphasise here that one cannot be too cautious in making this statement. One has to bear in mind that if, as is likely, transmutations of elements are an important source of stellar energy, the steady state of a star is consistent only with the equilibrium of the transmutations occurring in it. The reason for this is that if the

* Modified to take account of the known abundance of hydrogen (such calculations have been made by Eddington and Strömgren).

energy be liberated by non-equilibrium processes of transmutations then a star would be overstable because of the very high power of the temperature dependence of this mode of energy liberation. This conclusion has been reached by Steensholt and Sterne. If the transmutations then are to occur at equilibrium rates, the central temperatures must indeed be very much higher than is provided by the perfect gas hypothesis. But our earlier discussion now shows that if such high temperatures do at all exist in the interior of stars then it must be due to deviations from the known ideal gas laws in ways about which we have at present no precise information. It is conceivable, for instance, that at a very high critical density the atomic nuclei come so near one another that the nature of the interaction might suddenly change and be followed subsequently by a sharp alteration in the equation of state in the sense of giving a maximum density of which matter is capable. However, we are now entering a region of pure speculation, and it is best to conclude the discussion at this stage.

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