

Aspects of mutually unbiased bases in odd prime power dimensions

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(February 1, 2008)

We rephrase the Wootters-Fields construction [Ann. Phys., **191**, 363 (1989)] of a full set of mutually unbiased bases in a complex vector space of dimensions $N = p^r$, where p is an odd prime, in terms of the character vectors of the cyclic group G of order p . This form may be useful in explicitly writing down mutually unbiased bases for $N = p^r$.

In a complex vector space of dimension N , by a full set of mutually unbiased bases (MUB's) we mean a set of $N + 1$ orthonormal bases such that the modulus square of the scalar product of any member of one basis with any member of any other basis is equal to $1/N$. If we take $e^{(\alpha,k)}$ to denote the k^{th} vector in the α^{th} orthonormal basis, then having a full set of MUB's amounts to having a collection $e^{(\alpha,k)}$; $\alpha = 0, 1, \dots, N$; $k = 0, 1, \dots, N - 1$ of $N(N + 1)$, N -dimensional complex vectors satisfying

$$\begin{aligned} |\langle e^{(\alpha,k)}, e^{(\alpha',k')} \rangle|^2 &= \left| \sum_{l=0}^{N-1} (e_l^{(\alpha,k)})^* (e_l^{(\alpha',k')}) \right|^2 \\ &= \delta^{\alpha\alpha'} \delta^{kk'} + \frac{1}{N} (1 - \delta^{\alpha\alpha'}) ; \alpha, \alpha' = 0, 1, \dots, N ; k, k' = 0, 1, \dots, N - 1. \end{aligned} \quad (1)$$

Here $e_l^{(\alpha,k)}$ denotes the l^{th} component of the k^{th} vector belonging to the α^{th} orthonormal basis. Mutually unbiased bases thus generalise the properties of the eigenvectors of the familiar Pauli matrices $\sigma_x, \sigma_y, \sigma_z$.

Though the notion of a pair of mutually unbiased bases as corresponding to a pair of 'maximally non commuting' measurements was introduced by Schwinger [1] as early as 1960, explicit construction of a full set of MUB's for $N = p$ was first given by Ivanovic [2] and later by Wootters [3]. Subsequently Wootters and Fields [4] extended the construction in [3] to the case $N = p^r$ by making use of the properties of Galois fields [5]. In this work, Wootters and Fields, also clearly brought out the relevance of the MUB's for an optimal determination of the density matrix of an ensemble. It is this aspect of MUB's which underlies their usefulness in quantum estimation theory and in quantum cryptography [6]. A recent work by Bandyopadhyay et al [7] contains, among other interesting results, an explicit construction of the unitary matrices (analogues of the Pauli matrices in $N = p^r$ dimensions) whose eigenvectors provide a full set of MUB's.

The purpose of this Brief Report is to show that, by exploiting certain freedom inherent to the Wootters-Fields construction, the task of explicitly writing down the the full set of MUB's in odd prime power dimensions can be considerably simplified.

We begin by noting that for any N , one of the $N + 1$ orthonormal bases, say, the one corresponding to $\alpha = N$ may always be chosen to be the standard basis

$$e_l^{(N,k)} = \delta_{lk} , l, k = 0, 1, \dots, N - 1. \quad (2)$$

and we can, therefore, confine ourselves only to the remaining N orthonormal bases $e^{(m,k)}$ with both m and k running over $0, 1, \dots, N - 1$. These, of course, must not only be unbiased with respect to each other but must also be unbiased with respect to the standard basis. The latter requirement implies that $|e_l^{(m,k)}|$ should be equal to $1/\sqrt{N}$ for all m, k, l .

As noted above, explicit construction of a full set of MUB's has so far been possible for $N = p^r$, p a prime. For the case when p is odd, one has [4]

$$e_{\underline{l}}^{(\underline{m}, \underline{k})} = \frac{1}{\sqrt{N}} \omega^{\text{Tr}[\underline{m}\underline{l}^2 + \underline{k}\underline{l}]} ; \omega = e^{2\pi i/p}. \quad (3)$$

Here the symbols $\underline{m}, \underline{k}, \underline{l}$ which label bases, vectors in a given basis, and components of a given vector in a given basis respectively, stand for r -dimensional arrays $(m_0, m_1, \dots, m_{r-1})$ etc whose components take values in the set

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$0, 1, 2, \dots, p-1$ i.e. in the field \mathcal{Z}_p . Their boldfaced counterparts $\mathbf{m}, \mathbf{k}, \mathbf{l}$ which appear on the rhs of (3) belong to the Galois field $GF(p^r)$ i.e. they denote polynomials in x of degree r whose components in the basis $1, x, x^2, \dots, x^{r-1}$ are $(m_0, m_1, \dots, m_{r-1})$ etc. Thus $\underline{m} \longleftrightarrow \mathbf{m} \equiv m_0 + m_1 x + m_2 x^2 + \dots + m_{r-1} x^{r-1}$. The variable x is a root of a polynomial of degree r with coefficients in \mathcal{Z}_p and irreducible in \mathcal{Z}_p i.e. with no roots in \mathcal{Z}_p . The trace operation on the rhs of (3) is defined as follows

$$\text{Tr}[\mathbf{m}] = \mathbf{m} + \mathbf{m}^2 + \dots + \mathbf{m}^{p^r-1}, \quad (4)$$

and takes elements of $GF(p^r)$ to elements of \mathcal{Z}_p . On carrying out the trace operation in (3) one obtains

$$e_{\underline{l}}^{(\underline{m}, \underline{k})} = \frac{1}{\sqrt{N}} \omega^{\underline{m}^T \underline{q}(\underline{l})} \omega^{\underline{k}^T \underline{l}}. \quad (5)$$

The components of $\underline{q}(\underline{l})$ are given by

$$q_i(\underline{l}) = \underline{l}^T \beta_i \underline{l} \pmod{p}, \quad i = 0, 1, 2, \dots, r-1, \quad (6)$$

where the $r \times r$ matrices β_i , $i = 0, 1, \dots, r-1$ are obtained from the multiplication table of $(1, x, x^2, \dots, x^{r-1})$:

$$\begin{pmatrix} 1 \\ x \\ \vdots \\ x^{r-1} \end{pmatrix} (1 \ x \ \dots \ x^{r-1}) = \beta_0 + \beta_1 x + \beta_2 x^2 + \dots + \beta_{r-1} x^{r-1}. \quad (7)$$

(As shown by Wootters and Fields, (5) works for $p = 2$ as well if we replace ω by i in the first factor on the rhs and suspend mod p operation while calculating $q_i(\underline{l})$ using (6))

We may rewrite (5) in terms of extended arrays $(\underline{m}, \underline{k})$ and $(\underline{q}(\underline{l}), \underline{l})$ as

$$e_{\underline{l}}^{(\underline{m}, \underline{k})} = \frac{1}{\sqrt{N}} \omega^{(\underline{m}, \underline{k})^T (\underline{q}(\underline{l}), \underline{l})}, \quad (8)$$

from which it is immediately obvious that if we take \underline{l} to label the rows and $(\underline{m}, \underline{k})$ to label the columns (arranged in a lexicographical order) of an $N \times N^2$ matrix e then the l^{th} row of this matrix is given by

$$\frac{1}{\sqrt{N}} \chi^{(\underline{q}(\underline{l}), \underline{l})} \equiv \frac{1}{\sqrt{N}} \chi^{(q_0(\underline{l}))} \otimes \chi^{(q_1(\underline{l}))} \dots \otimes \chi^{(q_{r-1}(\underline{l}))} \otimes \chi^{(l_0)} \otimes \chi^{(l_1)} \otimes \dots \otimes \chi^{(l_{r-1})}, \quad (9)$$

where $\chi^{(l)}$; $l = 0, 1, \dots, p-1$ denote the character vectors of the cyclic group G of order p . The matrix e contains the full set of MUB's - the constituent orthonormal bases are obtained by chopping this matrix into strips of width N . Of course, to write this matrix down explicitly one needs to work out $\underline{q}(\underline{l})$ for each \underline{l} using (6). We now suggest a simpler way of achieving the same results with much less work. First, we notice that the rows of e can be stacked on top of each other on any order. We will take the first row to correspond to $\underline{l} = \underline{0}$ i.e. as $\chi^{(\underline{0}, \underline{0})}$. To determine the remaining rows we proceed as follows. Choose the irreducible polynomial $f(x)$ in such a way that x is a primitive element of $GF^*(p^r) \equiv GF(p^r) \setminus \{0\}$. Its powers x, x^2, \dots, x^{p^r-1} then give all the information we need to write the matrix e .

As an illustration, consider the case $p = 5, r = 1$. Here $GF^*(5) = \mathcal{Z}_5^* = \{1, 2, 3, 4\}$. it is easy to see that 3 is a primitive element and that its powers modulo 5 are

$$3 = 3, 3^2 = 4, 3^3 = 2, 3^4 = 1, \quad (10)$$

which gives $l = 3 \rightarrow q(l) = 4, l = 4 \rightarrow q(l) = 1, l = 2 \rightarrow q(l) = 4, l = 1 \rightarrow q(l) = 1$, and hence

$$e = \frac{1}{\sqrt{5}} \begin{pmatrix} \chi^{(0)} \otimes \chi^{(0)} \\ \chi^{(1)} \otimes \chi^{(1)} \\ \chi^{(4)} \otimes \chi^{(2)} \\ \chi^{(4)} \otimes \chi^{(3)} \\ \chi^{(1)} \otimes \chi^{(4)} \end{pmatrix}. \quad (11)$$

As another example, consider for instance $p = 3, r = 2$. In this case $f(x) = x^2 + x + 2$ is a polynomial of degree 2 irreducible over \mathcal{Z}_3 and such that x is a primitive element of the multiplicative abelian group $GF(3^2) \setminus \{0\}$ [8]. Computing the powers of x modulo $f(x)$ we obtain

$$x = 0 + 1x, x^2 = 1 + 2x, x^3 = 2 + 2x, x^4 = 2 + 0x, x^5 = 0 + 2x, x^6 = 2 + x, x^7 = 1 + x, x^8 = 1 + 0x, \quad (12)$$

which immediately gives the $\underline{l} \rightarrow \underline{q}(\underline{l})$ correspondences. Thus $x \equiv (0, 1) \rightarrow x^2 \equiv (1, 2); x^3 \equiv (1, 2) \rightarrow x^4 \equiv (2, 0)$ etc and we have

$$e = \frac{1}{\sqrt{9}} \begin{pmatrix} \chi^{(0)} \otimes \chi^{(0)} \otimes \chi^{(0)} \otimes \chi^{(0)} \\ \chi^{(1)} \otimes \chi^{(2)} \otimes \chi^{(0)} \otimes \chi^{(1)} \\ \chi^{(1)} \otimes \chi^{(2)} \otimes \chi^{(0)} \otimes \chi^{(2)} \\ \chi^{(1)} \otimes \chi^{(0)} \otimes \chi^{(1)} \otimes \chi^{(0)} \\ \chi^{(2)} \otimes \chi^{(1)} \otimes \chi^{(1)} \otimes \chi^{(1)} \\ \chi^{(2)} \otimes \chi^{(0)} \otimes \chi^{(1)} \otimes \chi^{(2)} \\ \chi^{(1)} \otimes \chi^{(0)} \otimes \chi^{(2)} \otimes \chi^{(0)} \\ \chi^{(2)} \otimes \chi^{(0)} \otimes \chi^{(2)} \otimes \chi^{(1)} \\ \chi^{(2)} \otimes \chi^{(1)} \otimes \chi^{(2)} \otimes \chi^{(2)} \end{pmatrix}. \quad (13)$$

Some interesting features which become explicit by examining the two factors on the rhs of (5) are listed below:

- The diagonal matrices $\Omega(\underline{m})$ with diagonal elements $\omega^{\underline{m}^T \underline{q}(\underline{l})}$ (\underline{l} taken as a row label) provide an $N = p^r$ dimensional unitary reducible representation of the direct product group $G^r = G \times G \times \cdots \times G$. This representation contains the trivial representation once together with half of the nontrivial irreducible representations which occur with multiplicity two.
- The diagonal matrices $\mathcal{R}(\underline{k})$ with diagonal elements $\omega^{\underline{k}^T \underline{l}}$ (\underline{l} taken as a row label) provide a $N = p^r$ dimensional unitary reducible representation of the direct product group $G^r = G \times G \times \cdots \times G$ which contains all the irreducible representations of once (the regular representation).
- The diagonal matrices $\Omega(\underline{m})\mathcal{R}(\underline{k})$ provide a $N = p^r$ dimensional unitary reducible representation of the direct product group $G^r \times G^r$ in which certain prescribed irreducible representations occur only once. This representation essentially yields the MUB's in odd prime power dimensions.

To conclude, we have shown that the freedom in the choice of the irreducible polynomial $f(x)$ in carrying out the computations in (6) and (7) can be profitably exploited to simplify the task by choosing to work with an $f(x)$ whose roots are primitive elements of $GF^*(p^r)$.

- [1] J. Schwinger, Proc. Nat. Acad. Sci. U.S.A **46**, 570 (1960).
- [2] I. D. Ivanovic, J. Phys. A**14**, 3241 (1981); J. Math. Phys. **24**, 1199 (1983).
- [3] W. K. Wootters, Found. Phys. **16**, 391 (1986).
- [4] W. K. Wootters and B. C. Fields Ann. Phys., **191**, 363 (1989).
- [5] See, for instance, R. Lidl and G. Pilz, *Applied Abstract Algebra*, (Springer Verlag, New York, Berlin, Heidelberg, Tokyo ; 1984).
- [6] H. Bechmann-Pasquinucci and A. Peres, Phys. Rev. Lett **85**, 3313 (2000).
- [7] S. Bandyopadhyay, P Oscar Boykin, V. Roychowdhury and F. Vatan, quant-ph/0103162.
- [8] Lists of irreducible polynomials for low values of p and r together with the order of their roots may be found in [5].