SINE TABLE IN ANCIENT INDIA

AMULYA KUMAR BAG

National Institute of Sciences of India, 1 Park Street, Calcutta 16

The construction of *jyā* table (or Indian sine table) was given great importance by Indian astronomers from fourth century A.D. onwards as it was required to calculate the planetary positions as accurately as possible. Technique of constructing the table has been described here as their probable method of computation. Greek Ptolemy (A.D. 150) gave previously similar sine table. This was further developed into tangent and co-tangent tables by the Arabs. The paper also contains a discussion on the priority or otherwise of the Indian and Greek origin of the sine table.

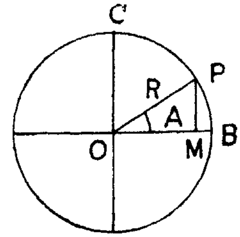
The construction of the *jyā* (*R* sine table) or the calculation of the values of the *jyās* of different arcs in a quadrant was given great importance by Indian astronomers as it was required to calculate the planetary positions as accurately as possible. For this the circumference of each quarter of a circle was divided into 24 equal parts, each covering 225 *liptās*[1] or *kalās* ($=3° 45'$) of the circumference. The value of this unit *jyā* and of its multiple up to 24 are given by many Indian astronomers. The scholars like Delambre,[2] Burgess,[3] Singh,[4] Naraharayya[5] and Ayyangar[6] have shown that the table was probably computed from a rule derived geometrically as given in the *Sūryasiddhānta* and *Āryabhaṭīya*. Gupta[7] has shown that Bhāskara I and succeeding scholars, namely Brahmagupta (A.D. 628), Vaṭeśvara (A.D. 904), Bhāskara II (A.D. 1150), Nārāyaṇa (A.D. 1356) and Gaṇeśa Daivajña (A.D. 1520) followed a more elegant algebraic formula to compute the *jyās*. The object of the present paper is to show that it was from the elementary trigonometrical relations that the *jyā* table of Varāhamihira (A.D. 505) and later scholars can be readily computed, though it is not known what method they followed.

Three trigonometrical functions (Fig. 1), namely *jyā* (PM), *kojyā* (OM) and *utkramajyā* (MB) for a small arc PB in the first quadrant as defined by Indian astronomers may be given with their modern equivalent as follows:

$$jy\bar{a}\ \mathrm{A} = \mathrm{PM} = \mathrm{R}\sin \mathrm{A}$$

$$kojy\bar{a}\ \mathrm{A} = \mathrm{OM} = \mathrm{R}\cos \mathrm{A}$$

$$utkramajy\bar{a}\ \mathrm{A} = \mathrm{MB} = \mathrm{R} - \mathrm{R}\cos \mathrm{A},$$



(Fig. 1)

where R = OP = radius.

The table for PM ($= 3° 45'$) and of its multiple up to 24 (i.e., $3° 45' \times 24 = 90°$) as given by different scholars are:

Serial No.	*Cāpa*	*Pauliśasiddhānta* (A.D. 400), vide Thibaut's ed. of *Pañcasiddhāntikā* (Ch. 4, v. 6–11) of Varāhamihira	*Sūryasiddhānta* (A.D. 400), Ch. 2, v. 16–33	*Āryabhaṭīya* of Āryabhaṭa (A.D. 499), Daśagītikā, v. 10	*Brāhmasphuṭasiddhānta* of Brahmagupta (A.D. 620), Spaṣṭādhikāra, v. 1–5	*Siddhānta Śekhara* of Śrīpati, Sphuṭādhyāya, v. 3–6	*Mahāsiddhānta* of Āryabhaṭa II (A.D. 950), Spaṣṭādhikāra, v. 4–5½	*Siddhāntaśiromaṇi* of Bhāskara II (A.D. 1150), Spaṣṭādhikāra, v. 2–6
1	3° 45′	7′ 51″	225′	225′	214′	223′	225′	225′
2	7° 30′	15′ 40″	449′	449′	427′	445′	449′	449′
3	11° 15′	23′ 25″	671′	671′	638′	666′	671′	671′
4	15° 0′	31′ 4″	890′	890′	846′	884′	890′	890′
5	18° 45′	38′ 34″	1105′	1105′	1056′	1098′	1105′	1105′
6	22° 30′	45′ 56″	1315′	1315′	1251′	1307′	1315′	1315′
7	26° 15′	53′ 5″	1520′	1520′	1446′	1510′	1520′	1520′
8	30° 0′	60′ 0″	1719′	1719′	1635′	1708′	1719′	1719′
9	33° 45′	66′ 40″	1910′	1910′	1817′	1898′	1910′	1910′
10	37° 30′	73′ 3″	2093′	2093′	1991′	2079′	2093′	2093′
11	41° 15′	79′ 7″	2267′	2267′	2156′	2252′	2267′	2267′
12	45° 0′	84′ 51″	2431′	2431′	2312′	2415′	2431′	2431′
13	48° 45′	90′ 13″	2585′	2585′	2458′	2568′	2585′	2585′
14	52° 30′	95′ 13″	2728′	2728′	2594′	2709′	2728′	2728′
15	56° 15′	99′ 46″	2859′	2859′	2719′	2839′	2859′	2859′
16	60° 0′	103′ 56″	2978′	2978′	2832′	2958′	2977′	2977′
17	63° 45′	107′ 38″	3084′	3084′	2933′	3063′	3084′	3084′
18	67° 30′	110′ 53″	3177′	3177′	3021′	3155′	3177′	3177′
19	71° 15′	113′ 38″	3256′	3256′	3096′	3234′	3256′	3256′
20	75° 0′	115′ 56″	3321′	3321′	3159′	3299′	3321′	3321′
21	78° 45′	117′ 43″	3372′	3372′	3207′	3347′	3372′	3372′
22	82° 30′	119′ 0″	3409′	3409′	3342′	3386′	3409′	3409′
23	86° 15′	119′ 45″	3431′	3431′	3263′	3408′	3431′	3431′
24	90° 0′	120′ 1″	3438′	3438′	3270′	3415′	3438′	3438′

It follows from above that in working out the values of the *jyā* table, circles of different radii, viz. $120'1''$, $3438'$, $3270'$, $3415'$, etc., were used by different workers. Here as the radius was expressed in minute, the respective *jyā* length was also expressed in minute. It is obvious from the definition that *jyā* $90° = R$.

Varāhamihira's *Pañcasiddhāntikā* gives the following trigonometrical results without mentioning how they were derived:

I. *vyāsārdhakṛtidhruvasaṃjñikā kṛtāṃśastataḥ sa meṣasya* |
dhruvakaraṇī meṣonā dvayastu rāśyoḥ padaṃ jyāḥ syuḥ ||
(*Pañcasiddhāntikā*, Ch. 4, v. 2)

'Square of the radius (R) is to be defined as constant; the one-fourth part of that (i.e. R^2) is the square of the Aries. The square root of the two quantities—the square of the Aries and the Aries lessened from the constant arc the *jyās* (of 30° and 60° respectively).'

In modern equivalents, this becomes

$$jy\bar{a}\ 30^\circ = \sqrt{\frac{R^2}{4}} = \frac{R}{2}$$

and

$$jy\bar{a}\ 60^\circ = \sqrt{R^2 - \frac{R^2}{4}} = \sqrt{\frac{3}{2}}R$$

II. *icchāṃśadviguṇonatribhajyayonā trayasya cāpajyā |*
ṣaṣṭiguṇā sā karaṇī tayā dhruvonāvaśeṣasya ||
(*Pañcasiddhāntikā*, Ch. 4, v. 5)

'Twice of any desired arc is subtracted from 90°; the *jyā* of the remainder is subtracted from the radius. The square root of the result multiplied by sixty (i.e. half the radius) is the *jyā* of that arc. By deducting that square (i.e. *jyā*^{2} A) from the constant (R^2), the square of the *kojyā* (i.e. *jyā* of the complementary arc) is obtained.'

That is,

$$(jy\bar{a}\ A)^2 = \frac{R}{2}\left[R - jy\bar{a}\left(\frac{\Pi}{2} - 2A\right)\right]$$

or

$$(jy\bar{a}\ A)^2 = \frac{R}{2}[R - kojy\bar{a}\ 2A] \quad \ldots \quad (1)$$

and

$$R^2 - (jy\bar{a}\ A)^2 = \left\{ jy\bar{a}\left(\frac{\pi}{2} - A\right)\right\}^2$$

or

$$jy\bar{a}^2\ A + jy\bar{a}^2\left(\frac{\pi}{2} - A\right) = R^2$$

or

$$jy\bar{a}^2\ A + kojy\bar{a}^2\ A = R^2. \quad \ldots \quad (2)$$

These two rules are equivalent to

$$\sin^2 A = \tfrac{1}{2}(1 - \cos 2A) \text{ and } \sin^2 A + \cos^2 A = 1$$

It might be pointed out that Varāhamihira (A.D. 505), who gave these trigonometrical formulae (1) and (2) along with the values of *jyās* of 30°, 60° and 90°, apparently deduced the table directly from the formulae and the values of *jyā* of 60° and 90°. Somayāji[8] and Gupta[9] pointed out the possibility of the application of the formulae in deducing the table but did not explain how it could be done. The deduction of the table may be illustrated from the following example using $\sqrt{3} = 1{\cdot}732$ and $\sqrt{2} = 1{\cdot}414$.

Let $R = 3438'$, then *jyā* $90^\circ = 3438'$, *jyā* $60^\circ = \sqrt{\frac{3}{2}}R = 2928'$. From the *jyā* values of 90° and 60°, the *jyā* values of 45° and 30°, and then of the corresponding half angles, viz. 22° 30′, 15°, 11° 15′, 7° 30′ and 3° 45′ can be gradually calculated with the help of the formulae (1) and (2). The values of the *jyās* of complementary angles, namely 67° 30′, 75°, 78° 30′, 86° 15′ can be calculated by (2). The *jyās* of half of these angles and again complements of these angles are calculated. By repeating the process 24 *jyās* at the intervals of 3° 45′ can be easily calculated. It has been verified that for other radii also, the table gives more or less correct values. There appears to be an error in the *Pauliśasiddhānta* and in the *Siddhānta Śekhara* of Śrīpatī with reference to the values of the radius. It should have been 120′ and 3416′ instead of 120′ 1″ and 3415′ respectively.

Bhāskara II[10] (A.D. 1150) gave the following two formulae in addition to the values for *jyā* of 30°, 45°, 60° and 90°.

$$jy\bar{a}^2\,A + kojy\bar{a}^2\,A = R^2$$

$$jy\bar{a}\,A/2 = \tfrac{1}{2}\,(jy\bar{a}^2\,A + utkramajy\bar{a}^2\,A)^{\frac{1}{2}}. \qquad .. \qquad .. \qquad (3)$$

The formula (3) follows from (2) if we substitute A for $2A$. Hence it will not be far from truth to state that Varāhamihira (*c.* A.D. 505) and possibly other scholars of his time prepared their *jyā* table using these formulae and the values of the *jyās* mentioned above. How the trigonometrical formulae and the value of *jyā* 60° and 30° were obtained is not known. Suggestions for this might be of particular interest and an attempt has been made in this paper to arrive at the formulae and the values of the *jyā* of 60° and 30°.

(1) $jy\bar{a}^2\ A + kojy\bar{a}^2\ A = R^2.$

The result follows directly if the radius is taken as the hypotenuse, the *jyā* as the perpendicular and *koṭijyā* as the base of the right-angled triangle.

(2) $kojy\bar{a}\ 2A = R - \dfrac{2\,jy\bar{a}^2\ A}{R}.$

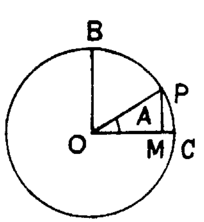
From definition,

$K = R - U$ (where $K = ko\d{t}ijy\bar{a} = OM$, $J = jy\bar{a} = PM$

$U = utkramajy\bar{a} = MC$ and $R =$ radius) (Fig. 2).

Now,

$$K^2 = R^2 - 2RU + U^2$$



(Fig. 2)

or $\quad R^2 - K^2 = 2RU - U^2$

or $\quad J^2 = 2RU - U^2$

or $\quad \dfrac{J^2 + U^2}{4} = \dfrac{RU}{2}$

or $$\frac{\sqrt{PM^2+MC^2}}{2} = \sqrt{\frac{R\times MC}{2}}$$

or $$\frac{PC}{2} = \sqrt{\frac{R\times MC}{2}}$$

or $$jy\bar{a}\,\frac{A}{2} = \sqrt{\frac{R\times utkramajy\bar{a}\ A}{2}}$$

or $$2\,jy\bar{a}^2\frac{A}{2} = R\times utkramajy\bar{a}\ A$$

or $$2\,jy\bar{a}^2\,A = R\times utkramajy\bar{a}\ 2A \text{ (substitue } 2A \text{ for } A\text{)}$$

or $$R(R-kojy\bar{a}\ 2A) = 2\,jy\bar{a}^2 A$$

or $$kojy\bar{a}\ 2A = R-\frac{2\,jy\bar{a}^2\,A}{R}.$$

In modern equivalents, it becomes,

$$\cos 2A = 1-2\sin^2 A.$$

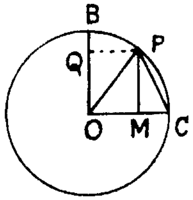
(3) *Values of sin 60°, sin 30°.*

Let PC = arc of 60°, i.e. $\angle POC = 60°$ (Fig. 3).
Again, $OP = OC = R$.
The perpendicular PM is the bisector of OC.

$$\therefore\ OM = R/2$$

and $$PM = \sqrt{R^2-\left(\frac{R}{2}\right)^2} = \sqrt{\frac{3}{2}}R.$$

$$\therefore\ jy\bar{a}\ 60° = PM = \sqrt{\frac{3}{2}}R.$$



(Fig. 3)

Again, $PQ = jy\bar{a}$ of the arc $BP = jy\bar{a}\ 30°$

or $$jy\bar{a}\ 30° = PQ = OM = \frac{R}{2}.$$

Ptolemy[11] (*c.* A.D. 150) first gave a table of chords within a circle of diameter 120 units. The arc ranges from ½ degree to 180 degrees at intervals of ½ degrees. Evidently it gives chord lengths for arcs increasing at an interval of ½ degrees. By Ptolemy, the chord lengths were given in terms of its diameter, whereas in India they were given in terms of its radius. With the help of this, when the length of the arc is known, the corresponding length of the chord can be calculated and vice versa. Depending on the similarity of Greek and Indian methods, Biot[12] opined that the Indian table of *jyā* was derived from that of Ptolemy. Burgess[13] expressed the view, 'it is rather difficult to calculate the sines given in the *Sūryasiddhānta* from Ptolemy's table of chords'. The agreement to a fair degree is actually detected by him by reading degrees and minutes instead of minutes and seconds in the values given in the *Pañcasiddhāntikā*.[14] Here corresponding results are shown by considering the arcs as multiples of 3° 45′ in Indian table of *jyā* and of 7° 30′ in Ptolemy's table of chords as follows:

Serial No.	*jyās* of the *Pauliśasiddhānta*	Ptolemy's chords	Serial No.	*jyās* of the *Pauliśasiddhānta*	Ptolemy's chords
1	7° 51′	7° 50′ 54″	13	90° 13′	90° 13′ 15″
2	15° 40′	15° 39′ 47″	14	95° 13′	95° 12′ 9″
3	23° 25′	23° 24′ 39″	15	99° 46′	99° 46′ 35″
4	31° 4′	31° 3′ 30″	16	103° 56′	103° 55′ 23″
5	38° 34′	38° 34′ 22″	17	107° 38′	107° 37′ 30″
6	45° 56′	45° 55′ 19″	18	110° 53′	110° 51′ 52″
7	53° 5′	53° 4′ 29″	19	113° 38′	113° 37′ 54″
8	60° 0′	60° 0′ 0″	20	115° 56′	115° 54′ 40″
9	66° 40′	66° 40′ 7″	21	117° 43′	117° 41′ 40″
10	73° 3′	73° 3′ 5″	22	118° 0′	118° 58′ 25″
11	79° 7′	79° 7′ 18″	23	119° 45′	119° 44′ 36″
12	84° 51′	84° 51′ 10″	24	120° 1′	120° 0′ 0″

From this it is seen that most of them agree to the nearest minute and a very few of them differ by a full minute.

Khalif Abbasid Al Mansoor (A.D. 712–775) founded a great centre of learning at Baghdad. He invited from Ujjain a scholar named Kaṅka at A.D. 770 for explaining to the Arabs the Hindu system of arithmetic and astronomy. By Khalif's order the *Brāhmasphuṭasiddhānta* that Kaṅka was using, was translated by Al-Fazāri into Arabic and was named *Sind Hind* or *Hind Sind*.[15] Al-Fazāri's work was in general use among the Arabs for a long time. An abridged edition was published by Muhammed ben Musa Al-Khowārizmi (*c*. A.D. 825). The latter also prepared another table based on Indian and Persian tables as also Ptolemy's astronomical works. Sachau[16] in the preface of Al-bernui's *India* pointed out that the Arabs learnt their astronomy from Brahmagupta's *Brāhmasphuṭasiddhānta* before they came to know about Ptolemy. Another work, *Khaṇḍakhādyaka* of Brahmagupta was translated by the name *Arkand* by the Arabs. Needham[17] is of the opinion that Indian work was taken over by the Arabs and by them transmitted to Europe. It is therefore quite plausible that the first impulse of preparing a sine table came to the Arabs either from India or from Greece. Ibn Jābir ibn Sinān Al-Battāni (*c*. A.D. 858–927) used sines and introduced concepts, from which the tangent and cotangent can be derived.[18] This table of Al-Battāni is undoubtedly an improvement over Indian and Greek chords. Abūl-Wefā (*c.* A.D. 980) computed the table of Ptolemy with much care. By fourteenth century A.D., it was through Peurback (*c.* A.D. 1460) and Regiomontanus (*c.* A.D. 1464) that the European scholars in general became well acquainted with it.[19]

Acknowledgement

I am grateful to Sri S. N. Sen and Prof. P. Rây for their kind interests in the preparation of the paper.

References

1 60 *vikalā* = 1 *kalā* or 1 *liptā* or 1 *liptikā* (1 minute),
60 *kalās* = 1 *bhāga*, 30 *bhāgas* = 1 *rāśi*, 12 *rāśis* = 1 *bhagan* (*Sūryasiddhānta*, Ch. 1, v. 28).

2 Delambre *Histoire de l'Astronomie*, **1**, p. 458.

3 Burgess, E., Translation of the *Sūryasiddhānta*, pp. 62-63, 335, Univ. of Calcutta, 1935.

4 Singh, A. N., Hindu Trigonometry; *Proceedings of the Benares Mathematical Society*, **1**, pp. 77–92, 1939.

5 Naraharayya, S. N., *Journal of the Indian Mathematical Society*, **15**, p. 113, Second Part, Notes and Questions, 1924-25.

6 Ayyangar, A. A. K., *Journal of the Indian Mathematical Society*, **15**, first part, pp. 121–6, 1924-25.

7 Gupta, R. C., 'Bhāskara I's Approximation of Sine', *Indian Journal of History of Science*, **2**, pp. 121–136, 1967.

8 *Vide* Srinivasiengar, C. N., *History of Mathematics in Ancient India*, pp. 55-56, Calcutta, 1967.

9 Gupta, R. C., 'Bhāskara I's Approximation to Sine', *Indian Journal of History of Science*, **2**, p. 121, 1967.

10 *Siddhāntaśiromaṇi*—Jyāotpatti, v. 4, 15.

11 Mathematical Syntaxis, Ch. 11.

12 Biot, M., *Journal des Savants*, p. 409, 1859.

13 Burgess, E., *vide* his translation of *Sūryasiddhānta*, p. 333, Univ. of Calcutta, 1935.

14 ———— *Indian Antiquary*, **20**, p. 228, 1891.

15 Smith, D. E., *History of Mathematics*, **1**, pp. 167-68, Dover publication. (There remains a difference of opinion whether *Sind Hind* was a translation of *Brāhmasphuṭasiddhānta* or *Sūryasiddhānta*.)

16 Sachau, E. C., *Alberuni's India*, **1**, Introduction, p. 31; also *ibid.*, **2**, p. 313.

17 Needham, J., *Science and Civilization in China*, **3**, p. 108, Cambridge, 1959.

18 Smith, D. E., *Hist. of Math.*, **2**, p. 608. Dover Publication.

19 *Ibid.*, p. 609.