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COHEN-MACAULAYNESS OF BLOW-UPS OF HOMOGENEOUS WEAK d-SEQUENCES

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ABSTRACT. Let R be a homogeneous Cohen-Macaulay algebra over a field, and let I be an ideal generated by a homogeneous weak d-sequence. We show, under reasonable conditions on the sequence, that the graded ring $\mathrm{gr}_M R[It]$ of the Rees algebra $R[It] = \bigoplus_{i\geq 0} I^i$ is Cohen-Macaulay. In particular we obtain the Cohen-Macaulayness of the blow-up ring $R[It]$.

In this note we prove a result about the Cohen-Macaulayness of the blow-up rings of certain homogeneous ideals generated by weak d-sequences with good depth properties. It is a generalization, in the graded case, of well-known results about the Cohen-Macaulayness of blow-up rings of ideals generated by d-sequences, proved by Huneke ([H1]) and Herzog-Simis-Vasconcelos ([HSV]). Huneke later introduced weak d-sequences in [H2] to study depths of powers of ideals of various determinantal varieties.

By making use of a sort of Gröbner basis technique following [HTU] and [RS], we filter the ring $\mathrm{gr}_M(R[It])$ and show the ideal of initial forms is Cohen-Macaulay via the standard exact sequence $0 \to K_1 \cap K_2 \to K_1 \oplus K_2 \to K_1 + K_2 \to 0$. The ideals which occur here involve essentially the related ideals and the initial quadratic relations of the given sequence.

Before stating the result, we will need some definitions. First we recall the definition of a quadratic sequence, which is a slight generalization of a weak d-sequence ([R]). Let H be a partially ordered set and $A \subset H$ an H-ideal (i.e. A is closed under $<$). If $\alpha \in H$ and lies just above A, i.e. $\beta < \alpha \Rightarrow \beta \in A$, we say (A, α) is a pair. Now let $\{x_\alpha\}_{\alpha\in H} \subset R$ be elements of a Noetherian ring R indexed by H. Let $I_A = (x_\alpha | \alpha \in A)$ and $I = I_H$. The sequence $\{x_\alpha\}$ is a *quadratic sequence* if for every pair (A, α) of H, there exists an H-ideal B so that $I_A \colon x_\alpha \cap I = I_B$ and $x_\alpha I_B \subset I_A I$.

In an attempt to mimic a d-sequence situation, which is linear, we consider a *linearization* of the weak d-sequence. By definition, this is just a bijective map of partially ordered sets $H \to \{1, 2, \ldots, n\}$. We will fix this linearization, once and for all, and define everything else with respect to this. We now write $x_1, \ldots, x_n$ for the linearized quadratic sequence. In the homogeneous case we

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require that this map be degree preserving, which just means that $\deg x_1 \leq \cdots \leq \deg x_n$.

The related ideals I_j, for $j = 1, \dots, n$, are the colon ideals

$$I_j = (x_1, \dots, x_{j-1}) \colon x_j.$$

In the polynomial ring $k[T_1, \dots, T_n]$ the *initial quadratic relations* are defined by

$$Q = (T_j T_k | x_k \in I_j,\ 1 \leq j \leq k \leq n).$$

The name will be justified shortly. We say the sequence is *squarefree* if Q is generated by squarefree monomials.

We call a linearized and degree-preserving quadratic sequence *standard* if it satisfies the condition

$$x_k \notin I_j \Rightarrow I_j \subset I_k.$$

Note that any d-sequence is standard since $I_1 \subset \cdots \subset I_n$; so is any straightening closed ideal in an ordinal Hodge algebra ([BST]).

Finally we will need the auxiliary ideals

$$Q_k = (T_j T_l \in Q | 1 \leq j \leq l < k)$$

and

$$Q'_k = Q_k + (T_j | T_j T_k \in Q,\ j < k) + (T_k^2 | T_k^2 \in Q).$$

Now let $S = R[T_1, \dots, T_n]$ be the polynomial ring over R. Recall the degree-lexicographic order on N_0^{n+1} is defined by $(\alpha_0, \dots, \alpha_n) < (\beta_0, \dots, \beta_n)$ if the first nonzero component from the left side of

$$\left(\sum_{i=0}^{n} \alpha_i - \sum_{i=0}^{n} \beta_i,\ \alpha_1 \beta_1,\ \dots,\ \alpha_n - \beta_n \right)$$

is negative. Think of $S = \bigoplus S_h$ as N_0^{n+1} graded. If we set $F_h S = \bigoplus_{g \geq h} S_g$, then $F = \{F_h S\}_{h \in N_0^{n+1}}$ is a filtration on S finer than its M-adic filtration (M is the irrelevant maximal ideal).

We have $\operatorname{gr}_F(R[It]) = \operatorname{gr}_F(S/J) \cong S/J_*$, where J_* is the ideal generated by the initial forms of elements of J. Under a slightly technical stability condition on the linearization, one has ([RS, Theorem 1.4])

$$J_* = I_1 T_1 + \cdots + I_n T_n + Q.$$

Theorem. *Let R be a homogeneous Cohen-Macaulay algebra over a field, and let I be an R-ideal generated by a stable, standard homogeneous quadratic sequence with $I_1 = 0$. If for $j = 1, \dots, n$*

$$\operatorname{depth} S/(I_j, Q'_j) \geq \dim S - j + 1,$$

then $\operatorname{gr}_M R[It]$ is a Cohen-Macaulay ring.

Corollary ([HTU, Theorem 1.6]). *Let R be a Cohen-Macaulay homogenous algebra over a field, and I an ideal generated by a homogeneous d-sequence $x_1, \dots, x_n$ with $\deg x_1 \leq \cdots \leq \deg x_n$. If grade $I > 0$ and $\operatorname{depth} R/I_j = \dim R - j + 1$ for $j = 1, \dots, n$, then $gr_M R[It]$ is Cohen-Macaulay.*

Proof. Since $x_1, \dots, x_n$ is a d-sequence, it is a stable, linearized degree-preserving quadratic sequence, $Q = 0$ and $I_1 = 0 \colon x_1 = 0 \colon I = 0$ since grade $I > 0$. □

It is not hard to check that a squarefree, standard weak d-sequence is stable. Hence one has the following

Corollary. *Let R be a homogeneous Cohen-Macaulay algebra over a field, and let I be an ideal generated by a standard, squarefree homogeneous weak d-sequence with $I_1 = 0$. If for $j = 1, \dots, n$, $\operatorname{depth} S/(I_j + Q'_j) \geq \dim S - j + 1$, then $R[It]$ is a Cohen-Macaulay algebra.*

Proof of Theorem. It is enough to show that S/J_* is Cohen-Macaulay.

Consider

$$J_* = I_1 T_1 + \cdots + I_n T_n + Q \subset S = R[T_1, \dots, T_n].$$

Write

$$J_* = L_1 T_1 + \cdots + L_n T_n$$

where $L_k = I_k + (T_j | T_j T_k \in Q, j \leq k)$.

We will show, by induction on k, that $\operatorname{depth} S/J_k \geq \dim S - k + 1$, where $J_k = L_1 T_1 + \cdots + L_k T_k$. This will prove the result since then $\operatorname{depth} S/J_* = \operatorname{depth} S/J_n \geq \dim S - n + 1 = \dim R + 1 = \dim S/J_*$ and hence S/J_* is Cohen-Macaulay.

If $k = 1$, then $J_1 = L_1 T_1 = 0$ since $I_1 = 0$ and so $\operatorname{depth} S/J_1 = \dim S$. Now suppose $k \geq 2$ and that by induction we have shown $\operatorname{depth} S/J_{k-1} \geq \dim S - k + 2$.

Claim. $J_k = (J_{k-1}, T_k) \cap (I_k, Q'_k)$.

Given the claim, we have an exact sequence

$$0 \to S/J_k \to S/(J_{k-1}, T_k) \oplus S/(I_k, Q'_k) \to S/(I_k, Q'_k, T_k) \to 0.$$

Since by assumption $\operatorname{depth} S/(I_k + Q'_k) \geq \dim S - k + 1$, by induction we have $\operatorname{depth} S/J_k \geq \dim S - k + 1$ as required.

Proof of Claim. ($\subset$) Suppose some $L_i T_i$ is not contained in (I_k, Q'_k), for some i, $1 \leq i \leq k$. By definition of Q'_k, I_i is not contained in I_k. Since I is standard, there is a quadratic relation $T_i T_k \in Q$, hence $T_i \in Q'_k$. This would imply $L_i T_i \subset Q'_k$, which is a contradiction.

($\supset$) It's enough to see that an element $aT_k \in (I_k, Q'_k)$ must lie in J_k. Consider two cases. First if $T_k^2 \notin Q$, then this is clear since then $a \in (I_k, Q'_k)$, as Q'_k contains no monomials involving T_k, and in the second case if $T_k^2 \in Q$, then $a \in (I_k, Q_k, T_k)$. In either case, $aT_k \in J_k$. This proves the claim and the theorem. □

Remark. Actually the proof shows that even more is true. Let π be a permutation of $\{1, 2, \dots, n\}$ and I a (stable) linearized degree-preserving homogeneous quadratic sequence with $I_1 = 0$ (not necessarily standard).

Call the sequence π-standard if $x_{\pi(i)} \notin I_{\pi(j)} \Rightarrow I_{\pi(i)} \subset L_{\pi(j)}$ and also

$$x_l \in I_{\pi(i)}, \quad 1 \leq l \leq \pi(i) \Rightarrow l = \pi(k), \quad 1 \leq k \leq j - 1.$$

(If π is the identity permutation, then π-standard is the same as standard.) Put $Q_k^\pi = (T_j T_l \in Q | j, l \in \pi(\{1, \dots, k-1\}))$ and $Q_k^{\pi'} = Q_k^\pi + (T_j | T_j T_{\pi(k)} \in Q, j \in \pi(\{1, \dots, k-1\})) + (T_{\pi(k)}^2 \in Q)$.

Then if the sequence if π-standard and satisfies $\operatorname{depth} S/(I_{\pi(j)} + Q'_j) \geq \dim S - j + 1$, then $\operatorname{gr}_M R[It]$ is Cohen-Macaulay.

One just applies the permutation π to the L's in the proof.

We conclude by mentioning two cases where the theorem readily applies. This gives very different proofs of these facts.

Example 1. *Straightening closed ideals in an ordinal Hodge Algebra.* In [EH] the Rees algebras of such ideals was shown to be Cohen-Macaulay. Indeed, these are always standard, squarefree weak d-sequences satisfying our depth condition. In particular this includes the fact that $R[I_n(X)t]$ is Cohen-Macaulay for $R = k[X]$, X a generic n by m matrix of variables. (See [EH] and [BST] for more details.)

Example 2. *Monomial curves in P^3 on a quadric.* These are the irreducible rational curves C parametrized by $s^d, s^a t^{d-a}, s^b t^{d-b}, t^d$. Let $I = I(C)$ be its homogeneous ideal in $R = k[x_0, \ldots, x_3]$. The case that C lies on a quadric has been recently considered in its context. Note that if it lies on a cone, it must be arithmetically Cohen-Macaulay and hence I is generated by a d-sequence of 3 elements. So we may it assume it lies on the hypersurface $x_0x_3 = x_1x_2$. $R[It]$ was shown to be Cohen-Macaulay in [S], [MS], and [HH]. In the first two works, the entire defining ideal of $R[It]$ had to be explicitly computed, while in the third one had to first establish that I is normally torsion free. As is well known (cf., e.g., [MS]), I is generated by a weak d-sequence, and the generators can be linearly ordered to give $I_2 = (x_0x_3 - x_1x_2)$, $I_j = (x_0, x_1)$, $3 \leq j \leq n$, $Q = (T_3, \ldots, T_{n-1})^2$. The sequence was checked stable in [RS]. Applying the permutation $\{1, 2, \ldots, n\} \to \{1, 2, n, n-1, \ldots, 3\}$, by the remark $\mathrm{gr}_M(R[I(C)t])$ is Cohen-Macaulay.

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