Initial ideals of tangent cones to Schubert varieties in orthogonal Grassmannians

K. N. Raghavan
Institute of Mathematical Sciences
C. I. T. Campus, Taramani
Chennai 600 113, INDIA
email: knr@imsc.res.in

Shyamashree Upadhyay
Chennai Mathematical Institute
Plot No. H1, SIPCOT IT Park
Padur Post, Siruseri 603 103, Tamilnadu, INDIA
email: shyama@cmi.ac.in

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Abstract

We compute the initial ideals, with respect to certain conveniently chosen term orders, of ideals of tangent cones at torus fixed points to Schubert varieties in orthogonal Grassmannians. The initial ideals turn out to be square-free monomial ideals and therefore Stanley-Reisner face rings of simplicial complexes. We describe these complexes. The maximal faces of these complexes encode certain sets of non-intersecting lattice paths.

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Contents

Introduction

This paper is a sequel to [9] and the fulfillment of the hope expressed there that the main result of that paper can be used to compute initial ideals, with respect to certain 'natural' term orders, of ideals of tangent cones (at torus fixed points) to Schubert varieties in orthogonal Grassmannians. Any such initial ideal turns out to be generated by square-free monomials and therefore the Stanley-Reisner face ring of a simplicial complex. We identify this complex (Theorem 1.8.1). The maximal faces of this complex encode a certain set of non-intersecting lattice paths (Remark 1.8.2).

The analogous problem for Grassmannians has been addressed in [7, 5, 6, 8] and for symplectic Grassmannians in [2]. Just as the ideals of tangent cones in those cases are generated respectively by determinants of generic matrices and determinants of generic symmetric matrices, so the ideals in the present case are generated by Pfaffians of generic skew symmetric matrices: see §1.5. The ideal generated by all Pfaffians of a fixed degree of a generic skew-symmetric matrix occurs as a special case: see §1.5.1. Initial ideals in the special case have been computed in [3, 4], but the term orders there are very different from ours: the Pfaffian generators are a Gröbner basis for those term orders but not for ours.

The present case of orthogonal Grassmannians features a novel difficulty not encountered with either Grassmannians or symplectic Grassmannians. Namely, when one tries, following the analogy with those cases, to compute the initial ideal from the knowledge of the Hilbert function (as obtained in [9]), it becomes evident that, in contrast to those cases, the natural generators of the ideal of a tangent cone—the Pfaffians mentioned above—do *not* form a Gröbner basis in any 'natural' term order: see Remark 1.9.1. Here what it means for a term order to be 'natural' is dictated by [9]: to each Pfaffian there is naturally associated a monomial which is a term in it, and a term order is *natural* if the initial term with respect to it of any Pfaffian is the associated monomial. This difficulty is overcome by the main technical result Lemma 4.2.1.

There is another naturally related question that asks if something slightly weaker continues to hold for orthogonal Grassmannians: namely, whether the initial ideals of a tangent cone with respect to natural term orders are all the same. This too fails: see Remark 1.9.2. In other words, the naturalness of a term order turns out not be a strong determiner, unlike for ordinary and symplectic Grassmannians.

This paper is organized as follows: the result is stated in §1 and proved in §4 after preparations in §2, 3. There is heavy reliance on the combinatorial definitions and constructions of [9]. Fortunately, however, only the statement and not the proof of the main theorem there is used.

1 The theorem

The whole of this section (except for §1.5.1, 1.9) is aimed towards the precise statement of our result, which appears in §1.8, after preparations in §1.1–1.6. For full details about the set up described, see [9]. In §1.9 the difficulty peculiar to orthogonal Grassmannians mentioned in the introduction is illustrated by means of an example.

1.1 Initial statement of the problem

Fix once for all a base field $\mathfrak{k}$ that is algebraically closed and of characteristic not equal to 2. Fix a natural number d, a vector space V of dimension $2d$, and a non-degenerate symmetric bilinear form $\langle\ ,\ \rangle$ on V. For k any integer, let $k^* := 2d + 1 - k$. Fix a basis $e_1, \ldots, e_{2d}$ of V such that

$$\langle e_i, e_k\rangle = \begin{cases} 1 & \text{if } i = k^* \\ 0 & \text{otherwise} \end{cases}$$

Denote by $\mathrm{SO}(V)$ the group of linear automorphisms of V that preserve the form $\langle\ ,\ \rangle$ and also the volume form. Denote by $\mathfrak{M}_d(V)'$ the closed sub-variety of the Grassmannian of d-dimensional subspaces consisting of the points corresponding to isotropic subspaces. The action of $\mathrm{SO}(V)$ on V induces an action on $\mathfrak{M}_d(V)'$. There are two orbits for this action. These orbits are isomorphic: acting by a linear automorphism that preserves the form but not the volume form gives an isomorphism. We denote by $\mathfrak{M}_d(V)$ the orbit of the span of $e_1, \ldots, e_d$ and call it the *(even) orthogonal Grassmannian.*

The *Schubert varieties* of $\mathfrak{M}_d(V)$ are defined to be the B-orbit closures in $\mathfrak{M}_d(V)$ (with canonical reduced scheme structure), where B is a Borel subgroup of $\mathrm{SO}(V)$. The problem that is tackled in this paper is this: given a point on a Schubert variety in $\mathfrak{M}_d(V)$, compute the initial ideal, with respect to some convenient term order, of the ideal of functions vanishing on the tangent cone to the Schubert variety at the given point. The term order is specified in §1.6, and the answer given in Theorem 1.8.1.

Orthogonal Grassmannians and Schubert varieties in them can, of course, also be defined when the dimension of the vector space V is odd. As is well known and recalled with proof in [9], such Schubert varieties are isomorphic to those in even orthogonal Grassmannians. The results of this paper would therefore apply also to them.

1.2 The problem restated

We take B to be the subgroup consisting of elements that are upper triangular with respect to the basis $e_1, \ldots, e_{2d}$. The subgroup T consisting of elements that are diagonal with respect to $e_1, \ldots, e_{2d}$ is a maximal torus of $\mathrm{SO}(V)$. The B-orbits of $\mathfrak{M}_d(V)$ are naturally indexed by its T-fixed points: each orbit contains one and only one such point. The T-fixed points of $\mathfrak{M}_d(V)$ are easily seen to be of the form $\langle e_{i_1}, \ldots, e_{i_d}\rangle$ for $\{i_1, \ldots, i_d\}$ in $I(d)$, where $I(d)$ is the set of subsets of $\{1, \ldots, 2d\}$ of cardinality d satisfying the following two conditions:

- for each k, $1 \leq k \leq d$, there does not exist j, $1 \leq j \leq d$, such that $i_k^* = i_j$—in other words, for each ℓ, $1 \leq \ell \leq 2d$, exactly one of ℓ and ℓ^* appears in $\{i_1, \ldots, i_d\}$;
- the parity is even of the number of elements of the subset that are (strictly) greater than d.

Let $I(d, 2d)$ denote the set of all subsets of cardinality d of $\{1, \ldots, 2d\}$. We use symbols v, w, ... to denote elements of $I(d, 2d)$ (in particular, those of $I(d)$). The members of v are denoted $v_1, \ldots, v_d$, with the convention that $1 \leq v_1 < \ldots < v_d \leq 2d$. There is a natural partial order on $I(d, 2d)$: $v \leq w$, if $v_1 \leq w_1, \ldots, v_d \leq w_d$.

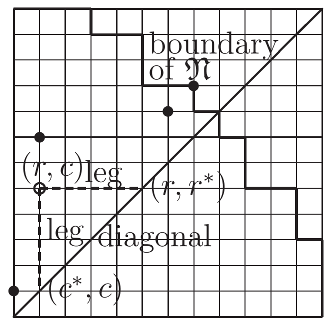
The point of the orthogonal Grassmannian $\mathfrak{M}_d(V)$ that is the span of e_{v_1}, ..., e_{v_d} for $v \in I(d)$ is denoted $\mathfrak{e}^v$. The B-orbit closure of $\mathfrak{e}^v$ is denoted $X(v)$. The point $\mathfrak{e}^v$ (and therefore the Schubert variety $X(v)$) is contained in the Schubert variety $X(w)$ if and only if $v \leq w$.

Our problem can now be stated thus: given elements $v \leq w$ of $I(d)$, find the initial ideal of functions vanishing on the tangent cone at $\mathfrak{e}^v$ to the Schubert variety $X(w)$. The tangent cone being a subvariety of the tangent space at $\mathfrak{e}^v$ to $\mathfrak{M}_d(V)$, we first choose a convenient set of co-ordinates for the tangent space. But for that we need to fix some notation.

1.3 Basic notation

Let an element v of $I(d)$ remain fixed. We will be dealing extensively with ordered pairs (r, c), $1 \leq r, c \leq 2d$, such that r is not and c is an entry of v. Let $\mathfrak{R}$ denote the set of all such ordered pairs, and set

$$\begin{aligned}
\mathfrak{N} &:= \{(r, c) \in \mathfrak{R} \,|\, r > c\} \\
\mathfrak{O}\mathfrak{R} &:= \{(r, c) \in \mathfrak{R} \,|\, r < c^*\} \\
\mathfrak{O}\mathfrak{N} &:= \{(r, c) \in \mathfrak{R} \,|\, r > c, r < c^*\} \\
&= \mathfrak{O}\mathfrak{R} \cap \mathfrak{N} \\
\mathfrak{d} &:= \{(r, c) \in \mathfrak{R} \,|\, r = c^*\}
\end{aligned}$$



The picture shows a drawing of $\mathfrak{R}$. We think of r and c in (r, c) as row index and column index respectively. The columns are indexed from left to right by the entries of v in ascending order, the rows from top to bottom by the entries of $\{1, \ldots, 2d\} \setminus v$ in ascending order. The points of $\mathfrak{d}$ are those on the diagonal, the points of $\mathfrak{O}\mathfrak{R}$ are those that are (strictly) above the diagonal, and the points of $\mathfrak{N}$ are those that are to the South-West of the poly-line captioned 'boundary of $\mathfrak{N}$'—we draw the boundary so that points on the boundary belong to $\mathfrak{N}$. The reader can readily verify that $d = 13$ and $v = (1, 2, 3, 4, 6, 7, 10, 11, 13, 15, 18, 19, 22)$ for the particular picture drawn. The points of $\mathfrak{O}\mathfrak{N}$ indicated by solid circles form a v-chain (see §1.7 below).

We will be considering *monomials*, also called *multisets*, in some of these sets. A *monomial*, as usual, is a subset with each member being allowed a multiplicity (taking values in the non-negative integers). The *degree* of a monomial has also the usual sense: it is the sum of the multiplicities in the monomial over all elements of the set. The *intersection* of a monomial in a set with a subset of the set has also the natural meaning: it is a monomial in the subset, the multiplicities being those in the original monomial.

We will refer to $\mathfrak{d}$ as the *diagonal*. For an element of $\alpha = (r, c)$ of $\mathfrak{R}$, we call (r, r^*) and (c, c^*) its *horizontal* and *vertical projections* (on the diagonal); they are denoted by $p_h(\alpha)$ and $p_v(\alpha)$ respectively. For (r, c) in $\mathfrak{O}\mathfrak{N}$, its vertical projection belongs to $\mathfrak{N}$ but not always so its horizontal projection. The term *projection* when not further qualified means either a vertical or horizontal projection.

1.4 The tangent space to $\mathfrak{M}_d(V)$ at $\mathfrak{e}^v$

Let $\mathfrak{M}_d(V) \subseteq G_d(V) \hookrightarrow \mathbb{P}(\wedge^d V)$ be the Plücker embedding (where $G_d(V)$ denotes the Grassmannian of all d-dimensional subspaces of V). For θ in $I(d, 2d)$, where $I(d, 2d)$ denotes the set of subsets of cardinality d of $\{1, \ldots, 2d\}$, let p_θ denote the corresponding Plücker coordinate. Consider the affine patch $\mathbb{A}$ of $\mathbb{P}(\wedge^d V)$ given by $p_v \neq 0$, where v is some fixed element of $I(d)$ ($\subseteq I(d, 2d)$). The affine patch $\mathbb{A}^v := \mathfrak{M}_d(V) \cap \mathbb{A}$ of the orthogonal Grassmannian $\mathfrak{M}_d(V)$ is an affine space whose coordinate ring can be taken to be the polynomial ring in variables of the form $X_{(r,c)}$ with $(r, c) \in \mathfrak{OR}$. Taking $d = 5$ and $v = (1, 3, 4, 6, 9)$ for example, a general element of $\mathbb{A}^v$ has a basis consisting of column vectors of a matrix of the following form:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ X_{21} & X_{23} & X_{24} & X_{26} & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ X_{51} & X_{53} & X_{54} & 0 & -X_{26} \\ 0 & 0 & 0 & 1 & 0 \\ X_{71} & X_{73} & 0 & -X_{54} & -X_{24} \\ X_{81} & 0 & -X_{73} & -X_{53} & -X_{23} \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -X_{81} & -X_{71} & -X_{51} & -X_{21} \end{pmatrix} \tag{1.4.1}$$

The origin of the affine space $\mathbb{A}^v$, namely the point at which all $X_{(r,c)}$ vanish, corresponds clearly to $\mathfrak{e}^v$. The tangent space to $\mathfrak{M}_d(V)$ at $\mathfrak{e}^v$ can therefore be identified with the affine space $\mathbb{A}^v$ with co-ordinate functions $X_{(r,c)}$.

1.5 The ideal I of the tangent cone to $X(w)$ at $\mathfrak{e}^v$

Fix elements $v \leq w$ of $I(d)$. Set $Y(w) := X(w) \cap \mathbb{A}^v$, where $X(w)$ is the Schubert variety indexed by w and $\mathbb{A}^v$ is the affine patch around $\mathfrak{e}^v$ as in §1.4. From [10] we can deduce a set of generators for the ideal I of functions on $\mathbb{A}^v$ vanishing on $Y(w)$ (see for example [9, §3.2.2]). We recall this result now.

In the matrix (1.4.1), columns are numbered by the entries of v, the rows by 1, ..., $2d$. For $\theta \in I(d)$, consider the submatrix given by the rows numbered $\theta \setminus v$ and columns numbered $v \setminus \theta$. Such a submatrix being of even size and skew-symmetric along the anti-diagonal, we can define its *Pfaffian* (see §3). Let f_θ denote this Pfaffian. We have

$$I = (f_\tau \mid \tau \in I(d), \tau \not\leq w)\,. \tag{1.5.1}$$

We are interested in the tangent cone to $X(w)$ at $\mathfrak{e}^v$ or, what is the same, the tangent cone to $Y(w) \subseteq \mathbb{A}^v$ at the origin. Observe that f_θ is a homogeneous polynomial of degree the v-degree of θ, where the *v-degree* of θ is defined as one half of the cardinality of $v \setminus \theta$. Because of this, $Y(w)$ itself is a cone and so equal to its tangent cone. The ideal of the tangent cone is therefore the ideal I in (1.5.1).

1.5.1 A special case

The ideal generated by all Pfaffians of a given degree r of a generic skew-symmetric $s \times s$ matrix occurs as a special case of the ideal I in (1.5.1): take $d = s$, $v = (1, \ldots, d)$, and $w = (2r-1, \ldots, d, 2d-2r+3, \ldots, 2d)$ (w consists of two blocks of consecutive integers). The initial ideals in this special case, with respect to certain term orders, have been computed in [3, 4]. The Pfaffian generators are a Gröbner basis for those orders unlike for ours: see §1.9.

1.6 The term order

We now specify the term order(s) $\rhd$ on monomials in the co-ordinate functions (of the tangent space at a torus fixed point) with respect to which the initial ideals in our theorem are to be taken.

Fix an element v of $I(d)$. Let $>_1$ and $>_2$ be total orders on $\mathfrak{OR}$ satisfying the following conditions. For both $i = 1$ and $i = 2$:

- $\alpha >_i \beta$ if $\alpha \in \mathfrak{ON}$, $\beta \in \mathfrak{OR} \setminus \mathfrak{ON}$, and the row indices of α and β are equal;
- $\alpha >_i \beta$ if $\alpha \in \mathfrak{ON}$, $\beta \in \mathfrak{ON}$, the row indices of α and β are equal, and the column index of α exceeds that of β.

In addition:

- $\alpha >_1 \beta$ (respectively $\alpha <_2 \beta$) if $\alpha \in \mathfrak{ON}$, $\beta \in \mathfrak{OR}$ and the row index of α is less than that of β.

Let $\rhd$ be one of the following term orders on monomials in $\mathfrak{OR}$ (terminology as in [1, pages 329, 330]):

- the homogeneous lexicographic order with respect to $>_1$;
- the reverse lexicographic order with respect to $>_2$.

1.6.1 A non-standard possibility for the term order

Here is another (somewhat non-standard) possibility for the term order $\triangleright$. We prescribe it in several steps. Let $\mathfrak{S}$ and $\mathfrak{T}$ be distinct monomials in $\mathfrak{O}\mathfrak{R}$.

- If $\deg \mathfrak{S} > \deg \mathfrak{T}$ then $\mathfrak{S} \triangleright \mathfrak{T}$.
- Suppose that $\deg \mathfrak{S} = \deg \mathfrak{T}$. Then look at the set of all projections (both vertical and horizontal, including multiplicities) on the diagonal of elements of $\mathfrak{S}$ and $\mathfrak{T}$—some of these projections may be in $\mathfrak{R}$ and not in $\mathfrak{N}$. Let $r_1 \geq \ldots \geq r_{2k}$ and $r'_1 \geq \ldots \geq r'_{2k}$ be respectively the row numbers of these projections for $\mathfrak{S}$ and $\mathfrak{T}$. If the two sequences are different, then $\mathfrak{S} \triangleright \mathfrak{T}$ if $r_j > r'_j$ for the least j such that $r_j \neq r'_j$.
- Suppose that the projections on the diagonal of $\mathfrak{S}$ and $\mathfrak{T}$ are the same. Consider the column numbers of elements in both $\mathfrak{S}$ and $\mathfrak{T}$ that give rise to the projection with the least row number (namely $r_{2k} = r'_{2k}$). Suppose $c_1 \geq \ldots \geq c_\ell$ and $c'_1 \geq \ldots \geq c'_\ell$ are these numbers respectively for $\mathfrak{S}$ and $\mathfrak{T}$. If these sequences are different, then let $\tilde{j}$ be the least integer j such that $c_j \neq c'_j$. The following three cases can arise:

  (a) Both $(r_{2k}, c_{\tilde{j}})$ and $(r_{2k}, c'_{\tilde{j}})$ are outside $\mathfrak{O}\mathfrak{N}$.

  (b) Exactly one of $(r_{2k}, c_{\tilde{j}})$ and $(r_{2k}, c'_{\tilde{j}})$ belongs to $\mathfrak{O}\mathfrak{N}$.

  (c) Both $(r_{2k}, c_{\tilde{j}})$ and $(r_{2k}, c'_{\tilde{j}})$ are inside $\mathfrak{O}\mathfrak{N}$.

  In case (a), we say that $\mathfrak{S} \triangleright \mathfrak{T}$ if $c_{\tilde{j}} < c'_{\tilde{j}}$, i.e., $(r_{2k}, c_{\tilde{j}})$ is more towards $\mathfrak{O}\mathfrak{N}$ than $(r_{2k}, c'_{\tilde{j}})$. In case (b), we say that $\mathfrak{S} \triangleright \mathfrak{T}$ if $(r_{2k}, c_{\tilde{j}}) \in \mathfrak{O}\mathfrak{N}$ and $(r_{2k}, c'_{\tilde{j}}) \notin \mathfrak{O}\mathfrak{N}$. In case (c), we say that $\mathfrak{S} \triangleright \mathfrak{T}$ if $c_{\tilde{j}} > c'_{\tilde{j}}$.

  If the sequences $c_1 \geq \ldots \geq c_\ell$ and $c'_1 \geq \ldots \geq c'_\ell$ are the same, then there is an equality of sub-monomials of $\mathfrak{S}$ and $\mathfrak{T}$ consisting of those elements with row numbers $r_{2k} = r'_{2k}$. We remove this sub-monomial from both $\mathfrak{S}$ and $\mathfrak{T}$ and then appeal to an induction on the degree.

This finishes the description of the term order $\triangleright$.

1.7 v-chains and $\mathfrak{O}$-domination

The description of the initial ideal in our theorem is in terms of $\mathfrak{O}$-domination of monomials. We now recall this notion from [9]. An element v of $I(d)$ remains fixed.

For elements $\alpha = (R, C)$, $\beta = (r, c)$ of $\mathfrak{O}\mathfrak{N}$ (or more generally of $\mathfrak{R}$), we write $\alpha > \beta$ if $R > r$ and $C < c$. A sequence $\alpha_1 > \ldots > \alpha_k$ of elements of $\mathfrak{O}\mathfrak{N}$ (or of $\mathfrak{N}$) is called a *v-chain*. The points indicated by solid circles in the picture in §1.3 form a v-chain. (For the statement of the theorem we need only consider v-chains in $\mathfrak{O}\mathfrak{N}$ but for the proof we will also need v-chains in $\mathfrak{N}$. The term 'v-chain' without further qualification means one in $\mathfrak{O}\mathfrak{N}$.)

To each v-chain C there is associated an element w_C (or $w(C)$) of $I(d)$: see [9, §2.2]. An element w of $I(d)$ *$\mathfrak{O}$-dominates* a v-chain C if $w \geq w(C)$; it *$\mathfrak{O}$-dominates* a monomial $\mathfrak{S}$ in $\mathfrak{O}\mathfrak{R}$ if it $\mathfrak{O}$-dominates every v-chain in $\mathfrak{S} \cap \mathfrak{O}\mathfrak{N}$.

1.8 The theorem

We are now ready to state our theorem. Let $\mathfrak{k}$ be a field, algebraically closed and of characteristic not 2. Let d be a positive integer and $\mathfrak{M}_d(V)$ the (even) orthogonal Grassmannian over $\mathfrak{k}$ (§1.1). Let $v \leq w$ elements of $I(d)$, $X(w)$ the Schubert variety in $\mathfrak{M}_d(V)$ corresponding to w, and $\mathfrak{e}^v$ the torus fixed point in $\mathfrak{M}_d(V)$ corresponding to v (§1.2). Let P denote the polynomial ring $\mathfrak{k}[X_\beta \mid \beta \in \mathfrak{O}\mathfrak{R}]$, the co-ordinate ring of the tangent space $\mathbb{A}^v$ to $\mathfrak{M}_d(V)$ at $\mathfrak{e}^v$ (§1.3, 1.4). Let I denote the ideal (1.5.1) in P of functions vanishing on the tangent cone to $X(w)$ at $\mathfrak{e}^v$ (§1.5). Let $\mathrm{in}_\triangleright I$ denote the initial ideal of I with respect to the term order $\triangleright$ (§1.6).

Theorem 1.8.1 *The initial ideal* $\mathrm{in}_\triangleright I$ *has a vector space basis over* $\mathfrak{k}$ *consisting of monomials in* $\mathfrak{O}\mathfrak{R}$ *not* $\mathfrak{O}$*-dominated by* w *(§1.7). In other words, the quotient ring* $P/\mathrm{in}_\triangleright I$ *is the Stanley-Reisner face ring of the simplicial complex with vertices* $\mathfrak{O}\mathfrak{R}$ *and faces the square-free monomials* $\mathfrak{O}$*-dominated by* w.

PROOF: The main theorem of [9] asserts that the dimension as a vector space of the graded piece of P/I of degree d equals the cardinality of the monomials in $\mathfrak{O}\mathfrak{R}$ of degree d that are $\mathfrak{O}$-dominated by w. Since P/I and $P/\mathrm{in}_\triangleright I$ have the same Hilbert function, the same is true with P/I replaced by $P/\mathrm{in}_\triangleright I$. It is therefore enough to show that every monomial in $\mathfrak{O}\mathfrak{R}$ that is not $\mathfrak{O}$-dominated by w belongs to $\mathrm{in}_{\rhd}I$, and this is proved in §4. □

Remark 1.8.2 The maximal faces of the simplicial complex, i.e., the square-free monomials in $\mathfrak{OR}$ maximal with respect to being $\mathfrak{O}$-dominated by w, encode a certain set of non-intersecting lattice paths: see [9, Part IV].

1.9 An example

Let v in $I(d)$ be fixed. To every element $\tau \geq v$ of $I(d)$ there is naturally associated a monomial in $\mathfrak{OM}$ ($\subseteq \mathfrak{OR}$). Namely, with terminology and notation as in [9], it is the result of the application of the map $\mathfrak{O}\phi$ to the standard monomial τ. This monomial occurs as a term in the Pfaffian f_τ defined in §1.5.

Remark 1.9.1 Suppose we have a term order $\succ$ on monomials in $\mathfrak{OR}$ such that, for every $\tau \geq v$ in $I(d)$, the initial term of the Pfaffian f_τ equals the monomial associated to τ as above: the term orders $\rhd$ of §1.6 are examples. It is natural to expect that, for $w \geq v$ fixed, the generators f_τ, τ in $I(d)$ such that $\tau \not\leq w$, of the ideal I (1.5.1) form a Gröbner basis with respect to $\succ$. The analogous statements for Grassmannians and symplectic Grassmannians are true [5, 2]. But this expectation fails rather spectacularly (i.e., even in the simplest examples), as we now observe.

Take $d = 5$ and $v = (1, 2, 3, 4, 5)$. Then the top half of the matrix (1.4.1) is the identity matrix and the bottom half looks like this:

$$\begin{pmatrix} a & b & c & d & 0 \\ e & f & g & 0 & -d \\ h & i & 0 & -g & -c \\ j & 0 & -i & -f & -b \\ 0 & -j & -h & -e & -a \end{pmatrix}$$

Consider the ideal generated by all Pfaffians of degree 2 of the above matrix. As observed in §1.5.1, this is the ideal I of (1.5.1) with $w = (3, 4, 5, 9, 10)$. There are 5 Pfaffians of degree 2 corresponding to the 5 values of τ in $I(d)$ such that $\tau \not\leq w$:

$$(1, 6, 7, 8, 9), \quad (2, 6, 7, 8, 10), \quad (3, 6, 7, 9, 10), \quad (4, 6, 8, 9, 10), \quad (5, 7, 8, 9, 10).$$

They are respectively (see Eq. (3.1.1))

$$di - cf + bg, \quad dh - ce + ag, \quad dj - be + af, \quad cj - bh + ai, \quad gj - fh + ei.$$

The monomials of $\mathfrak{O}\mathfrak{N}$ attached to the 5 elements τ above are respectively

$$di, \quad dh, \quad dj, \quad cj, \quad gj.$$

The ideal generated by these monomials does not contain any of the terms in the following element of I:

$$-h(di - cf + bg) + i(dh - ce + ag) = cfh - bgh - cei + agi. \tag{1.9.1}$$

So the Pfaffians f_τ above are not a Gröbner basis with respect to $\succ$.

On the other hand, the initial terms of the Pfaffians f_τ above with respect to the term order in [3] are respectively

$$bg, \quad ag, \quad af, \quad ai, \quad ei$$

The Pfaffians f_τ above are a Gröbner basis with respect to that term order [3].

Remark 1.9.2 The expectation in Remark 1.9.1 having failed, we could ask whether a weakening of it—also very natural—holds: are the initial ideals of a tangent cone to $X(w)$ with respect to various natural term orders all the same (namely, generated by monomials not $\mathfrak{O}$-dominated by w)? But this too fails as we now observe.

Consider the example discussed above. Identify $\mathfrak{O}\mathfrak{R} = \mathfrak{O}\mathfrak{N}$ with the variables $a, b, \ldots, j$. Consider the degree lexicographic order on monomials in these variables with respect to a total order on the variables in which d is bigger than a, b, c, e, f, g; and j is bigger than a, b, e, f, h, i. It is readily verified that this term order is natural in the sense that it satisfies the condition in Remark 1.9.1: there are 16 elements of $I(d)$: v, the 5 listed above, and 10 others the associated Pfaffians for which are respectively the 10 variables.

Now take a total order that looks like $d > j > a > \ldots$ (the rest can come in any order). The corresponding term order picks out agi as the initial term of the element of I' in Eq. (1.9.1), but the monomial agi is $\mathfrak{O}$-dominated by w as follows readily from the definitions.

2 New Forms of a v-chain

In this section, we construct new v-chains, called *new forms*, from a given one. New forms play a crucial role in the proof of the main Lemma 4.2.1. In fact, one may say that their construction, given in §2.2 below, is the main idea in the proof. A key property of new forms is recorded in §2.3. In §2.4 is described an association—not that of [9]—of an element y_C of $I(d)$ to a v-chain C. The elements y_C also play a crucial in the proof.

An element v of $I(d)$ remains fixed throughout.

2.1 Some conventions

We will often have to compare diagonal elements of $\mathfrak{R}$ (§1.3) with each other. With regard to such elements, the phrases *smaller than* and *greater than* (and correspondingly the symbols $<$ and $>$) mean respectively 'to the North-East of' and 'to the South-West of'. We use these phrases in their strict sense only: 'smaller than' means in particular 'not equal to'. This is consistent with the definition of the relation $>$ on $\mathfrak{R}$ in §1.7.

With regard to a v-chain (whether in $\mathfrak{ON}$ or in $\mathfrak{N}$), such terms as 'the first element', 'the last element', 'predecessor of a given element' have the obvious meaning: in $\alpha_1 > \ldots > \alpha_k$, the first element is α_1, the last α_k, the immediate predecessor of α_j is α_{j-1}, etc.

Two elements $\alpha > \beta$ of $\mathfrak{ON}$ are *intertwined* if their legs (see the picture in §1.3) intertwine, or, more precisely, the vertical projection of β dominates the horizontal projection of α. An *intertwined component* of a v-chain $\alpha_1 > \ldots > \alpha_m$ has the obvious meaning: it is a block $\alpha_i > \ldots > \alpha_j$ of consecutive elements such that, $\alpha_k > \alpha_{k+1}$ is intertwined for $i \leq k < j$, and $\alpha_{i-1} > \alpha_i$, $\alpha_j > \alpha_{j+1}$ are not intertwined (in case $i > 1$, $j < m$ respectively). Clearly a v-chain C can be decomposed as $C_1 > \ldots > C_\ell$ into its intertwined components. Observe that, in all intertwined components except perhaps the last, projections of all elements belong to $\mathfrak{N}$. A v-chain is *intertwined* if it consists of a single intertwined component.

Let F be an intertwined v-chain. We define $\operatorname{Proj} F$ to be the set (not multiset) of the projections of all its elements on the diagonal. Let λ be the smallest of all the projections. Set

$$\operatorname{Proj}^{\mathrm{e}} F := \begin{cases} \operatorname{Proj} F & \text{if } \operatorname{Proj} F \text{ has even cardinality} \\ \operatorname{Proj} F \setminus \{\lambda\} & \text{otherwise} \end{cases}$$

For a v-chain C with intertwined components $C_1 > \ldots > C_\ell$, set

$$\operatorname{Proj} C := \operatorname{Proj}^{\mathrm{e}} C_1 \cup \cdots \cup \operatorname{Proj}^{\mathrm{e}} C_{\ell-1} \cup \operatorname{Proj} C_\ell$$
$$\operatorname{Proj}^{\mathrm{e}} C := \operatorname{Proj}^{\mathrm{e}} C_1 \cup \cdots \cup \operatorname{Proj}^{\mathrm{e}} C_{\ell-1} \cup \operatorname{Proj}^{\mathrm{e}} C_\ell$$

For elements (R, C), (r, c) in $\mathfrak{N}$, we say that (R, C) *dominates* (r, c) if $R \geq r$ and $C \leq c$. If the elements belong to the diagonal, to say (R, C) dominates (r, c) is equivalent to saying $(R, C) \geq (r, c)$ (see the first paragraph above). Given v-chains $C : \mu_1 > \ldots > \mu_m$ and $D : \nu_1 > \ldots > \nu_n$ in $\mathfrak{N}$, we say that D dominates C if $n \geq m$ and ν_i dominates μ_i for i, $1 \leq i \leq m$.

2.2 The construction

Let E be a (non-empty) v-chain. The construction of a new form depends on two choices. The first of these is a *cut-off*, the choice of an element of E. Let us write E as $C > D$, where C is the part of E up-to and including the cut-off and D the rest of E. Of course, D can be empty—this happens if and only if the cut-off is the last element of E—but C is never empty.

Suppose such a cut-off is chosen. Let us write the v-chain E as $C_1 > \ldots > C_{\ell-1} > C_\ell > D_1 > D_2 > \ldots$, where $C_1 > \ldots > C_\ell$ is the decomposition of C into intertwined components, $C_\ell > D_1$ is the intertwined component containing C_ℓ of $C > D$ (with D_1 possibly empty), and $D_2 > \ldots$ is the decomposition of $D \setminus D_1$ into intertwined components. We will assume in the sequel that C_ℓ has at least two elements—one may also just say that there are no new forms of C obtained from the choice of this cut-off in case this condition isn't met.

The *new form* $\widetilde{E}$ of E is defined[1] to be $\widehat{C_1} > \ldots > \widehat{C_{\ell-1}} > \widetilde{C_\ell} > D_1 > \ldots$, where $\widehat{C_1}$, $\ldots$, $\widehat{C_{\ell-1}}$, and $\widetilde{C_\ell}$ are as described below. Note that the part D of E beyond the cut-off does not undergo any change. It will be obvious that (1) the vertical projection of the first element does not change in passing from C_j to $\widetilde{C_j}$ or $\widehat{C_j}$; (2) the horizontal projection of the last element gets no smaller in passing from C_j to $\widehat{C_j}$; and (3) the horizontal (respectively vertical) projection of the last element gets bigger (respectively no smaller)

[1] The new form $\widetilde{E}$ may not always be defined. As just remarked, if C_ℓ has only one element then $\widetilde{C_\ell}$ is not defined and so neither is $\widetilde{E}$. As we will see shortly, $\widetilde{C_\ell}$ is not defined more generally when $\operatorname{Proj} C_\ell$ has evenly many elements and contains no elements strictly in between the vertical and horizontal projections of the last element of C_ℓ.

in passing from C_ℓ to $\widetilde{C_\ell}$. We are therefore justified in writing $\widetilde{E}$ as $\widehat{C_1} > \ldots > \widehat{C_{\ell-1}} > \widetilde{C_\ell} > D_1 > \ldots$.

We first construct $\widetilde{C_\ell}$. In fact, we construct $\widetilde{F}$ for an arbitrary intertwined v-chain F with at least 2 elements (subject to a certain further condition as will be specified shortly). There are two cases according as the cardinality $\#\mathrm{Proj}\, F$ of $\mathrm{Proj}\, F$ is odd or even. Suppose first that it is odd. In this case no further choice is involved in the construction. Let (r_1, r_1^*), $\ldots$,(r_s, r_s^*), $\ldots$, (r_t, r_t^*) be the elements of $\mathrm{Proj}^{\mathrm{e}}\, F$ arranged in decreasing order, where (r_s, r_s^*) is the vertical projection of the last element of F. Then t is even; and, since there exists at least one horizontal projection that is also a vertical projection (because $\#\mathrm{Proj}\, F$ is assumed to be odd), we have

$$\begin{aligned} t - s + 1 &\leq \text{number of horizontal projections that are} \\ &\qquad\qquad \text{not vertical projections} \\ &< \text{number of horizontal projections} \\ &= \text{number of vertical projections} \\ &\leq s \end{aligned}$$

so that $2s - t$ is even and strictly positive. We define $\widetilde{F}$ to be the v-chain

$$(r_2, r_1^*) > \ldots > (r_{2s-t}, r_{2s-t-1}^*) > (r_{s+1}, r_{2s-t+1}^*) > \ldots > (r_t, r_s^*)$$

In case $s = t$, the 'second half' of $\widetilde{F}$, namely, $(r_{s+1}, r_{2s-t+1}^*) > \ldots > (r_t, r_s^*)$ is understood to be empty. Figure 2.2.1 above illustrates the construction.

In the case when $\#\mathrm{Proj}\, F$ is even, the construction of $\widetilde{F}$ is similar. The only difference is that (r_1, r_1^*), $\ldots$, (r_t, r_t^*) are now the elements in decreasing order of the set $\mathrm{Proj}\, F$ minus two elements, the last element and another that is smaller than (r_s, r_s^*)—if such an element does not exist, then $\widetilde{F}$ is not defined. The choice of such an element is the second of the two choices involved in the construction of the new form (the first being the cut-off). Observe that now $t - s + 2 \leq s$, so that $2s - t$ is again even and strictly positive.

To define $\widehat{C_1}$, $\ldots$, $\widehat{C_{\ell-1}}$, we define more generally $\widehat{F}$ for an arbitrary intertwined v-chain F both projections of all of whose elements belong to $\mathfrak{N}$. Let (r_1, r_1^*), $\ldots$, (r_t, r_t^*) be the elements in decreasing order of $\mathrm{Proj}^{\mathrm{e}}\, F$. We define $\widehat{F}$ to be the v-chain $(r_2, r_1^*) > \ldots > (r_t, r_{t-1}^*)$.

Proposition 2.2.1 *With notation as above,*

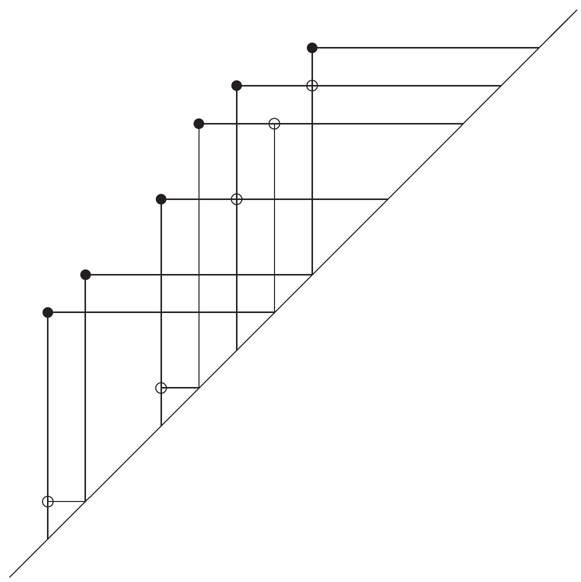


Figure 2.2.1: Illustration of the construction of $\widetilde{F}$ in the case when $\operatorname{Proj} F$ has odd cardinality: The solid circles indicate the points of the original v-chain F, the open circles those of $\widetilde{F}$.

1. *No two elements of* $\widetilde{C}$ *share a projection.*
2. $\operatorname{Proj} \widetilde{C}$ *has evenly many elements. It equals* $\operatorname{Proj}^{\mathrm{e}} C$ *if* $\operatorname{Proj} C$ *has oddly many elements.*
3. $\widetilde{C}$ *has strictly fewer elements than* C.

In particular, $\widetilde{E}$ *has strictly fewer elements than* E.

PROOF: (1) and (2) being clear from the definition of $\widetilde{C}$, we indicate a proof of (3). Using # to denote cardinality, we have

$$\#\operatorname{Proj} \widetilde{C} = \begin{cases} \#\operatorname{Proj}^{\mathrm{e}} C & \text{if } \#\operatorname{Proj} C \text{ is odd} \\ \#\operatorname{Proj}^{\mathrm{e}} C - 2 & \text{if } \#\operatorname{Proj} C \text{ is even} \end{cases}$$

Because of (1), $\#\widetilde{C} = \frac{\operatorname{Proj} \widetilde{C}}{2}$. Thus $\#\widetilde{C}$ equals the greatest integer smaller than $\frac{\#\operatorname{Proj} C}{2}$. But clearly $\frac{\#\operatorname{Proj} C}{2} \leq \#C$. □

2.2.1 An auxiliary construction

We now identify a certain sub-v-chain of the v-chain $\widetilde{F}$ constructed above. This auxiliary construction will be used in the proof of Lemma 2.3.5, the main ingredient in the proof of the key property of new forms stated in Proposition 2.3.2.

Let $F > D$ be an intertwined v-chain with $\widetilde{F}$ being defined. Let (r_1, r_1^*), $\ldots$, (r_s, r_s^*), $\ldots$, (r_t, r_t^*) be as in the construction of $\widetilde{F}$ in §2.2 above. Write $F > D$ as $F_1 > F_2$, where F_1 consists of all elements of F whose vertical projections belong to $\{(r_1, r_1^*), \ldots, (r_{2s-t}, r_{2s-t}^*)\}$ and F_2 is the complement in $F > D$ of F_1. Denote by $\ddot{F}_1$ the part $(r_2, r_1^*) > \ldots > (r_{2s-t}, r_{2s-t-1}^*)$ of $\widetilde{F}$. Consider the sub-v-chain S of $\widetilde{F}$ consisting of those elements (r_j, r_{s-t+j}^*), $s+1 \leq j \leq t$, such that (r_{s-t+j}, r_{s-t+j}^*) is the vertical projection of some element of F_2 (equivalently of $F_2 \setminus D$). We set $\ddot{F}_2$ to be $S > D$.

Lemma 2.2.2 *1. $\ddot{F}_1 > \ddot{F}_2$ is a sub-v-chain of $\widetilde{F} > D$ the inclusion being possibly strict.*

2. The projections of $\ddot{F}_1$ are even in number and all in $\mathfrak{N}$.

3. The legs of the elements of $\ddot{F}_1$ do not intertwine with one another. Nor does the horizontal leg of the last element of $\ddot{F}_1$ intertwine with the vertical leg of the first element of $\ddot{F}_2$.

4. The vertical projection of every element of F_1 is a projection (vertical or horizontal) of an element of $\ddot{F}_1$.

5. F_2 and $\ddot{F}_2$ are in bijective order preserving correspondence, where the corresponding elements have the same vertical projections (the correspondence is identity on D). Every element of $\ddot{F}_2$ has row index no smaller than that of the corresponding element of F_2: it is bigger for elements of $\ddot{F}_2$ not corresponding to elements of D (and of course equal for those corresponding to D).

Proof: (1) That $\ddot{F}_1 > \ddot{F}_2$ is a sub-v-chain is immediate from the construction. For an example when it is contained properly in $\widetilde{F}$, see Figure 2.2.1: the last but one open circle does not belong to $\ddot{F}_1 > \ddot{F}_2$.

(2) The number of projections of $\ddot{F}_1$ is $2s - t$ which is even since t is even. The horizontal projection of the last element of $\ddot{F}_1$ is (r_{2s-t}, r_{2s-t}^*) and this belongs to $\mathfrak{N}$ because $2s - t \leq s$ (since $s \leq t$).

(3) The first assertion is clear from the definition of $\ddot{F}_1$. The second too is clear: p_h(last element of $\ddot{F}_1$) $= (r_{2s-t}, r^*_{2s-t}) > (r_{2s-t+1}, r^*_{2s-t+1}) \geq$ p_v(first element of $\ddot{F}_2$).

(4) Clear from construction.

(5) Let F_2 be $\alpha_1 > \ldots > \alpha_k$ and $\ddot{F}_2$ be $\{\beta_1, \ldots, \beta_k\}$, where α_i, β_i have the same column index for $1 \leq i \leq k$. Then $\beta_1 > \ldots > \beta_k$, for, $\ddot{F}_2$ being part of $\widetilde{F} > D$ the β's form a v-chain in some order, and, their column indices being shared with the α's, the order $\beta_1 > \ldots > \beta_k$ is forced.

For the second part of the assertion, let $\alpha_1 > \ldots > \alpha_\ell$ be $F_2 \setminus D$, and let $R_1, \ldots, R_\ell$ be the respective row indices of $\alpha_1, \ldots, \alpha_\ell$. Then $r_t > R_\ell, \ldots, r_{t-i} > R_{\ell-i}$ for $1 \leq i \leq \ell$ (for the horizontal projection of the last element of F and possibly one more horizontal projection have been discarded from $\operatorname{Proj} F$ to obtain $(r_1, r^*_1), \ldots, (r_t, r^*_t)$). Also, if j be such that $(r_j, r^*_{s-t+j}) = \beta_i$ for some i, $1 \leq i \leq \ell$, then $j \leq t - (\ell - i)$ (strict inequality occurs when $\widetilde{F} > D$ properly contains $\ddot{F}_1 > \ddot{F}_2$). We thus have $r_j \geq r_{t-(\ell-i)} > R_{\ell-(\ell-i)} = R_i$, which is what we set out to prove. □

2.3 A key property of new forms

The main result of this subsection is Proposition 2.3.2 below. Invoked in its proof is Lemma 2.3.5 which is really where all the action takes place.

To a v-chain C of elements in $\mathfrak{O}\mathfrak{N}$, there is, as explained in [9, §2.2.2], an associated element w_C of $I(d)$. There is also a corresponding monomial $\mathfrak{S}_C$ in $\mathfrak{N}$ associated to C ([9, §5.3.3]).

Remark 2.3.1 In the statements and proofs of this section we need to refer to v-chains in monomials in $\mathfrak{N}$ (typically in $\mathfrak{S}_C$ where C is a v-chain in $\mathfrak{O}\mathfrak{N}$). Such v-chains are understood to be in $\mathfrak{N}$ (not necessarily restricted to be in $\mathfrak{O}\mathfrak{N}$).

Proposition 2.3.2 *Let E be a v-chain in $\mathfrak{O}\mathfrak{N}$ and $\widetilde{E}$ a new form of E. Then $w_{\widetilde{E}} \geq w_E$.*

Proof: By Lemmas 4.5 and 5.5 of [5], it is enough to show that every v-chain in $\mathfrak{S}_E$ is dominated by one in $\mathfrak{S}_{\widetilde{E}}$. Further, by [2, Lemma 5.15] (or, more precisely, its proof), it follows, from the symmetry about the diagonal of monomials attached to v-chains in $\mathfrak{O}\mathfrak{N}$, that it is enough to show that every v-chain in $\mathfrak{S}_E$ lying (weakly) above the diagonal (in other words, in $\mathfrak{O}\mathfrak{N} \cup \mathfrak{d}$)

is dominated by one in $\mathfrak{S}_{\widetilde{E}}$. We now make some observations after which it will only remain to invoke Lemmas 2.3.3 and 2.3.5 below.

Decompose E into intertwined components $C_1, \ldots, C_\ell - 1$, $C_\ell > D_1, \ldots$ as in the description of the construction of the new form $\widetilde{E}$. Let us call these the 'parts' of E (just for now). There is the corresponding decomposition of $\widetilde{E}$ into its 'parts' (this is the definition of the *parts* of $\widetilde{E}$): $\widehat{C_1}, \ldots \widehat{C_{\ell-1}}$, $\widetilde{C_\ell} > D_1, D_2, \ldots$. It is clear from the definitions of $\widehat{C_j}$ and $\widetilde{C_\ell}$ that each part of $\widetilde{E}$ is a union of intertwined components. In particular, as is immediate from the definition of connectedness in §5.3.2 of [9], each part (of E or $\widetilde{E}$) is a union of connected components. Thus we have

$$\mathfrak{S}_E = \mathfrak{S}_{C_1} \cup \cdots \cup \mathfrak{S}_{C_{\ell-1}} \cup \mathfrak{S}_{C_\ell > D_1} \cup \mathfrak{S}_{D_2} \cup \cdots$$

and

$$\mathfrak{S}_{\widetilde{E}} = \mathfrak{S}_{\widehat{C_1}} \cup \cdots \cup \mathfrak{S}_{\widehat{C_{\ell-1}}} \cup \mathfrak{S}_{\widetilde{C_\ell} > D_1} \cup \mathfrak{S}_{D_2} \cup \cdots$$

Further, since there are no intertwinings between parts, the following follow easily from the definition of the monomial attached to a v-chain:

- any v-chain G in $\mathfrak{S}_E$ can be decomposed as: $G_1 > \ldots > G_{\ell-1} > G_\ell > H_2 > \ldots$ where G_1 is a v-chain in $\mathfrak{S}_{C_1}, \ldots$, $G_{\ell-1}$ is a v-chain in $\mathfrak{S}_{C_{\ell-1}}$, G_ℓ is a v-chain in $\mathfrak{S}_{C_\ell > D_1}$, H_2 is a v-chain in $\mathfrak{S}_{D_2}, \ldots$;

- given v-chains G_1 in $\mathfrak{S}_{\widehat{C_1}}, \ldots$, $G_{\ell-1}$ in $\mathfrak{S}_{\widehat{C_{\ell-1}}}$, G_ℓ in $\mathfrak{S}_{\widetilde{C_\ell} > D_1}$, H_2 in $\mathfrak{S}_{D_2}$, $\ldots$, all lying weakly above the diagonal, these can be put together as $G_1 > \ldots > G_{\ell-1} > G_\ell > H_2 > \ldots$ to give a v-chain G in $\mathfrak{S}_{\widetilde{E}}$.

The proposition now follows from Lemmas 2.3.3 and 2.3.5 below. □

Lemma 2.3.3 *For an intertwined v-chain F both projections of all of whose elements belong to $\mathfrak{N}$, every v-chain in $\mathfrak{S}_F$ is dominated by one in $\mathfrak{S}_{\widehat{F}}$. (Observe that both $\mathfrak{S}_F$ and $\mathfrak{S}_{\widehat{F}}$ consist of diagonal elements.)*

PROOF: $\mathfrak{S}_F$ consists of the vertical projections elements of F in case $\#F$ is even, and of the vertical projections and the horizontal projection of the last element in case $\#F$ is odd. In any case $\mathfrak{S}_F$ consists of evenly many elements.

$\mathfrak{S}_{\widehat{F}}$ consists of all projections of all elements of F (in particular, $\mathfrak{S}_{\widehat{F}} \supseteq \mathfrak{S}_F$) in case the total number of such projections (considered as a set, not multiset) is even; and, in case that number is odd, it consists of all projections except the horizontal projection of the last element. In any case $\mathfrak{S}_{\widehat{F}}$ consists of evenly many elements.

Suppose that $\mathfrak{S}_{\widehat{F}} \not\supseteq \mathfrak{S}_F$. Then $\#F$ is odd, the total number of projections is odd, and $\mathfrak{S}_F \setminus \mathfrak{S}_{\widehat{F}} = \{$horizontal projection of the last element of $F\}$; in particular, $\#\mathfrak{S}_F = \#F + 1$. Since $\#\mathfrak{S}_{\widehat{F}} \geq \#F$ and $\#\mathfrak{S}_{\widehat{F}}$ is even, it follows that $\#\mathfrak{S}_{\widehat{F}} \geq \#F + 1$, which means that $\mathfrak{S}_{\widehat{F}}$ contains some projection not in $\mathfrak{S}_F$. Since any such projection is bigger than the horizontal projection of the last element of F, the lemma follows. □

Lemma 2.3.4 *Let $F > D$ be an intertwined v-chain with $\widetilde{F}$ being defined. Let F_1, F_2, $\ddot{F}_1$, $\ddot{F}_2$ be as in §2.2.1. Then*

1. *The elements in $\ddot{F}_1$ are all of type H in $\ddot{F}_1 > \ddot{F}_2$.*
2. *Vertical projections of elements of F_1 belong to $\mathfrak{S}_{\ddot{F}_1 > \ddot{F}_2}$.*

Proof: Statement (1) follows from (2) and (3) of Lemma 2.2.2. Statement (2) from (1) and Lemma 2.2.2 (4). □

Lemma 2.3.5 *Let $F > D$ be an intertwined v-chain with $\widetilde{F}$ being defined. Given a v-chain $\mu_1 > \mu_2 > \ldots$ in $\mathfrak{S}_{F>D}$, there exists a v-chain $\nu_1 > \nu_2 > \ldots$ in $\mathfrak{S}_{\widetilde{F}>D}$ that dominates it. If $\mu_1 > \mu_2 > \ldots$ lies weakly above the diagonal, then $\nu_1 > \nu_2 > \ldots$ can be chosen also to be so.*

Proof: Let F_1, F_2, $\ddot{F}_1$, $\ddot{F}_2$ be as defined in §2.2.1. We will show that there exists a v-chain $\nu_1 > \nu_2 > \ldots$ in $\mathfrak{S}_{\ddot{F}_1 > \ddot{F}_2}$ with the desired property. Since $\ddot{F}_1 > \ddot{F}_2$ is a sub-v-chain of $\widetilde{F} > D$ (Lemma 2.2.2 (1)), this will suffice (by either the proof of [9, Proposition 6.1.1 (1)] or [9, Corollary 6.1.2] and [5, Lemmas 4.5, 5.5]). For the same reasons as noted in the proof of Proposition 2.3.2, it is enough to assume that $\mu_1 > \mu_2 > \ldots$ lies weakly above the diagonal and find $\nu_1 > \nu_2 > \ldots$ that dominates it and lies weakly above the diagonal. Obviously, we may take without loss of generality $\mu_1 > \mu_2 > \ldots$ to be a maximal such v-chain.

The rest of the proof is divided into three parts:

- Enumerate the maximal v-chains $\mu_1 > \mu_2 > \ldots$ in $\mathfrak{S}_{F>D}$ lying weakly above the diagonal. There are two of these: see (*) and (**) below.

- Identify a certain v-chain (see (†) below) in $\mathfrak{S}_{\ddot{F}_1>\ddot{F}_2}$ and lying weakly above the diagonal and list its relevant properties.
- Show that the v-chain (†) dominates (*) in all cases and (**) in many cases. Find a v-chain (††) in $\mathfrak{S}_{\ddot{F}_1>\ddot{F}_2}$ and lying weakly above the diagonal that dominates (**) when (†) does not.

We start with the first part. Write $F > D$ as $\alpha_1 > \alpha_2 > \ldots$ and let k be the integer such that α_k is the last element of $F > D$ whose horizontal projection belongs to $\mathfrak{N}$: in other words, α_k is the immediate predecessor of what is called the critical element in [9, §5.3.4]. Of course such an element may not exist, and the proof below, interpreted properly, covers that case.

The v-chain $F > D$ being intertwined, its connected components (in the sense of [9, §5.3.2]) are determined by whether or not α_k is connected to its immediate successor: in either case, each element α_j for $j \geq k+2$ forms a component by itself, and the elements $\alpha_1, \ldots, \alpha_k$ are all in a single component. Consider the types of elements of $F > D$ as in [9, §5.3.4]. The possibilities for the sequence of these are listed in the following display. In these, the underlined type is that of the element α_k, the overlined type is that of either α_k or its immediate predecessor α_{k-1} according as whether k is odd or even, and the vertical bar indicates where the first disconnection occurs (either just after α_k or just after α_{k+1}):

$$\begin{array}{llllllll}
\text{Case I:} & \mathrm{V} \;\ldots\; \mathrm{V} & \overline{\underline{\mathrm{H}}}\,| & \mathrm{S} & \mathrm{S} & \mathrm{S} & \ldots \\
\text{Case II:} & \mathrm{V} \;\ldots\; \mathrm{V} & \overline{\underline{\mathrm{V}}} & \mathrm{V}\,| & \mathrm{S} & \mathrm{S} & \ldots \\
\text{Case III:} & \mathrm{V} \;\ldots\; \mathrm{V} & \overline{\mathrm{V}} & \underline{\mathrm{V}}\,| & \mathrm{S} & \mathrm{S} & \ldots \\
\text{Case IV:} & \mathrm{V} \;\ldots\; \mathrm{V} & \overline{\mathrm{V}} & \underline{\mathrm{V}} & \mathrm{S}\,| & \mathrm{S} & \ldots
\end{array}$$

That these possibilities are all follows readily from the definition of type.

For an element λ of a v-chain C (in $\mathfrak{O}\mathfrak{N}$), let $q_{C,\lambda}$ denote $p_v(\lambda)$ if λ is of type V or H and λ itself if it is of type S. It is easy to see (and in any case explicitly stated in [9, Proposition 5.3.4 (1)]) that $q_{C,\lambda} > q_{C,\lambda'}$ for (not necessarily consecutive) elements $\lambda > \lambda'$ in C. It follows that, in Cases II, III, and IV,

$$(*) \qquad q_{F>D,\alpha_1} > q_{F>D,\alpha_2} > \ldots$$

is the unique maximal v-chain in $\mathfrak{S}_{F>D}$ lying weakly above the diagonal; in Case I too it is a maximal v-chain but there is also another one, namely,

$$(**) \qquad p_v(\alpha_1) > p_v(\alpha_2) > \ldots > p_v(\alpha_k) > p_h(\alpha_k)$$

(if $p_h(\alpha_k)$ dominated α_j for some j, $k < j$, it would contradict the disconnection between α_k and α_{k+1}: recall that α_k and α_{k+1} are intertwined). This finishes our first task of determining the maximal v-chains in $\mathfrak{S}_{F>D}$ that lie weakly above the diagonal.

Next we identify a certain v-chain (see (†) below) in $\mathfrak{S}_{\ddot{F}_1>\ddot{F}_2}$ that will have the desired property in almost all cases. Let e be the integer such that F_1 is $\alpha_1 > \ldots > \alpha_e$ (and F_2 is $\alpha_{e+1} > \ldots$). Let $\beta_{e+1} > \ldots$ be the counterparts in $\ddot{F}_2$ respectively of $\alpha_{e+1} > \ldots$, the correspondence $\alpha \leftrightarrow \beta$ being as in Lemma 2.2.2 (5):

(a) The vertical projections of α_j and β_j are equal for $j = e+1, e+2, \ldots$. And the row index of β_j is no less than that of α_j (Lemma 2.2.2 (5)).

Let f be the largest integer, $f \geq e$, such that β_f is of type V or H in $\ddot{F}_1 > \ddot{F}_2$: if either α_{e+1} does not exist or β_{e+1} is of type S, then $f := e$ and β_e is taken to be the last element of $\ddot{F}_1$ (this is not to say that the cardinality of $\ddot{F}_1$ is e). Consider the subset Z of $\mathfrak{S}_{\ddot{F}_1>\ddot{F}_2}$ consisting of contributions of elements up to and including β_f and only those contributions that are not smaller than $p_v(\beta_{f+1})$ (equivalently β_{f+1}): if β_{f+1} does not exist, then this condition is vacuous. In other words, Z consists of (1) the vertical projections of all elements of $\ddot{F}_1 > \ddot{F}_2$ up to and including β_f; and (2) the horizontal projections of all elements of $\ddot{F}_1 > \ddot{F}_2$ of type H except perhaps of β_f itself: the horizontal projection of β_f does not belong to Z if it is smaller than $p_v(\beta_{f+1})$ (even if β_f should be of type H). Letting the elements of Z arranged in order be $\gamma_1 > \ldots > \gamma_g$, we have the following v-chain in $\mathfrak{S}_{\ddot{F}_1>\ddot{F}_2}$:

$$(\dagger) \qquad \gamma_1 > \ldots > \gamma_g > \beta_{f+1} > \beta_{f+2} > \ldots$$

We claim:

(i) $p_v(\alpha_1)$, $\ldots$, $p_v(\alpha_f)$ belong to Z. (So $g \geq f$.)

(ii) The horizontal projection of α_{f+1} does not belong to $\mathfrak{N}$. That is, $f \geq k$ with k as defined earlier.

(iii) The types of α_{f+2}, α_{f+3}, $\ldots$ in $F > D$ are all S.

(iv) The type of α_{f+1} in $F > D$ is either V or S. If it is V, then $f = k$ and we are in Case II (in the enumeration of types listed above).

(v) The critical element of $\ddot{F}_1 > \ddot{F}_2$ (if it exists) is either β_f or β_{f+1}.

(vi) If $g \not\geq f+1$ (observe that $g \geq f$ always by (i)), then e is even.

(vii) If $g \not\geq f+1$ and f is odd, then β_f is of type H (in $\ddot{F}_1 > \ddot{F}_2$) and α_{f+1} is of type S (in $F > D$, if α_{f+1} exists).

PROOF: (i) If $j \leq e$ (i.e., if α_j belongs to F_1), then $p_v(\alpha_j)$ belongs to Z by Lemma 2.3.4 (2); if $e < j \leq f$, then $p_v(\alpha_j) = p_v(\beta_j)$ (see (a) above) and so belongs to Z.

(ii) On the one hand, $p_h(\beta_{f+1}) \notin \mathfrak{N}$, for β_{f+1} is of type S. On the other hand, the row index of β_{f+1} is at least that of α_{f+1} (see (a) above).

(iii) and (iv) follow from combining (ii) with the enumeration of cases of types of elements of $F > D$ above (Cases I–IV).

(v) This follows from the definition of type and the choice of f: an element of type S cannot precede the critical element; an element of type V cannot succeed the critical element.

(vi) Suppose that e is odd. The contributions to $\mathfrak{S}_{\ddot{F}_1 > \ddot{F}_2}$ of elements of $\ddot{F}_1$ include $p_v(\alpha_1), \ldots, p_v(\alpha_e)$ and are evenly many in number (Lemma 2.3.4 (1)); Z contains all of these (Lemma 2.2.2 (3)) in addition to $p_v(\beta_{e+1}), \ldots, p_v(\beta_f)$, so $g \geq (e+1) + (f-e) = f+1$. Thus e is even.

(vii) By (vi), e is even. Since f is odd, it follows that $f \geq e+1$. We first show that h is odd, where β_h is the first element of the connected component of $\ddot{F}_1 > \ddot{F}_2$ that contains β_f. Consider a connected component of $\ddot{F}_2 > D$ contained entirely within $\{\beta_{e+1}, \ldots, \beta_{f-1}\}$ (if any should exist) (if $f = e+1$, then $\{\beta_{e+1}, \ldots, \beta_{f-1}\}$ is understood to be empty). If its cardinality is odd, then its last element, say β_i, has type H (this follows from the definition of type: by choice of f, the type can only be V or H), and $p_h(\beta_i)$ is bigger than $p_v(\beta_{i+1})$ (for otherwise β_{i+1} will be forced to have type S ([9, Proposition 5.3.4 (1) and (3)]), a contradiction to the definition of f); and Z would contain $p_h(\beta_i)$ in addition to the elements in (i), a contradiction. Thus all such components have even cardinality. This implies that $h - e$ is odd, and, since e is even (by (vi)), that h is odd.

Since β_{f+1} is of type S (by choice of f), it is the last element in its connected component and the component has odd cardinality. Since h and f are odd, this component can only be $\{\beta_{f+1}\}$. This means that β_f is the last element in its connected component, and so of type H: its type is either V or H by choice of f, and further because $f - h + 1$ is odd its type is H.

If $p_h(\beta_f) \geq p_v(\beta_{f+1})$, then $g \geq f+1$, for Z would contain $p_h(\beta_f)$ in addition to the elements in (i). So $p_h(\beta_f) < p_v(\beta_{f+1})$. Since β_{f+1} is not connected to β_f (as was just shown), it follows that $R' \leq R^*$ where R, R' are the row indices of β_f, β_{f+1}. Letting r, r' be the row indices of α_f, α_{f+1}, we have, by (a) above, $r' \leq R' \leq R^* \leq r^*$. This means that α_{f+1} is not connected to α_f and so is of type S (see (ii) above). □

The second part of the proof (of the lemma) being over, we start on the third. We first show that (†) dominates (*). From (a) above and (iii) of the claim, it follows that $q_{\ddot{F}_1>\ddot{F}_2,\beta_{f+2}} = \beta_{f+2} > q_{\ddot{F}_1>\ddot{F}_2,\beta_{f+3}} = \beta_{f+3} > \ldots$ dominates $q_{\ddot{F}_1>\ddot{F}_2,\alpha_{f+2}} = \alpha_{f+2} > q_{\ddot{F}_1>\ddot{F}_2,\alpha_{f+3}} = \alpha_{f+3} > \ldots$. From (i) of the claim it follows that $\gamma_1 > \ldots > \gamma_g > q_{\ddot{F}_1>\ddot{F}_2,\beta_{f+1}}$ dominates $q_{F>D,\alpha_1} > \ldots > q_{F>D,\alpha_{f+1}}$ if either $q_{\ddot{F}_1>\ddot{F}_2,\beta_{f+1}}$ dominates $q_{F>D,\alpha_{f+1}}$ (which fails by (a) only when α_{f+1} has type V) or $g \geq f+1$ (by the definition of Z and (a)). Suppose that α_{f+1} has type V. It follows from (iv) of the claim that f is odd, and so, from (vii) of the claim, that $g \geq f+1$. Thus (†) dominates (*).

Now assume that the types of the elements of $F > D$ are as in Case I and that $\mu_1 > \mu_2 > \ldots$ is (**). If $f \geq k+1$, then (†) dominates (**), for (†) contains $p_v(\alpha_1)$, …, $p_v(\alpha_k)$, $p_v(\alpha_{k+1})$ (see (i) of the claim), and $p_v(\alpha_{k+1}) \geq p_h(\alpha_k)$ (for $F > D$ is intertwined); so assume that $f = k$ (by (ii), we have $f \geq k$ always). If $g \geq f+1 = k+1$, then again (†) dominates (**) for similar reasons: Z contains $p_v(\alpha_1)$, …, $p_v(\alpha_k)$, and it also contains g elements that dominate $p_h(\alpha_k)$: $p_v(\beta_{k+1}) = p_v(\alpha_{k+1}) \geq p_h(\alpha_k)$ for $F > D$ is intertwined. So assume that $g = f = k$ ($g \geq f$ always by (i)). Since we are in Case I, k is odd (and hence so is f). By (vii), β_f is of type H and the following v-chain is in $\mathfrak{S}_{\ddot{F}_1>\ddot{F}_2}$:

$$(\dagger\dagger) \quad p_v(\alpha_1) > \ldots > p_v(\alpha_e) > p_v(\alpha_{e+1})(= p_v(\beta_{e+1})) > \\ \ldots > p_v(\alpha_f)(= p_v(\beta_f)) > p_h(\beta_f)$$

This v-chain dominates (**) by (a) above. □

2.4 The element y_E attached to a v-chain E

Let E be a v-chain in $\mathfrak{O}\mathfrak{N}$. From $\mathrm{Proj}^{\mathrm{e}}\, E$ we can get an element y_E of $I(d, 2d)$ by the following natural process (see the proof of [5, Proposition 4.3]): the column indices of elements of $\mathrm{Proj}^{\mathrm{e}}\, E$ occur as members of v; these are replaced by the row indices to obtain y_E.

Proposition 2.4.1 *$y_E \geq v$ and y_E belongs to $I(d)$.*

Proof: Think of y_E as being the result of a series of operations done starting with v. Let $x \in I(d)$ be such that $x \geq v$. Suppose $(r, c) \in \mathfrak{O}\mathfrak{N}$ is such that c occurs and r does not in x. Let x' be the result of replacing c and r^* in x by r and c^*. Then, clearly, either $r > r^*$ in which case $r^* \leq d < d+1 \leq c^*$ and $c \leq d < d+1 \leq r$, or $r < r^*$ in which case $c < r \leq d < d+1 \leq r^* < c^*$. In either case $x' \geq x \geq v$ and x' belongs to $I(d)$.

The proposition follows easily, as we now show, from the observation just made. Consider the elements of $\mathrm{Proj}^{\mathrm{e}} E$ that are not in $\mathfrak{N}$. These can only be horizontal projections, each of some unique element of E. Pair these up, each with the vertical projection of the corresponding element of E (all vertical projections belong to $\mathrm{Proj}^{\mathrm{e}} E$). Since $\mathrm{Proj}^{\mathrm{e}} E$ has even cardinality, there are evenly many elements left (all in $\mathfrak{N}$) after the elements not in $\mathfrak{N}$ are paired up as prescribed. Pair these up in some arbitrary way. If (r, r^*) and (c^*, c) are the horizontal and vertical projections of an element (r, c) in $\mathfrak{O}\mathfrak{N}$, we can think of replacing r^* by r and c by c^* as the single operation described in the previous paragraph in going from x to x'. It should now be clear that y_E is obtained from v by a series of operations, each of which is like the one described in the above paragraph. $\square$

In fact, we have

Proposition 2.4.2 *$y_E \geq w_E$, where w_E is the element of $I(d)$ attached as in [9, §2.2.2] to E.*

Proof: The strategy is similar to that of the proof of Proposition 2.3.2. There corresponds to y_E ([5, Proposition 4.3]) a subset $\mathfrak{S}_{y_E}$ of $\mathfrak{N}$ that is 'distinguished' in the sense of [5, §4]. (Furthermore, the subset is symmetric about the diagonal and contains evenly many diagonal elements [9, Proposition 5.2.1].)

We first give an explicit description of $\mathfrak{S}_{y_E}$. Let the elements of $\mathrm{Proj}^{\mathrm{e}} E$ arranged in decreasing order be

$$(r_1, r_1^*), \ldots, (r_u, r_u^*), \ldots, (r_t, r_t^*)$$

where u is such that (r_u, r_u^*) but not (r_{u+1}, r_{u+1}^*) belongs to $\mathfrak{N}$, or, equivalently, $r_u > r_u^*$ but $r_{u+1} < r_{u+1}^*$. Throughout this proof, we use i and j consistently to denote integers in the range $1, \ldots, u$ and $u+1, \ldots, t$ respectively.

Clearly (r_j, r_j^*) are all horizontal projections. Let $p(j)$ be such that $(r_j, r_{p(j)}^*)$ belongs to E: all the column indices of elements of E must appear as column indices also in $\mathrm{Proj}^{\mathrm{e}}\, E$, for no vertical projection is left out in $\mathrm{Proj}^{\mathrm{e}}\, E$. Then $(r_{u+1}, r_{p(u+1)}^*) > \ldots > (r_t, r_{p(t)}^*)$ is a v-chain and $p(u+1) < \ldots < p(t)$.

Let σ denote the function $\{u+1, \ldots, t\} \to \{1, \ldots, u\}$ defined inductively as follows:

- $\sigma(t)$ is largest possible such that $r_t > r_{\sigma(t)}^*$;
- $\sigma(t-1)$ is largest possible in $\{1, \ldots, t\} \setminus \{\sigma(t)\}$ such that $r_{t-1} > r_{\sigma(t-1)}^*$;

$\vdots$

- $\sigma(j)$ is largest possible in $\{1, \ldots, t\} \setminus \{\sigma(t), \sigma(t-1), \ldots, \sigma(j+1)\}$ such that $r_j > r_{\sigma(j)}^*$.

Such a choice of σ is possible. Indeed,

1. $\sigma(t) \geq p(t),\ \ldots,\ \sigma(j) \geq p(j),\ \ldots,\ \sigma(u+1) \geq p(u+1)$;
2. If $\sigma(j) > p(j)$, then $\sigma(j-1) \geq p(j)$ (for $r_{j-1} > r_j > r_{p(j)}^*$).

We have

$$\mathfrak{S}_{y_E} = \{(r_j, r_{\sigma(j)}^*), (r_{\sigma(j)}, r_j^*) \mid u+1 \leq j \leq t\} \bigcup \{(r_i, r_i^*) \mid 1 \leq i \leq u,\ \nexists\ j \text{ with } i = \sigma(j)\}$$

Next we draw some conclusions from the above description of $\mathfrak{S}_{y_E}$:

(a) If $E_1 > \ldots > E_\ell$ be the decomposition of E into intertwined components, then $\mathfrak{S}_{y_E} = \mathrm{Proj}^{\mathrm{e}}\, E_1 \cup \cdots \cup \mathrm{Proj}^{\mathrm{e}}\, E_{\ell-1} \cup \mathfrak{S}_{y_{E_\ell}}$.

(b) Vertical projections of all elements preceding the critical element belong to $\mathfrak{S}_{y_E}$.

(c) If there exists an element α in E_ℓ of type H (there is at most one such element) and $p_h(\alpha)$ belongs to $\mathrm{Proj}^{\mathrm{e}}\, E$, then $p_h(\alpha) \in \mathfrak{S}_{y_{E_\ell}}$.

(d) For each α in E there exists a unique element β in $\mathfrak{S}_{y_E}$ that shares its column index with α. This element lies on or above the diagonal and its row index is no smaller than that of α. If E is $\alpha_1 > \alpha_2 > \ldots$, then the corresponding elements form a v-chain $\beta_1 > \beta_2 > \ldots$ in $\mathfrak{S}_{y_E}$.

(e) Suppose that α is the critical element of E and $\beta \neq p_v(\alpha)$ where β is the corresponding element in $\mathfrak{S}_{y_E}$ (see (d)). Then $p(j) = \sigma(j)\ \forall\ j$ and $\operatorname{Proj} E = \operatorname{Proj}^{\mathrm{e}} E$.

(f) Let α be the critical element of E. If α has type V, its horizontal projection $p_h(\alpha)$ belongs to $\operatorname{Proj}^{\mathrm{e}} E$ (in other words $p_h(\alpha) = (r_{u+1}, r^*_{u+1})$), and $\sigma(j) = p(j)\ \forall\ j$, then the only elements of $\operatorname{Proj}^{\mathrm{e}} E \cap \mathfrak{N}$ smaller than $p_v(\alpha)$ are the vertical projections of elements of E (evidently of those beyond the critical element).

PROOF: (a) Observe that the critical element $(r_{u+1}, r^*_{p(u+1)})$ belongs to E_ℓ (for the critical element is intertwined with all its successors). Since $\sigma(j) \geq p(j)$ for all j and $p(u+1) < \ldots < p(t)$, the conclusion follows.

(b) This is because $\{\sigma(t), \ldots, \sigma(u+1)\} \subseteq \{p(u+1), p(u+1)+1, \ldots, t\}$.

(c) Let $p_h(\alpha) = (r_s, r^*_s)$. Since α is not connected to (but is intertwined with) any of its successors, we have $r_j \not> r^*_s\ \forall\ j$, so $s \notin \{\sigma(u+1), \ldots, \sigma(t)\}$. And clearly $s \leq u$, so the conclusion follows.

(d) Since $p_v(\alpha) \in \operatorname{Proj}^{\mathrm{e}} E$, the existence and uniqueness of β is clear from the description of $\mathfrak{S}_{y_E}$ above. Also clear from the description is that the only elements below the diagonal in $\mathfrak{S}_{y_E}$ are those with column indices r^*_j, but $p_v(\alpha) = (r_i, r^*_i)$ for some i ($p_v(\alpha) \in \mathfrak{N}$ surely), so β lies on or above the diagonal.

To see that the row index of β is no smaller than that of α, first note that this is clear if $\beta = p_v(\alpha)$. If α precedes the critical element, then $\beta = p_v(\alpha)$ by (b). So suppose that $\alpha = (r_j, r^*_{p(j)})$ and further that $p(j) = \sigma(j')$ for some j', $u+1 \leq j' \leq t$ (if no such j' exists, then again $\beta = p_v(\alpha)$ by the description of $\mathfrak{S}_{y_E}$). Then $p(j) \geq p(j')$ (for $\sigma(j') \geq p(j')$), so $j \geq j'$ (for $p(u+1) < \ldots < p(t)$). Since $\beta = (r_{j'}, r^*_{\sigma(j')})$, it follows that $r_{j'} \geq r_j$, i.e., β has no smaller row index than that of α.

Finally, that β_1, β_2, …form a v-chain follows readily by combining the assertion just proved with the distinguishedness of $\mathfrak{S}_{y_E}$.

(e) The assumption that $\beta \neq p_v(\alpha)$ implies that $p_v(\alpha)(= (r_{p(u+1)}, r^*_{p(u+1)}))$ does not belong to $\mathfrak{S}_{y_E}$, which means $p(u+1) = \sigma(j)$ for some j. If $j > u+1$, we have $\sigma(j) \geq p(j) > p(u+1)$ (see (1) above), a contradiction, so $p(u+1) = \sigma(u+1)$. By (2) above, it follows that $p(j) = \sigma(j)$ for all j.

Suppose that $\operatorname{Proj} E$ has oddly many elements. Let i be such that (r_i, r^*_i) is the vertical projection of the last element, say λ, of E. Since $p_h(\lambda) \notin \operatorname{Proj}^{\mathrm{e}} E$, it follows that $i > p(t)$ (note that $(r_{p(t)}, r^*_{p(t)})$ is the vertical projection of the element of E with horizontal projection (r_t, r_t^*)). Since $r_t > r > r_i^*$, where r denotes the row index of λ, we have $\sigma(t) \geq i > p(t)$ contradicting the previous assertion.

(f) Note that $(r_{p(u+1)}, r_{p(u+1)}^*)$ is the vertical projection of α (by the definition of p). Suppose that there exists (r_i, r_i^*) with $i > p(u+1)$ that is not the vertical projection of any element of E, i.e., there does not exist j with $i = p(j)$. Then (r_i, r_i^*) is a horizontal projection, evidently of some predecessor of α. If $r_{u+1} < r_i^*$, then α is not connected with that predecessor, therefore neither to its immediate predecessor, and so of type S (rather than V as assumed). We may therefore assume that $r_{u+1} > r_i^*$. Now, if $i = \sigma(j)$ for some $j > u+1$, then $\sigma(j) \neq p(j)$, a contradiction; if not, then it follows from the definition of σ that $\sigma(u+1) \geq i > p(u+1)$, again a contradiction. (It is easy to construct counter-examples to the assertion with the critical element being the last element of E and its horizontal projection being not in $\mathrm{Proj}^{\mathrm{e}}\, E$, in which case the hypothesis that $\sigma(j) = p(j)$ for all j is vacuously satisfied.) □

We are finally ready for the proof of the proposition. By [5, Lemmas 4.5, 5.5], it is enough to show that every v-chain in $\mathfrak{S}_E$ is dominated by one in $\mathfrak{S}_{y_E}$. Let $E_1 > \ldots > E_\ell$ be the decomposition of E into intertwined components. Take a v-chain C in $\mathfrak{S}_E$. As observed in the proof of Proposition 2.3.2, C is just a concatenation of v-chains $C_1, \ldots, C_\ell$ with C_j being a v-chain in $\mathfrak{S}_{E_j}$. We have already seen in Lemma 2.3.3 that there exist v-chains $D_1, \ldots, D_{\ell-1}$ in $\mathrm{Proj}^{\mathrm{e}}\, E_1, \ldots, \mathrm{Proj}^{\mathrm{e}}\, E_{\ell-1}$ respectively dominating $C_1, \ldots, C_{\ell-1}$. In the light of (a) above, we'd be done if we can find D_ℓ in $\mathfrak{S}_{y_{E_\ell}}$ dominating C_ℓ, for then the concatenation $D_1 > \ldots > D_{\ell-1} > D_\ell$ would be a v-chain in $\mathfrak{S}_{y_E}$ dominating C. As in the proof of Lemma 2.3.5, we may reduce to the case when C_ℓ lies weakly above the diagonal (this follows from the proof of [2, Lemma 5.15] and the symmetry about the diagonal of monomials attached to v-chains).

We now show that such a chain D_ℓ exists. In fact, let us show: for an intertwined v-chain F and $\mu_1 > \mu_2 > \ldots$ a maximal v-chain in $\mathfrak{S}_F$ lying weakly above the diagonal, there exists $\nu_1 > \nu_2 > \ldots$ in $\mathfrak{S}_{y_F}$ lying weakly above the diagonal that dominates $\mu_1 > \mu_2 > \ldots$. The goal being analogous to that of Lemma 2.3.5, we adopt the notation and arguments from the first of the three parts of that proof. There are two possibilities for $\mu_1 > \mu_2 > \ldots$, namely (*) and (**) as in the proof of that lemma.

First consider (**). If $p_h(\alpha_k)$ belongs to $\mathrm{Proj}^{\mathrm{e}}\, F$, then (**) is contained in $\mathfrak{S}_{y_F}$ by (b) and (c) above. If not, then α_k is the last element of F, so that all projections of F belong to $\mathfrak{N}$. In this case, $\mathfrak{S}_{y_F} = \mathrm{Proj}^{\mathrm{e}}\, F = \mathfrak{S}_{\widehat{F}}$, and we're done by invoking Lemma 2.3.3.

Now consider the v-chain (*). Because of (b) and (d) above, it follows that the v-chain $\beta_1 > \beta_2 > \ldots$ as in (d) dominates (*) except in the following situation: the critical element α_{k+1} has type V and $\beta_{k+1} \neq p_v(\alpha_{k+1})$. So assume that we are in this situation (which means that the types of elements of F are as in Case II on page 21 and in particular that k is odd). Assertions (e) and (f) above apply.

The elements $p_v(\alpha_1), \ldots, p_v(\alpha_k)$ belong to $\mathfrak{S}_{y_F}$ (by (b)). If there is one other element in $\mathfrak{S}_{y_F}$ that dominates $p_v(\alpha_{k+1})$, then these elements together form a v-chain $\gamma_1 > \ldots > \gamma_{k+1}$ in $\mathfrak{S}_{y_F}$ that dominates $p_v(\alpha_1) > \ldots > p_v(\alpha_k) > p_v(\alpha_{k+1})$, and $\gamma_1 > \ldots > \gamma_{k+1} > \beta_{k+2} > \beta_{k+3} > \ldots$ dominates (*), and we're done. So assume that this is not the case. From (e) and (f) above it follows that $\mathrm{Proj}\, F$ consists precisely of $p_v(\alpha_1), \ldots, p_v(\alpha_k)$ and both projections of $\alpha_{k+1}, \alpha_{k+2}, \ldots$, and so of an odd number (because k is odd), contradicting (e). □

3 Pfaffians and their Laplace-like expansions

This section can be read independently of the rest of the paper. We define here the *Pfaffian* of a matrix of even size that is skew-symmetric along the anti-diagonal and show that it satisfies a Laplace-like expansion formula similar to the one for the determinant. In fact we define the Pfaffian by such a formula: see Eq. (3.1.1). We then show that it is independent of the choice of the integer involved in the expansion and that it is a square root of the determinant (Corollary 3.2.2). The expansion formula is used crucially in the proof of the main Lemma 4.2.1 in §4.

3.1 The Pfaffian defined by a Laplace-like expansion

Let n be a non-negative integer. For k an integer, define $k^* = 2n + 1 - k$. Let $A = (a_{ij})$ be a $2n \times 2n$ matrix that is skew-symmetric along the anti-diagonal, meaning that $a_{ij} = -a_{j^*i^*}$ for $1 \leq i, j \leq 2n$. We will be considering submatrices of A. Let $A_{r,c}$ denote the submatrix obtained by deleting the row numbered r and the column numbered c; $A_{r_1r_2,c_1c_2}$ the submatrix obtained by deleting rows numbered r_1, r_2 and column numbers c_1, c_2; and so forth. Let D, $D_{r,c}$, $D_{r_1r_2,c_1c_2}$, ... denote respectively the determinants of A, $A_{r,c}$, $A_{r_1r_2,c_1c_2}$, ...

We define the *Pfaffian* Q of the matrix A by induction on n: for $n = 0$, set $Q := 1$; for $n \geq 1$, set

$$Q := \sum_{j=1}^{2n} (-1)^{m+j^*} \operatorname{sgn}(mj)\, a_{m,j^*}\, Q_{mj,j^*m^*} \tag{3.1.1}$$

where m is a fixed integer, $1 \leq m \leq 2n$; Q_{mj,j^*m^*} is the Pfaffian of the submatrix A_{mj,j^*m^*}; and, for natural numbers i and j,

$$\operatorname{sgn}(ij) := \begin{cases} 1 & \text{if } i < j \\ -1 & \text{if } i > j \\ 0 & \text{if } i = j \end{cases}$$

(Q_{mj,j^*m^*} is not defined when $j = m$ but this does not matter since $\operatorname{sgn}(mj) = 0$ then). To see that the expression (3.1.1) is independent of the choice of m, proceed by induction on n. If p is another choice, then, by the induction hypothesis, Q_{mj,j^*m^*} equals

$$\sum_{k=1}^{2n} (-1)^{p+k^*} \operatorname{sgn}(pm)\operatorname{sgn}(pj)\operatorname{sgn}(k^*j^*)\operatorname{sgn}(k^*m^*)\operatorname{sgn}(pk)\, a_{p,k^*} Q_{pmjk,k^*j^*m^*p^*}$$

and, similarly, Q_{pk,k^*p^*} equals

$$\sum_{j=1}^{2n} (-1)^{m+j^*} \operatorname{sgn}(mj)\operatorname{sgn}(pm)\operatorname{sgn}(mk)\operatorname{sgn}(k^*j^*)\operatorname{sgn}(j^*p^*)\, a_{m,j^*} Q_{pmjk,k^*j^*m^*p^*}$$

so that, irrespective of whether m or p is chosen, we get

$$Q = \textstyle\sum_{j,k=1}^{2n} (-1)^{m+j^*+p+k^*} \operatorname{sgn}(mj)\operatorname{sgn}(pm)\operatorname{sgn}(pj)\operatorname{sgn}(k^*j^*)\operatorname{sgn}(k^*m^*) \cdot \\ \operatorname{sgn}(pk)\, a_{m,j^*}\, a_{p,k^*}\, Q_{pmjk,k^*j^*m^*p^*}.$$

Since

$$(-1)^{m+j^*} \operatorname{sgn}(mj) a_{m,j^*} Q_{mj,j^*m^*}$$

is symmetric in m and j (for we have $(-1)^m = -(-1)^{m^*}$, $(-1)^{j^*} = -(-1)^j$, $\mathrm{sgn}(mj) = -\mathrm{sgn}(jm)$, $a_{m,j^*} = -a_{j,m^*}$, and, obviously, $Q_{mj,j^*m^*} = Q_{jm,m^*j^*}$), the summation in equation (3.1.1) can be taken over m:

$$Q = \sum_{m=1}^{2n} (-1)^{m+j^*} \mathrm{sgn}(mj)\, a_{m,j^*}\, Q_{mj,j^*m^*} \tag{3.1.2}$$

Corollary 3.1.1 *The number of terms in the Pfaffian of a generic* $2n \times 2n$ *matrix skew-symmetric along the anti-diagonal is* $(2n-1)\cdot(2n-3)\cdot\cdots\cdot 3\cdot 1$. By convention we take this number to be 1 when $n = 0$ (in analogy with the convention $0! = 1$). □

3.2 Pfaffians and determinants

Proposition 3.2.1 *For integers* a, j, k *such that* $1 \le a, j, k \le 2n$ *and* $a \ne j$, $a \ne k$,

$$D_{aj,k^*a^*} = (-1)^{n-1} Q_{aj,j^*a^*} Q_{ak,k^*a^*}.$$

Proof: Proceed by induction. Writing the Laplace expansion for D_{aj,k^*a^*} along row k of A_{aj,k^*j^*}, we get

$$D_{aj,k^*a^*} = \sum_{i=1}^{2n} (-1)^{k+i^*} \mathrm{sgn}(ak)\mathrm{sgn}(jk)\mathrm{sgn}(i^*k^*)\mathrm{sgn}(i^*a^*)\, a_{k,i^*} D_{ajk,i^*k^*a^*}.$$

Writing the Laplace expansion for $D_{ajk,i^*k^*a^*}$ along column j^* of $A_{ajk,i^*k^*a^*}$, we get

$$D_{ajk,i^*k^*a^*} = \textstyle\sum_{\ell=1}^{2n} (-1)^{\ell+j^*} \mathrm{sgn}(a\ell)\mathrm{sgn}(j\ell)\mathrm{sgn}(k\ell)\mathrm{sgn}(i^*j^*)\mathrm{sgn}(k^*j^*)\cdot \\ \mathrm{sgn}(j^*a^*)\, a_{\ell,j^*} D_{ajk\ell,i^*k^*j^*a^*}.$$

By the induction hypothesis,

$$D_{ajk\ell,i^*k^*j^*a^*} = (-1)^{n-2} Q_{ajk\ell,\ell^*k^*j^*a^*} Q_{ajki,i^*k^*j^*a^*}$$

Substituting this into the expression for $D_{ajk,i^*k^*a^*}$ and the result in turn into the expression for D_{aj,k^*a^*}, and rearranging terms—we have replaced $\mathrm{sgn}(i^*k^*)$ by $\mathrm{sgn}(ki)$ and $(-1)^{n-2}\mathrm{sgn}(jl)$ by $(-1)^{n-1}\mathrm{sgn}(lj)$—we get

$$D_{aj,k^*a^*} = (-1)^{n-1}\cdot \\ \left(\textstyle\sum_{i=1}^{2n}((-1)^{k+i^*}\mathrm{sgn}(ak)\mathrm{sgn}(jk)\mathrm{sgn}(i^*j^*)\mathrm{sgn}(i^*a^*))\, \mathrm{sgn}(ki) a_{k,i^*} Q_{ajki,i^*k^*j^*a^*}\right) \\ \left(\textstyle\sum_{\ell=1}^{2n}((-1)^{\ell+j^*}\mathrm{sgn}(a\ell)\mathrm{sgn}(k\ell)\mathrm{sgn}(k^*j^*)\mathrm{sgn}(j^*a^*))\mathrm{sgn}(\ell j) a_{l,j^*} Q_{ajk\ell,\ell^*k^*j^*a^*}\right)$$

By equations (3.1.1) and (3.1.2), the factors in the second and third lines of the above display are respectively Q_{aj,j^*a^*} and Q_{ak,k^*a^*}, so we are done. □

Corollary 3.2.2 $D = (-1)^n Q^2$.

PROOF: Put $j = k$ in the proposition. □

4 The proof

We are now ready to prove our result (Theorem 1.8.1). Lemma 4.2.1 is the technical result that enables the proof. Its proof uses the results of §2, 3.

Notation is fixed as in §1.8.

4.1 Setting it up

Our goal is to prove:

> *Every monomial in $\mathfrak{O}\mathfrak{R}$ that is not $\mathfrak{O}$-dominated by w occurs as an initial term with respect to the term order $\triangleright$ of an element of the ideal I of the tangent cone.*

As explained in §1.8, putting this assertion together with the main result of [9] yields Theorem 1.8.1.

Let I' be the ideal generated by f_τ, $\tau \in I(d)$, $v \leq \tau \not\leq w$. Since $I' \subseteq I$, and since a monomial in $\mathfrak{O}\mathfrak{R}$ that is not $\mathfrak{O}$-dominated by w contains, by the definition of $\mathfrak{O}$-domination (§1.7), a v-chain in $\mathfrak{O}\mathfrak{R}$ that is not $\mathfrak{O}$-dominated by w, it suffices to prove the following (after which it will follow that $I' = I$):

> *Every v-chain that is not $\mathfrak{O}$-dominated by w occurs as the initial term of an element of I'.*

Putting $j = 1$ in Lemma 4.2.1 below yields this, so it suffices to prove that lemma.

4.2 The main lemma

Fix a v-chain $A : \alpha_1 > \ldots > \alpha_m$ that is not $\mathfrak{O}$-dominated by w. Let j be an integer, $1 \leq j \leq m$. Define A_j to be the sub-v-chain $\alpha_1 > \ldots > \alpha_j$. Set

$$\Gamma_j := \begin{cases} \operatorname{Proj}^{\mathrm{e}} A_j & \text{if } \#\operatorname{Proj} A_j \text{ is odd} \\ \operatorname{Proj}^{\mathrm{e}} A_j \setminus \{p_v(\alpha_j), p_h(\alpha_j)\} & \text{if } \#\operatorname{Proj} A_j \text{ is even} \end{cases}$$

See §2.1 for the definition of Proj and $\operatorname{Proj}^{\mathrm{e}}$. Observe that

> (‡) if $\#\operatorname{Proj} A_{j-1}$ is even (equivalently $\operatorname{Proj} A_{j-1} = \operatorname{Proj}^{\mathrm{e}} A_{j-1}$), then $\Gamma_j = \operatorname{Proj}^{\mathrm{e}} A_{j-1}$, no matter whether $\#\operatorname{Proj} A_j$ is even or odd.

Γ_j being a subset of even cardinality, say $2q_j$, of the diagonal elements of $\mathfrak{O}\mathfrak{R}$, it defines an element of $I(d)$. The corresponding Pfaffian we denote by f_j. The degree of f_j is q_j and the number of terms in f_j is, by Corollary 3.1.1, $n_j := (2q_j - 1) \cdot (2q_j - 3) \cdot \cdots \cdot 3 \cdot 1$. By convention, $n_j = 1$ when $q_j = 0$.

Lemma 4.2.1 *Let $A : \alpha_1 > \ldots > \alpha_m$ be a v-chain not $\mathfrak{O}$-dominated by w. For every integer j, $1 \leq j \leq m$, there exists a homogeneous element F_j of the ideal I' such that*

1. *For a monomial occurring with non-zero coefficient in F_j, consider the set (counted with multiplicities) of the projections on the diagonal of the elements of $\mathfrak{O}\mathfrak{R}$ that occur in the monomial. This set is the same for every such monomial.*
2. *The sum of the initial n_j terms (with respect to the term order $\rhd$) of F_j is $f_j X_{\alpha_j} \cdots X_{\alpha_m}$.*

Consider any fixed monomial (occurring with non-zero coefficient) in F_j other than one in $f_j X_{\alpha_j} \cdots X_{\alpha_m}$. From (1) and (2) it follows that, given an integer b, $j \leq b \leq m$, there exists precisely one X_{δ_b} occurring in the monomial with the row index of δ_b being that of α_b.

3 *There exists b for which $\delta_b \neq \alpha_b$ and, for the largest b of this kind, either $\delta_b \notin \mathfrak{O}\mathfrak{N}$ or the column index of δ_b is less than that of α_b.*

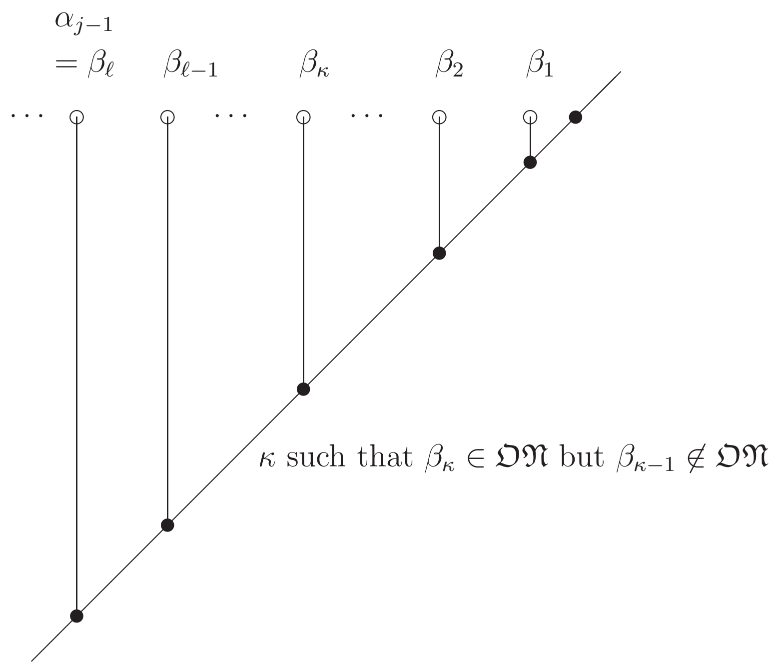
Proof: Proceed by an induction on m and then another (in reverse) on j. Let us suppose that we know the result for j and prove it for $j - 1$. The proof below covers also the base cases for the induction. Consider $\operatorname{Proj} A_{j-1}$.

Suppose first that its cardinality $\#\mathrm{Proj}\,A_{j-1}$ is odd. Write A as $C > D$ with $C = A_{j-1}$ and D being $\alpha_j > \ldots > \alpha_m$. Observe that the last intertwined component of C has at least two elements. Let $\widetilde{A}$ be the new form of A constructed as in §2.2. Since $\widetilde{A}$ has fewer elements than A (Proposition 2.2.1) and is not $\mathfrak{O}$-dominated by w (Proposition 2.3.2), the induction hypothesis applies to $\widetilde{A}$. Apply it with $k = \#\widetilde{C} + 1$ in place of j in the statement of the lemma. If F is the element in I' as in its conclusion, set $F_{j-1} = X_{\alpha_{j-1}} F$.

We claim that F_{j-1} has the desired properties. That it satisfies (1) is clear. We now observe that it satisfies (2). Since $\mathrm{Proj}\,\widetilde{A}_{k-1} = \mathrm{Proj}\,\widetilde{C}$ has evenly many elements (Proposition 2.2.1), it follows (observation (‡) above) that Γ_k (calculated for $\widetilde{A} : \widetilde{C} > D$) equals $\mathrm{Proj}^{\mathrm{e}}\,\widetilde{C} = \mathrm{Proj}\,\widetilde{C}$. On the other hand, $\Gamma_{j-1} = \mathrm{Proj}^{\mathrm{e}}\,C = \mathrm{Proj}\,\widetilde{C}$ (since $\mathrm{Proj}\,A_{j-1}$ is odd, by Proposition 2.2.1). So F_{j-1} satisfies (2). That F_j satisfies (3) is readily verified.

Now suppose that $\#\mathrm{Proj}\,A_{j-1}$ is even. Apply the induction hypothesis with j and let F_j be as in its conclusion. The base case $j - 1 = m$ needs to be treated separately here, as follows. Let y_A be the element of $I(d)$ defined as in §2.4. We take F_j to be the Pfaffian f_{y_A} attached to y_A (see §1.5). That F_j belongs to I' follows from Propositions 2.4.1 and 2.4.2. The rest of the proof is the same for the induction step as well as the base case.

From the observation (‡) above, it follows that $\Gamma_j = \mathrm{Proj}\,A_{j-1}$. Here is a picture of Γ_j (the solid circles denote elements of Γ_j):



Applying to f_j the Laplace-like expansion formula (3.1.1) for Pfaffians, we see that the sum of its initial n_{j-1} terms, the next n_{j-1} terms, … are (up to sign factors) $g_\kappa X_{\beta_\kappa}$, $g_{\kappa+1} X_{\beta_{\kappa+1}}$, …, $g_{\ell-1} X_{\beta_{\ell-1}}$, $g_\ell X_{\alpha_{j-1}}$, …, where g_i is the Pfaffian associated to $\Gamma_j \setminus \{p_v(\beta_i), p_h(\beta_i)\}$, so that the corresponding initial terms of F_j are $g_\kappa X_{\beta_\kappa} X_{\alpha_j} \cdots X_{\alpha_m}$, $g_{\kappa+1} X_{\beta_{\kappa+1}} X_{\alpha_j} \cdots X_{\alpha_m}$, …, $g_{\ell-1} X_{\beta_{\ell-1}} X_{\alpha_j} \cdots X_{\alpha_m}$, $g_\ell X_{\alpha_{j-1}} X_{\alpha_j} \cdots X_{\alpha_m}$, …. We will now modify F_j (by subtracting from it elements of I') so as to kill the terms $g_\kappa X_{\beta_\kappa} X_{\alpha_j} \cdots X_{\alpha_m}$, …, $g_{\ell-1} X_{\beta_{\ell-1}} X_{\alpha_j} \cdots X_{\alpha_m}$. But of course this needs to be done carefully in order that the resulting element of I' has the desired properties.

Write A as $C > D$ where $C = A_{j-1}$ and D is $\alpha_j > \ldots > \alpha_m$. We may assume that the last intertwined component of C consists of at least two elements, for otherwise F_j itself without further modification has the desired properties (we can take F_{j-1} to be F_j). We may further assume that there is some element of $\operatorname{Proj} A_{j-1}$ that is strictly in between the vertical and horizontal projections of α_{j-1}, for otherwise again we can take F_{j-1} to be F_j. Consider the new forms of A as in §2.2. In their construction there is the choice involved of a diagonal element strictly in between the vertical and horizontal projections of the last element of C. We can choose this element to be the vertical projection of β_i where $\kappa \leq i \leq \ell-1$. Corresponding to each choice we get a new form which let us denote $\widetilde{A}(i)$ $(= \widetilde{C}(i) > D)$. Since $\widetilde{A}(i)$ has fewer elements than A (Proposition 2.2.1) and is not $\mathfrak{O}$-dominated by w (Proposition 2.3.2), the induction hypothesis applies to $\widetilde{A}(i)$. Apply it with $k = \#\widetilde{C}(i) + 1$ in place of j in the statement of the lemma. Let $F(i)$ in I' be as in its conclusion. Set $F_{j-1} = F_j - \sum_{i=\kappa}^{\ell-1} F(i) X_{\beta_i}$.

It remains only to verify that F_{j-1} has the desired properties. Since $\operatorname{Proj} \widetilde{A}(i)_{k-1} = \operatorname{Proj} \widetilde{C}(i)$ has evenly many elements (Proposition 2.2.1), it follows (observation (‡) above) that Γ_k (calculated for $\widetilde{A}(i) : \widetilde{C}(i) > D$) equals $\operatorname{Proj}^{\mathrm{e}} \widetilde{C}(i) = \operatorname{Proj} \widetilde{C}(i)$. From the definition of $\widetilde{C}(i)$ and observation (‡), it follows that $\operatorname{Proj} \widetilde{C}(i)$ is $\Gamma_j \setminus \{p_v(\beta_i), p_h(\beta_i)\}$. So the sum of the initial n_{j-1} terms of $F(i)$ is $g_i X_{\alpha_j} \cdots X_{\alpha_m}$. That F_{j-1} has the desired properties can now be readily verified. □

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