Space, Time and Relativity

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1. Introduction

Special Relativity is now a hundred years old, and General Relativity is just ten years younger. Even the general literate public probably knows that these two theories of physics – STR and GTR – profoundly altered previous conceptions and understanding of space and time in physics. We will try to describe these changes beginning with earlier ideas which had served physics for almost 300 years, and seeing how they had to be modified in the light of accumulating experience.

2. Newtonian Space and Time, Inertial Frames

In his great work titled the 'Principia' published in 1685, Newton began by expressing in clear terms the natures of space and time as he understood them. The importance of his enunciation of their properties cannot be overestimated, because they provided something tangible which could be used as a basis for further work, and equally importantly which could be examined and criticised in a fruitful manner. Quoting from Einstein:

"... what we have gained up till now would have been impossible without Newton's clear system".

Newton was carrying forward what Galileo had begun. Let us start by reminding you of the implicit and explicit assumptions underlying Galilean–Newtonian physics, allowing for some overlaps in the interests of clarity.

1) In principle, there are infinitely many observers, in various states of relative motion, each with a global space-time reference frame. Each of them can assign space time coordinates to every event.

2) For all of them, space is the same: absolute, infinite and Euclidean; time is the same: absolute (and independent of space), linear, flowing uniformly.

In these statements, as Einstein says, 'absolute' means the following:

"... physically real ... independent in its physical properties, having a physical effect, but not itself influenced by physical conditions".

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3) Simultaneity of spatially separated events is absolute, the same for all observers.

4) Distance – spatial separation – between simultaneous events is absolute, the same for all observers.

5) Time lapse or interval between any two events is absolute, the same for all observers.

6) There exist special observers – called inertial observers with inertial reference frames – with respect to whom (sufficiently well) isolated bodies are unaccelerated, so Newton's 1st Law holds. In these special frames, there are no inertial forces (see later), and the laws of mechanics take their simplest form.

7) Inertial frames can in principle be identified by 6): isolated bodies move uniformly in them.

We should emphasize the importance of inertial frames. In a short essay on Kepler, Einstein says that while working on the laws of planetary motion in the Copernican framework, Kepler faced the problem of choosing a suitable reference frame in astronomical space with respect to which planetary orbits could be described. The sun could be chosen as the origin, but what about the directions of the axes? Einstein then says:

"This was Kepler's answer: The apparent motions of the

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– *Einstein*

planet Mars are known with great accuracy, including the time of its revolution about the sun (the "Martian year"). It is probable that at the end of each Martian year Mars is at the same spot in (planetary) space. If we limit ourselves for the time being to these points in time, then the planet Mars represents for them a fixed point in planetary space, a point that may be used in triangulation .... This is how Kepler gained the basis for formulating the three fundamental laws with which his name will remain associated for all time to come".

Even in the usual class room derivation of Kepler's laws starting with Newton's Law of Motion – the 2nd Law – and universal gravitation, we have to use some coordinate system which we believe is inertial, and this is always taken to be a frame at rest with respect to the 'fixed stars'. Einstein put it this way:

"We may look upon the principle of inertia as established, to a high degree of approximation, for the space of our planetary system, provided that we neglect the perturbations due to the sun and planets".

From points 6) and 7) above we thus have (we hope!) in principle a method to identify inertial frames. In Galilean–Newtonian physics the way in which bodies acted upon one another was by action at a distance. In fact the only known fundamental force was gravitation described by the instantaneous inverse square law. So on a sufficiently isolated body, we may assume no true forces act, and in any inertial frame it moves uniformly in a straight line.

So let us accept that a frame with the sun as origin and axes which are stationary with respect to the fixed stars is indeed inertial. If K is an inertial frame, and a frame K' moves uniformly in a straight line with respect to K, then K' is also inertial. Essentially this was Galileo's discovery. The relationships between space and time variables in two inertial frames K, K' are given by the equations of Galilean relativity. In the simplest case, for relative velocity v in the x direction, they are:

$$K \to K' \ : \ x' = x - vt, \ y' = y, z' = z, \ t' = t. \qquad (1)$$

These are relatively simple mathematical equations. In the general case they are best written in three-dimensional vector notation:

$$K \to K' \ : \ \mathbf{x}' = R\,\mathbf{x} - \mathbf{v}t + \mathbf{a}, \ t' = t + b. \qquad (2)$$

Here R is a (proper) three-dimensional rotation, $\mathbf{v}$ the relative velocity, $\mathbf{a}$ a shift in the spatial origin, and b a shift in the zero of time. Equations(2) describe a general element of the Galilei group $\mathcal{G}$, and involve ten independent real parameters. So the set of all inertial frames in Galilean–Newtonian physics is an infinite ten-parameter family, connected pairwise by transformations of the form (2). The associated relativity principle says that all inertial frames are equivalent for the description of mechanical phenomena, gravity included; Newton's three Laws of Motion hold in all of them. One consequence of the Galilean transformation equations (1),(2) is the Newtonian law for addition of velocities.

If we imagine a noninertial frame, i.e., one which is accelerated with respect to (any) inertial frame, Newton's 1st Law will not hold in it.

3. Noninertial Frames, Inertial Forces

If now we imagine a noninertial frame, i.e., one which is accelerated with respect to (any) inertial frame, Newton's 1st Law will not hold in it. Even isolated bodies will move with acceleration, and not uniformly in a straight line. We say that there are 'inertial forces' acting even on isolated bodies, causing them to accelerate. On a general body there are both 'true forces' due to other bodies acting at a distance, and inertial forces due to the frame being noninertial. The latter – since they are purely kinematic in origin – cause mass independent accelerations. The most familiar examples, due to uniform rotation with respect to an inertial frame, are centrifugal and Coriolis forces and accelerations.

Inertial forces are kinematic in origin; they cause mass independent accelerations.

Both because of its rotation and its revolution around the sun, a frame rigidly attached to the earth is noninertial. However in comparison, say, to the acceleration due to gravity on the earth's surface, the noninertiality is quite small. The corresponding centrifugal accelerations are:

- due to earth's rotation about its own axis: $v^2/R \sim 3.4 \times 10^{-3} g$ at equator;
- due to earth's orbital revolution: $v^2/R \sim 6 \times 10^{-4} g,\ g \sim 981$ cm/sec^2.

So for many practical purposes – short distance projectile motion, chemistry, atomic physics, ... – a laboratory on the earth is a good approximation to an inertial frame.

Should one similarly 'subtract away' the sun's rotation about the galactic centre to get an 'even better' inertial frame? Here the centrifugal acceleration is:

- Sun's motion about galactic centre: $v^2/R \sim 2.4 \times 10^{-11} g$,

but by this stage of precision it is time to move on to new concepts and revise our ideas.

We said that since inertial forces are kinematic in origin, they cause mass independent accelerations. From Galileo's experiment in Pisa, and later Newton's experiments with pendula, it was known that gravity is similar: it too causes mass independent accelerations. Thus by suitable choice of units, for all bodies the inertial and the gravitational masses are the same:

$$m_{\text{inertial}} \equiv m_{\text{in}} = m_{\text{gravitational}} \equiv m_{\text{gr}}. \tag{3}$$

In Galilean–Newtonian mechanics this was accepted as a fact, so gravity was curiously similar to inertial forces in causing the same accelerations in all bodies. But again in the Galilean–Newtonian framework we have (we hope!) in principle a way to tell them apart – in a non-inertial frame, a (sufficiently well) isolated body experiences (by definition) only inertial forces, not gravity. So by transforming to an inertial frame these inertial forces could be reduced to zero; but gravity, if initially present, would remain nonzero.

4. Electricity, Magnetism, STR

Now let us look briefly at the later historical development, over about two centuries, in the form of a time capsule:

G–N mechanics Gravity 1685	electricity, magnetism action at a distance ~1800	Maxwell's Equations field theory ~1865	STR 1905

(4)

Now to some comments. Evidently Maxwell accepted the Newtonian concepts of space and time, which implied Newton's law for addition of velocities. But his electromagnetic field equations predicted a definite speed, $c = 3 \times 10^{10}$cm/sec, for waves which he identified with light. Therefore his equations, unlike mechanics, could not be invariant under Galilean transformations; they could not be valid in all the inertial frames of mechanics. They could only hold in a subset of inertial frames, at rest relative to one another, and which could then define absolute rest. In any inertial frame outside this set, the speed of light would differ from c and even be direction dependent. Pictorially we have the situation as shown in *Figure* 1.

Maxwell thus thought that electromagnetic experiments could disclose departures from absolute rest defined by the aether. Thus basically, mechanics would obey the Galilean principle of relativity while electromagnetism would not, leading to a clash between them.

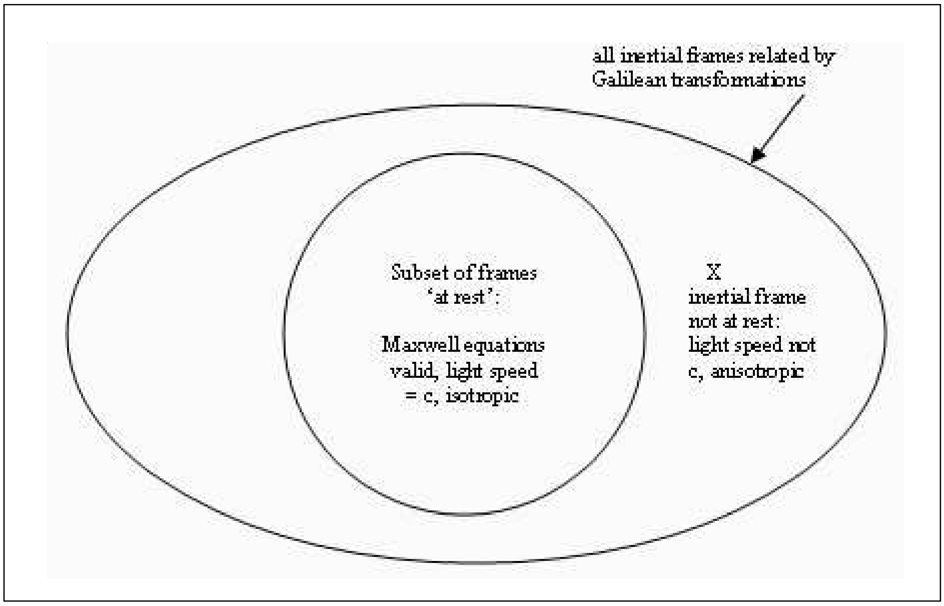

Figure 1.
This figure is not 'drawn to scale': The outer oval encloses a ten-fold infinity of inertial frames. The inner circle encloses a seven-fold infinity of inertial frames at rest with respect to one another.

As is well known, the Michelson–Morley experiment gave a null result, not the one predicted by theory. All efforts to measure the speed of the earth with respect to the aether failed. This impasse was resolved by Einstein's STR of 1905, leading to a new view of space and time, a new understanding of the harmony between mechanics and electromagnetism. The Maxwell equations were invariant under the Lorentz transformations, not the earlier Galilean transformations, so Einstein went on to modify mechanics to make it also Lorentz invariant. The two branches of physics were then ruled by a common new principle of relativity, leading to a deep unification of ideas.

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In place of the elementary Galilean equations (1) we now have the Lorentz transformation equations.

$$K \to K' \; : \; x' = \frac{(x - vt)}{\sqrt{(1 - v^2/c^2)}}, \; t' = \frac{(t - vx/c^2)}{\sqrt{(1 - v^2/c^2)}},$$

$$y' = y, \; z' = z. \tag{5}$$

This can be extended to a general ten parameter inhomogeneous Lorentz transformation or Poincaré transformation which replaces equation (2). The important consequences, the physical features of STR, are:

{1} The idea of globally defined inertial reference frames is retained, but these frames are related by Poincaré transformations, not by the elements of $\mathcal{G}$. We have a new ten-parameter relativity group, the Poincaré group $\mathcal{P}$.

{2} In each inertial frame, space is three-dimensional, infinite and Euclidean, while time flows uniformly.

{3} Under a nontrivial change of inertial frame corresponding to Lorentz transformations such as in equation(5), the separation of space-time into space and time gets altered; simultaneity, lengths and time intervals are also altered. But the concepts of absolute past and future are preserved.

{4} The equations of mechanics have to be modified to be invariant under Poincaré transformations, to be in harmony with Maxwell's field equations of electromagnetism.

5. From STR to GTR – Principle of Equivalence

We can say that the story so far was the first stage in the process of revising the Galilean–Newtonian ideas of space and time. Now let us look at the second stage, the road from STR to GTR, of course going only a short distance along this road! It is clear that STR is the correct language to handle electromagnetism, and all of mechanics except Newtonian gravity. In terms of the entries in equation (4), STR could handle all that stands to its left except gravity.

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In 1907 while Einstein was preparing a review article on STR, two beautiful ideas occurred to him:

"Is it conceivable that the principle of relativity also applies to systems that are accelerated relative to each other?" – *Einstein*

(a) there seems to be no compelling reason why the physical equivalence of reference frames should be limited to those having only uniform relative motions. Indeed in his review he asks:

" Is it conceivable that the principle of relativity also applies to systems that are accelerated relative to each other?"

(b) Galileo's discovery of the equality of inertial and gravitational masses, equation (3), was not a mere curiosity but had deep physical significance – gravitational accelerations and inertial accelerations (felt in any noninertial frame), both being independent of mass, were of the same nature. Thus while the latter could be made to vanish by going to any inertial frame, the former (in a limited space time region of uniform field) could also be made to vanish by going to a suitable uniformly accelerated noninertial frame. He called this 'the happiest thought of my life': that a person falling freely in the earth's gravitational field would feel no weight at all! And then he expanded it to cover all physical phenomena in his celebrated Principle of Equivalence:

".. we shall therefore assume the *complete physical equivalence* of a gravitational field and a corresponding acceleration of the reference system. This assumption extends the principle of relativity to the uniformly accelerated translational motion of the reference system". – *Einstein*

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Let us explore this idea a little bit, first against the background of Newtonian space time, and then see what further changes are brought about by STR.

The truth of the statement that in free fall we are weightless can be seen in the following way. When we stand on the earth's surface we do feel our weight, the force exerted on our bodies by the earth's gravitational field. But we certainly do not feel the gravitational pull of the sun, about $6 \times 10^{-4} g$ as we saw earlier, because (along with the earth) we are in 'free fall' as far as the sun is concerned. Here 'free fall' includes not just hurtling vertically downward but also motion in any other orbit in a given gravitational field, as happens with astronauts in a spaceship orbiting the earth. In Einstein's own words:

"For experiments upon the earth tell us nothing of the fact that we are moving about the sun with a velocity of approximately 30 km a second."

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This insight into the meaning of equation (3) can thus be expressed as a mechanical Principle of Equivalence:

Mechanical phenomena in an inertial frame K in which there is no gravity, when viewed from a uniformly accelerated (noninertial) frame K', would appear the same as if K' were inertial but a uniform gravitational field was present.

Here is yet another expression of the same idea:

If in an inertial frame K there is a uniform gravitational field, then by going over to a noninertial frame K' which has a compensating uniform acceleration with respect to K, we can reduce gravity in K' to zero.

We should appreciate that in these statements we are transcending – going beyond – Galilean relativity since noninertial frames are involved. Under Galilean transformations a nonzero gravitational field remains nonzero and cannot be eliminated. We should next appreciate that in his Principle of Equivalence Einstein went beyond the above limited statements and replaced 'mechanical phenomena' by 'all physical phenomena'. At that time this essentially meant including electromagnetic phenomena. Let us stress again that it is the nature of gravity that forces us to go beyond Galilean relativity.

The content of Einstein's Principle of Equivalence can be conveyed in a simplified yet effective manner as follows.

		I: Gravity affects matter not light	II: Gravity affects matter and light
1	Inertial frame, no gravity	Unaccelerated Straight [a]	Unaccelerated Straight [e]
2	Uniformly accelerated frame, no gravity	Accelerated Bent [b]	Accelerated Bent [f]
3	Inertial frame, uniform gravity	Accelerated Straight [c]	Accelerated Bent [g]
4	Freely falling frame, uniform gravity	Unaccelerated Bent [d]	Unaccelerated Straight [h]

We consider two options – I : light is unaffected by gravity; II : light is affected by gravity. Then we consider the behaviours of matter and of light in four different physical situations and reference frames: inertial frame without gravity; uniformly accelerated frame without gravity; inertial frame with uniform gravity; and freely falling frame in uniform gravitational field. We can then see the table given above emerging:

In each box, [a] to [h], we have given first the nature of motion of matter, then the properties of light rays. The second row arises from the first by purely kinematic reasoning as there is no gravity. That is why [a] = [e] and [b] = [f]. When we bring in gravity, since light behaves differently in Option I and in Option II, we see why [c] $\neq$ [g] and [d] $\neq$[h]. The fourth row arises from the third by 'cancellation of gravity by suitable acceleration'. Now we can see that in Option I, boxes [a], [b], [c], [d] are all distinct, so the behaviours of matter plus light can distinguish among the frames 1, 2, 3 and 4. But with Option II, which is Einstein's Principle of Equivalence, acceleration can mimic and even cancel gravity! So [f] = [g]; and [e] = [h] because of the cancellation in the frame 4. Matter and light both experience gravity, only two combinations of behaviours are realized; we cannot distinguish frame 1 from 4, or 2 from 3!

6. Gravity and STR

After having thus glimpsed Einstein's reinterpretation of gravitation in the Newtonian space-time framework – giving it an almost kinematical meaning – let us see what happens when STR and its lessons on the natures of space and time are brought in. In STR, simultaneity of events and instantaneous spatial separations of bodies are both nonabsolute notions, they both change from frame to frame. So also the isolation of bodies and action at a distance lose meaning. The spirit of STR is that all interactions must be local and be transmitted with finite speed. So the Newtonian way of trying to distinguish between gravitational and inertial accelerations is lost. In fact Einstein criticized the idea of inertial frames already in Newton's scheme in this way:

"The weakness of the principle of inertia lies in this, that it involves an argument in a circle: a mass moves without acceleration if it is sufficiently far from other bodies; we know that it is sufficiently far from other bodies only by the fact that it moves without acceleration."

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So this leaves us no choice but to accept the following: when we go from Newtonian mechanics to STR, and then bring in gravity, there is no way to separate the total mass independent accelerations of bodies into unique gravitational and inertial parts, to disentangle them. So we cannot reach or identify global inertial frames. With inertial frames lost, it becomes meaningless to say that a particular frame is noninertial! We cannot say whether mass independent accelerations are due to 'real' gravity or to inertial forces or how much of each, we cannot draw a distinction between them. So, we abolish the distinction, and give up the idea of global inertial frames. In greater detail we can say the following.

In Newtonian mechanics, inertial frames were globally well defined, and were definitely useful. So in any gen-

In general, different local inertial frames do not all combine or mesh together to form a single globally well-defined inertial frame.

eralization, we expect that inertial frames should find a place with possible modifications or limits of range of validity. When we try to bring STR and gravitation together, the spirit of the former tells us: Reference frames are only locally definable. Leaving aside all other forces, as well as interactions between test bodies, all bodies experience the same accelerations at the same space-time locations. Travelling alongside any such 'freely falling' body, we can create a local reference frame in the sense of STR in some neighbourhood, defined upto a Lorentz transformation. In such a frame, all other nearby bodies are also unaccelerated, so it is locally an inertial frame. In a limited space-time region viewed from such a frame – there is no gravity, it has been 'reduced to zero'. More precisely, locally over a region of uniform field:

- Nongravitational phenomena in freely falling frame $\equiv$ phenomena in inertial frame in absence of gravity.

In a general local but not freely falling frame, all bodies experience common accelerations. We can call it gravitational or inertial, we agree as a convention to call it gravity.

You can sense that in trying to reconcile STR with Newtonian gravity, Einstein found that he had to transcend both of them! Inertial frames remained only as a local concept; in general, different local inertial frames do not all combine or mesh together to form a single globally well-defined inertial frame.

7. Non-Euclidean Geometry

We have come quite far from the initial Newtonian ideas of absolute space, absolute time and inertial frames. But there was still a long way to go before Einstein could complete his GTR. Already in 1907 and 1911, arguing on the basis of his Principle of Equivalence of uniform gravity and uniform acceleration, he had shown that light bends in a gravitational field; its frequency is red shifted in passing from regions of lower to regions of higher gravitational potential; clocks run slow and even light slows down in regions of lower gravitational potential. However the calculation of the deflection of light was later found to be incorrect by a factor of 2.

Light bends in a gravitational field; its frequency is red shifted in passing from regions of lower to regions of higher gravitational potential; clocks run slow and even light slows down in regions of lower gravitational potential.

Then in 1912 he switched from uniform acceleration of the reference frame to uniform rotation, and found (using STR) that in such a frame the laws of geometry are not Euclidean! This was something new, and by a leap of the imagination he was led to this idea – the description of the geometry of space-time is a description of gravitation. To pursue this idea and to create a new comprehensive theory of gravitation generalizing Newton's theory involved a heroic struggle which ended only in November 1915. Along the way he learnt – with much struggle – that in the process of reconciling STR and gravitation, space time coordinates lose their immediate physical meanings in terms of measurements with standard rods and clocks. They become no more than markers like telephone numbers, only helping us distinguish distinct events from one another. Just as there are in general no globally defined privileged inertial reference frames, so also in general there are no privileged coordinate systems either. This meant that the new theory of gravitation should treat all possible choices of coordinate systems on an equal footing, favouring none. By another leap of the imagination, this invariance of the laws of gravitation under arbitrary coordinate changes became for him an expression of the Principle of Equivalence.

The action of gravity on light and matter was in effect conceived of as a two-step process. First, all sources of energy and momentum – both matter and light – act as sources of the gravitational field and influence the geometry of space time. Second, matter and light move and

"It is contrary to the mode of thinking in science to conceive of a thing .... which acts itself, but which cannot be acted upon."
– Einstein

propagate in space time following its geometrical rules! To realize all this Einstein had to master and extend the existing mathematics of Riemannian geometry. At the end, space time was not as in Newton's view a passive unchangeable stage on which other physical phenomena were played out, it was an active participant in the proceedings. In Einstein's words:

"It is contrary to the mode of thinking in science to conceive of a thing .... which acts itself, but which cannot be acted upon."

There can be no unmoved mover!

At the end of the story, as we saw with the inertial frame concept, STR also reappears as a locally valid concept in GTR. Its familiar descriptions of space and time and the action of Lorentz transformations are all valid locally in freely falling frames in limited space time regions or patches, but they cannot be all put together seamlessly to give a global system. From a physical point of view, all nongravitational phenomena must obey the rules of STR, so this theory acts as a restrictive principle. Once a theory passes the STR test, it can more or less automatically be woven into the fabric of GTR.

8. Concluding Comments

As we mentioned at the beginning, STR and GTR are close to a century old. Many major developments have occurred since then, both on the theoretical side and in experiment. Inspired by the beauty of GTR, physicists like Hermann Weyl and Theodore Kaluza and Oskar Klein suggested generalizations of great mathematical beauty. Though their specific proposals turned out to be untenable, their ideas survived and have inspired later work. Einstein himself and Schrödinger too tried to extend GTR in various ways, though without significant success.

We hope the account given here, though incomplete, will inspire our readers and create the confidence to pursue the study of GTR and of the various later developments upto recent times.

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Suggested Reading

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Philosophy is written in this grand book – I mean the universe – which stands continually open to our gaze, but it cannot be understood unless one first learns to comprehend the language and interpret the characters in which it is written. It is written in the language of mathematics, and its characters are triangles, circles, and other geometrical figures, without which it is humanly impossible to understand a single word of it.

Galileo