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THE $2\frac{1}{2}$-POST-NEWTONIAN EQUATIONS OF HYDRODYNAMICS AND RADIATION REACTION IN GENERAL RELATIVITY

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ABSTRACT

In this paper the equations of hydrodynamics in the $2\frac{1}{2}$-post-Newtonian approximation to general relativity are derived. In this approximation all terms of $O(c^{-5})$ are retained consistently with Einstein's field equations; it is also the approximation in which terms representing the reaction of the fluid to the emission of gravitational radiation by the system first make their appearance.

The paper is in four parts. In Part I (by S. C.) the lowest-order terms in the metric coefficients are derived which are consequences of the imposition of the Sommerfeld radiation-condition at infinity. It is shown (following an early investigation of Trautman) that these terms are of $O(c^{-5})$ in g_{00}, of $O(c^{-6})$ in $g_{0\alpha}$, and of $O(c^{-5})$ in $g_{\alpha\beta}$. Unique expressions are obtained for these terms. They are found to be purely of Newtonian origin.

In Part II (by S. C. and F. P. E.) the equations of motion governing the fluid in the $2\frac{1}{2}$-post-Newtonian approximation are derived. In addition to the coefficients already determined, these equations depend on a knowledge of the term of $O(c^{-7})$ in g_{00}. This term is determined by an explicit appeal to the field equation. It is further shown that this approximation brings no change to the density $(c^2\rho u^0\sqrt{-g})$ and the linear momentum (π_α) that are conserved in the second post-Newtonian approximation.

In Part III (by S. C.) it is shown that the terms of $O(c^{-5})$ in the equations of motion contribute principally to the dissipation of the energy and the angular momentum that are conserved in the second post-Newtonian approximation. The rates of dissipation of energy and of angular momentum that are predicted are in exact agreement with the expectations of the linearized theory of gravitational radiation.

Finally, in Part IV (by S. C. and F. P. E.) the energy, $\Theta^{00} - c^2\rho u^0\sqrt{-g}$, to be associated with the $2\frac{1}{2}$-post-Newtonian approximation is derived by evaluating the (0, 0)-component of the Landau-Lifshitz complex and the conserved density in the $3\frac{1}{2}$-post-Newtonian approximation.

PART I

THE LOWEST-ORDER TERMS IN THE METRIC COEFFICIENTS THAT DERIVE FROM THE OUTGOING-RADIATION CONDITION AT INFINITY

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I. INTRODUCTION

In two earlier papers (Chandrasekhar 1965 and Chandrasekhar and Nutku 1969; these two papers will be referred to hereafter as Papers I and II, respectively) the equations of hydrodynamics governing a perfect fluid were derived in the first and the second post-Newtonian approximations to the equations of general relativity. In these two approximations the equations of motion include all terms of orders c^{-2} and c^{-4}, respectively, that arise in a systematic solution of Einstein's field equations in an expansion[1] in inverse powers of c. The orders to which the different metric coefficients have to be known in order to obtain the equations of motion, as well as the corresponding conserved quantities, are set out in Table 1 in § II below (see also Table 1 in Paper II).

It is known that no term representing the reaction of the fluid to the emission of

[1] Precisely, the expansion is in powers of v/c, $\sqrt{U}/c$, $\sqrt{(p/\rho)}/c$, and $\sqrt{\Pi}/c$, all of which are considered to be of the first order of smallness and comparable.

gravitational radiant energy appears in the first or the second post-Newtonian approximation; also, that on the standard linearized theory of gravitational radiation, the reaction of the fluid to the emission of gravitational radiation by the system must first manifest itself in the next "$\frac{1}{2}$-approximation" by the explicit appearance in the equations of motion of certain "damping terms" of $O(c^{-5})$. The question, how the succession of terms of higher even powers of $1/c$, that follow one another on the standard post-Newtonian schemes, gets broken at precisely the order c^{-5}, has concerned many authors including Infeld (1938; see also Infeld and Plebanski 1960), Hu (1947), Trautman (1958*a*, *b*), and Peres (1960), all in the framework of the original theory of Einstein, Infeld, and Hoffmann of n mass points. But the results of these authors were inconsistent with one another, inconclusive, or incomplete.

More recently, in the framework of the hydrodynamics of a perfect fluid, Thorne (1969*b*) has, in the context of his exact treatment of the nonradial oscillations of highly relativistic neutron stars (Thorne 1969*a*), shown (developing certain ideas of Burke; see below) that in the limit of weak fields his exact results on the rate of emission of gravitational radiation are in accord with the expectations of the linearized theory. More generally, Thorne has concluded that the *secular effects* of the emission of quadrupole gravitational radiation, in the lowest order, can be formally included by modifying the Euler equation of Newtonian hydrodynamics,

$$\rho \frac{dv_\alpha}{dt} + \frac{\partial p}{\partial x_\alpha} - \rho \frac{\partial U}{\partial x_\alpha} = 0 , \tag{1}$$

in the manner

$$\rho \frac{dv_\alpha}{dt} + \frac{\partial p}{\partial x_\alpha} - \rho \frac{\partial U}{\partial x_\alpha} - \frac{2G}{15c^5} \rho \left(x_\alpha \frac{d^5 I_{\mu\mu}}{dt^5} - 3x_\mu \frac{d^5 I_{\mu\alpha}}{dt^5} \right) = 0 , \tag{2}$$

where

$$I_{\alpha\beta} = \int_V \rho x_\alpha x_\beta dx \tag{3}$$

denotes the moment-of-inertia tensor. However, as Thorne has himself emphasized, the inclusion of the damping terms of $O(c^{-5})$ in the Newtonian equation of motion overlooks terms of the same order and, indeed, lower-order terms which describe nonsecular effects (of greater importance over shorter intervals of time).

A somewhat more general framework for Thorne's ideas (but less explicit in important details) has been described by Burke (1969).

In this paper, we shall develop an alternative approach to this same problem of the reaction of the fluid to the emission of gravitational radiation that is in harmony with the post-Newtonian scheme as set out in Papers I and II and in a related paper (Chandrasekhar 1969*b;* this paper will be referred to hereafter as Paper III) on the conservation laws in general relativity. It should, however, be stated that the present paper derives its basic ideas from Trautman's (1958*a*, *b*) discussion of this same problem in the framework of the original theory of Einstein, Infeld, and Hoffmann. In that discussion Trautman failed to get agreement with the predictions of the linearized theory of gravitation; but this disagreement arose, as we shall see, from a simple oversight. When this is corrected, Trautman's procedure (as applied and extended in this paper in the framework of hydrodynamics) becomes consistent with the predictions of the linearized theory. And to this writer it appears that Trautman's approach to this problem is the simplest and the most direct that has been devised so far.

II. PRELIMINARY CONSIDERATIONS

First it is useful to recall why the post-Newtonian schemes as normally developed for obtaining the equations of motion as a series in inverse powers of c automatically generate an even series.

The starting point of all post-Newtonian schemes is provided by initial values

$$g_{00} = 1 - 2U/c^2 + \ldots , \quad g_{0\alpha} = 0 + \ldots , \quad \text{and} \quad g_{\alpha\beta} = -\delta_{\alpha\beta} + \ldots , \tag{4}$$

where the term $-2U/c^2$ in g_{00} is demanded by the principle of equivalence. The first iteration of Einstein's field equations with the initial values (4) leads to an improvement in the metric coefficients by determining further terms of $O(c^{-4})$ in g_{00}, of $O(c^{-3})$ in $g_{0\alpha}$, and of $O(c^{-2})$ in $g_{\alpha\beta}$—it is these "improvements" that lead to the equations of motion in the first post-Newtonian approximation. Had we supposed, in developing the expansion for the metric coefficients, that terms of $O(c^{-3})$ in g_{00}, of $O(c^{-2})$ in $g_{0\alpha}$, and of $O(c^{-1})$ in $g_{\alpha\beta}$ occur, then we should have found that these terms satisfy *homogeneous* equations (unlike the terms of one higher order which satisfy *inhomogeneous* equations); they can, consequently, be set equal to zero by a suitable choice of gauge.[2] The same phenomenon will be repeated when we proceed to the second post-Newtonian approximation: the terms of $O(c^{-5})$ in g_{00}, of $O(c^{-4})$ in $g_{0\alpha}$, and of $O(c^{-3})$ in $g_{\alpha\beta}$ will again satisfy homogeneous equations; and again, they can be set equal to zero by a suitable choice of gauge. By induction it follows that we shall continue to skip the "odd" steps indefinitely if we continue the scheme of iterations without any modifications. The question arises: How are we to break this chain by providing a nonzero source that will provide "starting values" for a first nontrivial odd step (even as the principle of equivalence originally provided the "source" $-2U/c^2$ in g_{00} for starting the even series)?

Clearly, the reason why the standard post-Newtonian scheme fails to provide a source for a nontrivial odd step is that nowhere in the scheme do we impose on the solutions the Sommerfeld radiation-condition, namely, that at "infinity" there is only outgoing radiation. This condition cannot, however, be applied in any straightforward manner to the solutions obtained in a "slow-motion" approximation as the post-Newtonian approximations are, for these approximations, based as they are on the assumption that $v/c < 1$, require that the operation of $\partial/\partial x_0$ $(=\partial/c\partial t)$ on any quantity lowers its order by one. This last fact implies that the solutions obtained on the basic assumptions of the post-Newtonian scheme can be valid only in the *near zone* where $r < ct$. What is required, then, is a "matching" (in the sense of Thorne and Burke) of the solutions appropriate to the near zone with those appropriate to the *far zone* (where the Sommerfeld condition is to be imposed). A way in which this matching can be "painlessly" accomplished is the principal burden of Trautman's 1958 papers.

As we have already stated, an oversight in Trautman's paper will have to be corrected. But his (uncorrected) considerations are still useful (and, in the author's opinion, necessary) for answering the two main questions: (1) Why is there no nontrivial 1½-post-Newtonian approximation?; and (2) What are the lowest orders (in powers of c^{-1}) in which the outgoing-radiation condition at infinity modifies the metric coefficients in the near zone?

We start with the equation which provides the basis for the standard linearized theory of gravitational radiation. Letting

$$\gamma^{ik} = \eta^{ik} - \mathfrak{g}^{ik} = \eta^{ik} - g^{ik}\sqrt{-g} , \tag{5}$$

where η^{ik} is the diagonal Minkowskian metric, $(+1, -1, -1, -1)$, and imposing on γ^{ik} the de Donder condition

$$\gamma^{ik}{}_{,k} = 0 , \tag{6}$$

we have the equation

$$\Box\gamma^{ik} = \frac{16\pi G}{c^4} T^{ik} \quad \left(\Box = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right), \tag{7}$$

[2] By a choice of gauge we mean here the same as in Paper II (n. 1 on p. 60), namely, the choice of the four arbitrary functions that naturally occur in the solutions of the field equations in each successive approximation.

where T^{ik} is ascribed (in the first instance) its Newtonian value:

$$T^{00} = \rho c^2 \,, \quad T^{0\alpha} = \rho c v_\alpha \,, \quad \text{and} \quad T^{\alpha\beta} = \rho v_\alpha v_\beta + p\delta_{\alpha\beta} \,. \tag{8}$$[3]

In writing equation (7), we do not foreclose the possibility that its use, together with the imposed gauge on γ^{ik} and the choice of T^{ik}, may be inconsistent with the solutions already derived (in a different gauge) in the first and the second post-Newtonian approximations. Indeed, as we shall see in § III below, for consistency T^{ik} must be replaced by the Landau-Lifshitz complex Θ^{ik}. On this account, it should be emphasized that our present objective is only to obtain some preliminary answers to the two questions which we have raised.

The solution of equation (7) which satisfies the outgoing-radiation condition is

$$\gamma^{ik}(x, t) = -\frac{4G}{c^4} \int_V \frac{dx'}{|x - x'|} T^{ik}(x', t - |x - x'|/c) \,. \tag{9}$$

By this choice of the particular solution of equation (7), expressed in terms of the "retarded potentials," we have automatically excluded the possibility of any incoming radiation at infinity. Also, it should be noted that equation (9) represents an *exact* solution of equation (7) which satisfies the Sommerfeld radiation-condition at infinity (albeit eq. [7] is itself not an exact consequence of the field equations).

We now expand $T^{ik}(x', t - |x - x'|/c)$ as a series in inverse powers of c in a manner that is appropriate for the near zone. Thus

$$\begin{aligned} \gamma^{ik}(x, t) = &-\frac{4G}{c^4} \int_V \frac{T^{ik}(x', t)}{|x - x'|} dx' \\ &+ \frac{4G}{c^4} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!c^n} \frac{\partial^n}{\partial t^n} \int_V T^{ik}(x', t) |x - x'|^{n-1} dx' \,. \end{aligned} \tag{10}$$

Considering first the (0, 0)-component of equation (10) and writing out the first few terms explicitly, we have

$$\begin{aligned} \gamma^{00} = &-\frac{4U}{c^2} - \frac{2G}{c^4} \frac{\partial^2}{\partial t^2} \int_V \rho(x', t) |x - x'| dx' \\ &+ \frac{4G}{c^3} \frac{\partial}{\partial t} \int_V \rho(x', t) dx' + \frac{2G}{3c^5} \frac{\partial^3}{\partial t^3} \int_V \rho(x', t) |x - x'|^2 dx' \,. \end{aligned} \tag{11}$$

We recognize that the terms of $O(c^{-2})$ and $O(c^{-4})$ in the solution (11) are the same as those that occur in the solutions for the metric coefficients in the Newtonian and in the first post-Newtonian approximations.[4] We also observe that the law of the conservation of mass, in its Newtonian form

$$\frac{\partial}{\partial t} \int_V \rho(x', t) dx' = 0 \,, \tag{12}$$

[3] The convention regarding the indices is the same as in Papers I and II: Latin indices take the values 0, 1, 2, and 3, and Greek indices take only the values 1, 2, and 3 referring to the spatial coordinates; and the summation convention will be restricted to their respective ranges. Also, x_0 will be replaced by ct when the notation of ordinary *Cartesian tensors* is used; and when the notation of Cartesian tensors is used, the Greek indices will *always* be written as subscripts; and the summation over repeated Greek (Cartesian) indices will also be assumed.

[4] However, in making the "identifications," it must be borne in mind that the solutions in the post-Newtonian approximations are expressed in a different gauge.

already ensures that

$$\underset{3}{\gamma}^{00} = 0\,. \tag{13}$$

(By the numeral 3 below γ^{00} we indicate that we are referring to the term of $O(c^{-3})$ in the series expansion of γ^{00} in the inverse powers of c. This notation for referring to the terms of different orders in the expansion of a quantity will be adopted in the rest of this paper.)

More precisely than equation (12), we know from the results of the post-Newtonian approximations that

$$\frac{\partial}{\partial t}\int_V \rho(x', t)dx' = O(c^{-2}) \tag{14}$$

and is a function of time only. Again from the laws of the conservation of mass and of linear momentum (in their Newtonian forms) it follows that the term of $O(c^{-5})$, already present in the solution (11), namely,

$$\frac{\partial^3}{\partial t^3}\int_V \rho(x', t)|x - x'|^2 dx' = \frac{\partial^3}{\partial t^3}\int_V \rho(x', t)(|x|^2 + |x'|^2 - 2x_\mu x'_\mu)dx' = \frac{d^3 I_{\mu\mu}}{dt^3}, \tag{15}$$

is also a function of time only. From the results (14) and (15) we may conclude that the lowest-order nonvanishing odd term in γ^{00} is of $O(c^{-5})$:

$$\underset{5}{\gamma}^{00} \neq 0 \qquad \text{and is a function of time only}\,. \tag{16}$$

Considering next the $(0, \alpha)$-component of equation (10), we have

$$\begin{aligned}\gamma^{0\alpha} = &-\frac{4G}{c^3}\int_V \frac{\rho(x', t)v_\alpha(x', t)}{|x - x'|}dx' - \frac{2G}{c^5}\frac{\partial^2}{\partial t^2}\int_V \rho(x', t)v_\alpha(x', t)|x - x'|dx' \\ &+ \frac{4G}{c^4}\frac{\partial}{\partial t}\int_V \rho(x', t)v_\alpha(x', t)dx' + \frac{2G}{3c^6}\frac{\partial^3}{\partial t^3}\int_V \rho(x', t)v_\alpha(x', t)|x - x'|^2 dx'\,.\end{aligned} \tag{17}$$

The terms of $O(c^{-3})$ and $O(c^{-5})$ on the right-hand side of equation (17) occur in the solutions for the metric coefficients in the first and in the second post-Newtonian approximations. On the other hand, since we know from the post-Newtonian results that

$$\frac{\partial}{\partial t}\int_V \rho(x', t)v_\alpha(x', t)dx' = O(c^{-2})\,, \tag{18}$$

we may conclude without any ambiguity that

$$\underset{4}{\gamma}^{0\alpha} = 0 \qquad \text{while} \qquad \underset{6}{\gamma}^{0\alpha} \neq 0\,. \tag{19}$$

Turning finally to the (α, β)-component of equation (10), we have

$$\gamma^{\alpha\beta}(x, t) = -\frac{4G}{c^4}\int_V \frac{T^{\alpha\beta}(x', t)}{|x - x'|}dx' + \frac{4G}{c^5}\frac{\partial}{\partial t}\int_V T^{\alpha\beta}(x', t)dx'\,. \tag{20}$$

Therefore, the lowest-order nonvanishing odd term in $\gamma^{\alpha\beta}$ is

$$\underset{5}{\gamma}^{\alpha\beta} = \frac{1}{c^5} f_{\alpha\beta}(t)\,. \tag{21}$$ [5]

[5] We are not writing here the explicit form for $f_{\alpha\beta}(t)$ suggested by the solution (20) since we shall find in § III below that the suggested form is in fact incorrect.

The Newtonian forms of the laws of the conservation of mass and of linear momentum have thus allowed us to infer that

$$\underset{3}{\gamma}^{00} = \underset{2}{\gamma}^{0\alpha} = \underset{3}{\gamma}^{\alpha\beta} = 0$$

and (22)

$$\underset{5}{\gamma}^{00} = \frac{1}{c^5} F(t) , \quad \underset{5}{\gamma}^{\alpha\beta} = \frac{1}{c^5} f_{\alpha\beta}(t) , \quad \text{and} \quad \underset{4}{\gamma}^{0\alpha} = 0 ,$$

though it should be noted that the present considerations do not enable us to specify $F(t)$ and $f_{\alpha\beta}(t)$ in a unique manner.

TABLE 1

INFORMATION ON THE METRIC COEFFICIENTS THAT IS NEEDED IN THE VARIOUS APPROXIMATIONS

Equations of Motion	Orders of the Metric Coefficients Needed		
	$g_{\alpha\beta}$	$g_{0\alpha}$	g_{00}
Newtonian	$-\delta_{\alpha\beta}$	0	$1-2U/c^2$*
½-post-Newtonian	0†	0†	0†
1-post-Newtonian	$-2U\delta_{\alpha\beta}c^{-2}$	$P_\alpha c^{-3}$	$2(U^2-2\Phi)c^{-4}$
1½-post-Newtonian	0†	0†	$Q_{00}{}^{(5)}c^{-5}$‡
2-post-Newtonian	$(\ldots)c^{-4}$	$(\ldots)c^{-5}$	$(\ldots)c^{-6}$
2½-post-Newtonian§	$Q_{\alpha\beta}{}^{(5)}c^{-5}$‡	$Q_{0\alpha}{}^{(6)}c^{-6}$‡	$Q_{00}{}^{(7)}c^{-7}$
3-post-Newtonian	$(\ldots)c^{-6}$‖	. . .	. . .
3½-post-Newtonian	$Q_{\alpha\beta}{}^{(7)}c^{-7}$‖	. . .	. . .

* The term $-2U/c^2$ in g_{00} is demanded by the principle of equivalence.

† These terms vanish by virtue of the laws of the conservation of mass and of linear momentum.

‡ These are the lowest-order terms in the metric coefficients that derive from the imposition of the Sommerfeld radiation-condition at infinity. (All the other terms in the table are obtained by solving the field equations.)

§ This is the approximation in which radiative-reaction terms first appear.

‖ These terms are needed to determine the energy in the one lower approximation.

The foregoing results for γ^{ij} when expressed in terms of the metric coefficients g_{ij} (with the aid of eqs. [41] below) imply

$$\underset{3}{g}_{00} = \underset{2}{g}_{0\alpha} = \underset{1}{g}_{\alpha\beta} = 0 , \tag{23}$$

and

$$\underset{5}{g}_{00} = \frac{1}{c^5} Q_{00}{}^{(5)}(t) \quad \text{and} \quad \underset{4}{g}_{0\alpha} = \underset{3}{g}_{\alpha\beta} = 0 . \tag{24}$$

The vanishing of the terms listed in (23) establishes why there is no nontrivial ½-post-Newtonian approximation. Also, even though we do not as yet know precisely what $Q_{00}{}^{(5)}$ is, the fact that it is a function of time only is sufficient to ensure that by a gauge transformation (involving only the function W in the transformation equations given in Paper II, eq. [39]) among the coefficients (24), we can reduce $Q_{00}{}^{(5)}$ to zero. In other words, there is also no nontrivial 1½-post-Newtonian approximation. These results are in agreement with the known expectation that the terms representing radiation reaction must first manifest themselves in the 2½-post-Newtonian approximation. In Table 1 the conclusions to which we have presently arrived are summarized.

III. THE UNIQUE SPECIFICATION OF $Q_{00}{}^{(5)}$, $Q_{0\alpha}{}^{(6)}$, AND $Q_{\alpha\beta}{}^{(5)}$

The preliminary considerations of § II have indicated that the lowest-order terms in the metric coefficients that result from the imposition of the Sommerfeld radiation-condition at infinity are $O(c^{-5})$ in g_{00}, $O(c^{-6})$ in $g_{0\alpha}$, and $O(c^{-5})$ in $g_{\alpha\beta}$. We shall denote these terms by

$$\underset{5}{g_{00}} = \frac{1}{c^5} Q_{00}{}^{(5)}, \quad \underset{6}{g_{0\alpha}} = \frac{1}{c^6} Q_{0\alpha}{}^{(6)}, \quad \text{and} \quad \underset{5}{g_{\alpha\beta}} = \frac{1}{c^5} Q_{\alpha\beta}{}^{(5)}. \tag{25}$$

Our problem now is to obtain unique expressions for these coefficients. For this purpose, we start with the field equation written in terms of the Landau-Lifshitz complex Θ^{ik}. We have (cf. Paper III, eqs. [31] and [32])

$$(\mathfrak{g}^{ik}\mathfrak{g}^{lm} - \mathfrak{g}^{il}\mathfrak{g}^{km})_{,l,m} = \frac{16\pi G}{c^4}\Theta^{ik}. \tag{26}$$

If in equation (26) we let Θ^{ik} have the value $\Theta^{ik}{}_{(1)}$ determined in the first post-Newtonian approximation and the metric coefficients have the values $\mathfrak{g}^{ik}{}_{(2)}$ appropriate to the *second* post-Newtonian approximation, then the two sides of the equation will be balanced with respect to the order of the terms that are retained on each side; and, moreover, the equation will be satisfied to that order.

Let γ^{ik} denote the lowest-order term in $\mathfrak{g}^{ik}$ that derives from the imposition of the boundary condition at infinity. From our considerations of § II, we expect them to be

$$\underset{5}{\gamma^{00}}, \quad \underset{6}{\gamma^{0\alpha}}, \quad \text{and} \quad \underset{5}{\gamma^{\alpha\beta}}. \tag{27}$$

Now make the substitution (cf. eq. [5])

$$\mathfrak{g}^{ik} = \mathfrak{g}^{ik}{}_{(2)} - \gamma^{ik} \tag{28}$$

on the left-hand side of equation (26) and linearize with respect to γ^{ik}: the terms that will thus be ignored will be of orders higher than any that are retained. We shall then obtain

$$\begin{aligned}(\mathfrak{g}^{ik}{}_{(2)}\mathfrak{g}^{lm}{}_{(2)} - \mathfrak{g}^{il}{}_{(2)}\mathfrak{g}^{km}{}_{(2)} - \mathfrak{g}^{ik}{}_{(2)}\gamma^{lm} - \mathfrak{g}^{lm}{}_{(2)}\gamma^{ik} + \mathfrak{g}^{il}{}_{(2)}\gamma^{km} + \mathfrak{g}^{km}{}_{(2)}\gamma^{il})_{,l,m} \\ = \frac{16\pi G}{c^4}\Theta^{ik}.\end{aligned} \tag{29}$$

So long as the $\gamma^{\cdot\cdot}$'s are of the orders specified in (27)—and we shall presently verify that they are—we may consistently replace the $\mathfrak{g}^{\cdot\cdot}{}_{(2)}$'s, that occur as factors of the $\gamma^{\cdot\cdot}$'s, by the corresponding Minkowskian coefficients. Also, we may impose on the $\gamma^{\cdot\cdot}$'s a gauge independently of the one chosen in the solution for the $\mathfrak{g}^{\cdot\cdot}{}_{(2)}$'s: there will be no conflict since the $\mathfrak{g}^{\cdot\cdot}{}_{(2)}$'s do not include any terms of the orders of the $\gamma^{\cdot\cdot}$'s. And we shall find it convenient to impose on the $\gamma^{\cdot\cdot}$'s the same de Donder condition (6). On these assumptions and restrictions, equation (29) becomes

$$(\mathfrak{g}^{ik}{}_{(2)}\mathfrak{g}^{lm}{}_{(2)} - \mathfrak{g}^{il}{}_{(2)}\mathfrak{g}^{km}{}_{(2)})_{,l,m} + \Box\gamma^{ik} = \frac{16\pi G}{c^4}\Theta^{ik}. \tag{30}$$

The terms in the $\mathfrak{g}^{\cdot\cdot}{}_{(2)}$'s on the left-hand side of equation (30) do not include any of the orders specified in (27); and, moreover, as we have already noted, these terms (derived from the $\mathfrak{g}^{\cdot\cdot}{}_{(2)}$'s), when consistently expanded, will just make up $16\pi G\Theta^{ik}{}_{(1)}/c^4$. Accordingly, $\Theta^{ik}{}_{(1)}$ provides the "source" for both the near-zone $\mathfrak{g}^{\cdot\cdot}{}_{(2)}$ and the far-zone γ^{ik}; and we conclude that γ^{ik} is determined by the equation

$$\Box\gamma^{ik} = \frac{16\pi G}{c^4}\Theta^{ik}{}_{(1)}. \tag{31}$$

This equation differs from equation (7) only by $\Theta^{ik}{}_{(1)}$ having replaced T^{ik}; and this difference, as it will appear, is crucial for the unique specification of γ^{ik}. Notice also that by virtue of the property,

$$\Theta^{ik}{}_{,i} = \Theta^{ik}{}_{,k} = 0, \tag{32}$$

of the Landau-Lifshitz complex, equation (31) is consistent with the de Donder condition imposed on the $\gamma^{\cdot\cdot}$'s (to the order required).

We may now treat equation (31) in the same way as we treated equation (7) in § II. Thus, by considering the (0, 0)-component of equation (31), we shall obtain (cf. eq. [11])

$$\gamma^{00} = \frac{4G}{c^5}\frac{\partial}{\partial t} \int \Theta^{00}{}_{(1)}(x', t)dx' + \frac{2G}{3c^7}\frac{\partial^3}{\partial t^3} \int \Theta^{00}{}_{(1)}(x', t)|x - x'|^2 dx' , \tag{33}$$

where we have not written the terms of even order. (It may be recalled here that the dominant term in Θ^{00} is ρc^2.)

For $\Theta^{00}{}_{(1)}$ determined in the first post-Newtonian approximation

$$\frac{\partial}{\partial t} \int \Theta^{00}{}_{(1)}(x', t)dx' = O(c^{-2}) . \tag{34}$$

The first term on the right-hand side of equation (33) is, therefore, of $O(c^{-7})$ in contrast to the second term which is of $O(c^{-5})$. We thus obtain without any ambiguity that

$$\underset{5}{\gamma^{00}} = \frac{2G}{3c^5}\frac{\partial^3}{\partial t^3} \int_V \rho(x', t)|x - x'|^2 dx' . \tag{35}$$

Considering next the (0, α)-component of equation (31), we similarly obtain (cf. eq. [17], but recall that now the dominant term in $\Theta^{0\alpha}$ is $\rho c v_\alpha$)

$$\gamma^{0\alpha} = \frac{4G}{c^5}\frac{\partial}{\partial t} \int \Theta^{0\alpha}{}_{(1)}(x', t)dx' + \frac{2G}{3c^7}\frac{\partial^3}{\partial t^3} \int \Theta^{0\alpha}{}_{(1)}(x', t)|x - x'|^2 dx' . \tag{36}$$

For $\Theta^{0\alpha}{}_{(1)}$ determined in the first post-Newtonian approximation,

$$\frac{\partial}{\partial t} \int \Theta^{0\alpha}{}_{(1)}(x', t)dx' = O(c^{-3}) . \tag{37}$$

The first term on the right-hand side of equation (36) is, therefore, of $O(c^{-8})$ in contrast to the second which is of $O(c^{-6})$. We thus obtain, again, without any ambiguity that

$$\underset{6}{\gamma^{0\alpha}} = \frac{2G}{3c^6}\frac{\partial^3}{\partial t^3} \int_V \rho(x', t)v_\alpha(x', t)|x - x'|^2 dx' . \tag{38}$$

Finally, considering the (α, β)-component of equation (31), we obtain

$$\underset{5}{\gamma^{\alpha\beta}} = \frac{4G}{c^5}\frac{\partial}{\partial t} \int \Theta^{\alpha\beta}(x', t)dx' , \tag{39}$$

where it will suffice for our present purposes to substitute for $\Theta^{\alpha\beta}$ its *Newtonian* expression (cf. Paper III, eqs. [11] and [12])

$$\Theta^{\alpha\beta} = \Theta_{\alpha\beta} = \rho v_\alpha v_\beta + p\delta_{\alpha\beta} + t_{\alpha\beta} ,$$

where

$$t_{\alpha\beta} = \frac{1}{16\pi G}\left[4\frac{\partial U}{\partial x_\alpha}\frac{\partial U}{\partial x_\beta} - 2\delta_{\alpha\beta}\left(\frac{\partial U}{\partial x_\mu}\right)^2\right] . \tag{40}$$

It is important to observe that the expression (39) for $\gamma^{\alpha\beta}$ differs from that which follows from equation (20) by the fact that $\Theta^{\alpha\beta}$ *now includes the gravitational contribution to the total stress;* and this difference, it will appear, is decisive. The "oversight" in Trautman's treatment to which we have referred earlier occurs precisely here: he has $T^{\alpha\beta}$ where he should have had $\Theta^{\alpha\beta}$.

The expressions (35), (38), and (39) that we have found for the $\gamma^{\cdot\cdot}$'s can be transformed into expressions for the corresponding $g_{\cdot\cdot}$'s with the aid of the formulae

$$\underset{n}{g_{00}} = \tfrac{1}{2}(\underset{n}{\gamma^{00}} + \underset{n}{\gamma^{\alpha\alpha}})\,, \quad \underset{n}{g_{0\alpha}} = -\underset{n}{\gamma^{0\alpha}}\,,$$

and

$$\underset{n}{g_{\alpha\beta}} = \underset{n}{\gamma^{\alpha\beta}} + \tfrac{1}{2}\delta_{\alpha\beta}(\underset{n}{\gamma^{00}} - \underset{n}{\gamma^{\mu\mu}})\,, \tag{41}$$

which follow from the general relation (Trautman 1958*b*, eq. [27])

$$\underset{n}{g_{ij}} = \eta_{ik}\eta_{jl}(\underset{n}{\gamma^{kl}} - \tfrac{1}{2}\eta^{kl}\eta_{rs}\underset{n}{\gamma^{rs}})\,. \tag{42}$$

We thus obtain

$$Q_{00}{}^{(5)} = \tfrac{1}{3}G\frac{\partial^3}{\partial t^3}\int\rho(x',t)\,|x-x'|^2dx' + 2G\frac{\partial}{\partial t}\int\Theta_{\mu\mu}dx\,, \tag{43}$$

$$Q_{0\alpha}{}^{(6)} = -\tfrac{2}{3}G\frac{\partial^3}{\partial t^3}\int\rho(x',t)v_\alpha(x',t)\,|x-x'|^2dx'\,, \tag{44}$$

and

$$Q_{\alpha\beta}{}^{(5)} = 4G\frac{\partial}{\partial t}\int\Theta_{\alpha\beta}dx + \delta_{\alpha\beta}\Big[\tfrac{1}{3}G\frac{\partial^3}{\partial t^3}\int\rho(x',t)\,|x-x'|^2dx' - 2G\frac{\partial}{\partial t}\int\Theta_{\mu\mu}dx\Big]\,. \tag{45}$$

It is satisfactory in many ways that the expressions which we have found for *the lowest-order terms in the metric coefficients in the near zone, which reflect the outgoing-radiation condition at infinity, are purely of Newtonian origin* in the sense that they do not involve explicitly any quantity defined in the higher approximations.

a) Alternative Expressions for $Q_{00}{}^{(5)}$, $Q_{0\alpha}{}^{(6)}$, and $Q_{\alpha\beta}{}^{(5)}$ in Terms of the Moment-of-Inertia Tensor

We have already seen in § II (eq. [15]) how by expanding $|x-x'|^2$ and making use of the Newtonian laws of the conservation of mass and of linear momentum, we can write

$$\frac{\partial^3}{\partial t^3}\int_V\rho(x',t)\,|x-x'|^2dx' = \frac{d^3I_{\mu\mu}}{dt^3}\,. \tag{46}$$

Similarly, we can transform the integral expression $Q_{0\alpha}{}^{(6)}$ to give

$$\frac{\partial^3}{\partial t^3}\int_V\rho(x',t)v_\alpha(x',t)\,|x-x'|^2dx' = \frac{d^3}{dt^3}\int_V\rho v_\alpha|x|^2dx$$
$$-2x_\mu\frac{d^3}{dt^3}\int_V\rho(x',t)v_\alpha(x',t)x'_\mu dx'\,. \tag{47}$$

The further simplification of the expressions (43)–(45) depends on the following lemmas which derive from the tensor virial theorem and the associated definitions and relations (for a brief account of this theory, see Chandrasekhar 1969*a*, chap. 2).

LEMMA 1: *The tensor*

$$t_{\alpha\beta} \equiv -\tfrac{1}{2}\rho\mathfrak{B}_{\alpha\beta}\ (\text{mod div})\,, \tag{48}$$[6]

where

$$\mathfrak{B}_{\alpha\beta}(x) = G\int_V\rho(x')\frac{(x_\alpha-x'_\alpha)(x_\beta-x'_\beta)}{|x-x'|^2}dx' \tag{49}$$

denotes the tensor potential.

[6] As defined in Paper III (§ IV), two functions are said to be equal modulo divergence if they differ by the divergence of a vector which vanishes sufficiently rapidly at infinity that their integrals over the whole space (assuming that they exist) are equal.

PROOF: Clearly,

$$4\frac{\partial U}{\partial x_\alpha}\frac{\partial U}{\partial x_\beta} = -2\frac{\partial U}{\partial x_\alpha}\frac{\partial}{\partial x_\beta}\nabla^2\chi \equiv 2\nabla^2 U\frac{\partial^2\chi}{\partial x_\alpha \partial x_\beta}\ (\text{mod div})$$
$$\equiv -8\pi G\rho\frac{\partial^2\chi}{\partial x_\alpha \partial x_\beta}\ (\text{mod div})\ . \tag{50}$$

Also (cf. Paper III, eq. [49]),

$$-2\delta_{\alpha\beta}\left(\frac{\partial U}{\partial x_\mu}\right)^2 \equiv -8\pi G\rho U\delta_{\alpha\beta}\ (\text{mod div})\ . \tag{51}$$

From equations (50) and (51) and the known relation

$$\mathfrak{V}_{\alpha\beta} = \frac{\partial^2\chi}{\partial x_\alpha \partial x_\beta} + U\delta_{\alpha\beta}\ , \tag{52}$$

the result stated follows.

LEMMA 2: *For $\Theta_{\alpha\beta}$ defined as in equation (40),*

$$\int\Theta_{\alpha\beta}dx = \tfrac{1}{2}\frac{d^2 I_{\alpha\beta}}{dt^2}\ . \tag{53}$$

PROOF: From Lemma 1 and the definition of $\Theta_{\alpha\beta}$ it follows

$$\begin{aligned}\int\Theta_{\alpha\beta}dx &= \int(\rho v_\alpha v_\beta + p\delta_{\alpha\beta} + t_{\alpha\beta})dx\\ &= \int[\rho(v_\alpha v_\beta - \tfrac{1}{2}\mathfrak{V}_{\alpha\beta}) + p\delta_{\alpha\beta}]dx\\ &= 2\mathfrak{T}_{\alpha\beta} + \mathfrak{W}_{\alpha\beta} + \mathfrak{P}\delta_{\alpha\beta}\ ,\end{aligned} \tag{54}$$

where

$$\mathfrak{P} = \int_V p dx \tag{55}$$

and $\mathfrak{T}_{\alpha\beta}$ and $\mathfrak{W}_{\alpha\beta}$ denote the kinetic-energy and the potential-energy tensors, respectively. The result stated now follows from the standard form of the tensor virial theorem.

LEMMA 3:

$$\frac{d}{dt}\int_V \rho v_\alpha x_\beta dx = \tfrac{1}{2}\frac{d^2 I_{\alpha\beta}}{dt^2}\ . \tag{56}$$

PROOF: The conservation of angular momentum ensures the symmetry of the integral on the left-hand side with respect to α and β. Therefore, we may write instead

$$\tfrac{1}{2}\frac{d}{dt}\int_V \rho(v_\alpha x_\beta + v_\beta x_\alpha)dx = \tfrac{1}{2}\frac{d}{dt}\int_V \rho\frac{d}{dt}(x_\alpha x_\beta)dx = \tfrac{1}{2}\frac{d^2 I_{\alpha\beta}}{dt^2}\ , \tag{57}$$

and this is the result stated.

By making use of the relations (46) and (47) and the foregoing lemmas, the expressions (43)–(45) for the metric coefficients can be brought to the forms

$$Q_{00}{}^{(5)} = \tfrac{4}{3}G\frac{d^3 I_{\mu\mu}}{dt^3}\ , \tag{58}$$

$$Q_{0\alpha}{}^{(6)} = \tfrac{2}{3}Gx_\mu\frac{d^4 I_{\mu\alpha}}{dt^4} - \tfrac{2}{3}G\frac{d^3}{dt^3}\int\rho v_\alpha|x|^2 dx\ , \tag{59}$$

and

$$Q_{\alpha\beta}{}^{(5)} = 2G\frac{d^3 I_{\alpha\beta}}{dt^3} - \tfrac{2}{3}G\delta_{\alpha\beta}\frac{d^3 I_{\mu\mu}}{dt^3}. \tag{60}$$

An important consequence of the formula (60) for $Q_{\alpha\beta}{}^{(5)}$ is

$$Q_{\mu\mu}{}^{(5)} = 0\,, \tag{61}$$

i.e., $Q_{\alpha\beta}{}^{(5)}$ is traceless. We shall see that *this traceless character of $Q_{\alpha\beta}{}^{(5)}$ is essential for the consistency of the whole development.*

Further consequences of the foregoing formulae are

$$\frac{dQ_{00}{}^{(5)}}{dt} = 2\frac{\partial Q_{0\alpha}{}^{(6)}}{\partial x_\alpha} \quad \text{and} \quad \frac{\partial Q_{0\alpha}{}^{(6)}}{\partial x_\beta} = \frac{\partial Q_{0\beta}{}^{(6)}}{\partial x_\alpha}. \tag{62}$$

The first of these relations is equivalent to the de Donder condition that was imposed on the $\gamma^{\cdot\cdot}$'s in obtaining the solutions, and the second is an expression of the conservation of the Newtonian angular momentum.

PART II

THE EQUATIONS OF MOTION IN THE 2½-POST-NEWTONIAN APPROXIMATION

S. Chandrasekhar and F. Paul Esposito

IV. THE SOLUTION FOR $Q_{00}{}^{(7)}$

In Part I, we have seen how the metric coefficients $g_{0\alpha}$ and $g_{\alpha\beta}$, to orders required for the 2½-post-Newtonian approximation, can be deduced by supplementing the solutions obtained in the second post-Newtonian approximation by terms of $O(c^{-6})$ in $g_{0\alpha}$ and of $O(c^{-5})$ in $g_{\alpha\beta}$—terms which are required by the outgoing-radiation condition at infinity. But the equations of motion in the 2½-post-Newtonian approximation cannot be written down without a knowledge of the term of $O(c^{-7})$ in g_{00}. Our earlier considerations have determined $Q_{00}{}^{(5)}$; this is not sufficient for our purposes. We need to know "$Q_{00}{}^{(7)}$"; and to determine it an appeal must be made to the field equation

$$R_{00} = -\frac{8\pi G}{c^4}(T_{00} - \tfrac{1}{2}g_{00}T)\,. \tag{63}$$

We shall presently see how our knowledge of $Q_{00}{}^{(5)}$, $Q_{0\alpha}{}^{(6)}$, and $Q_{\alpha\beta}{}^{(5)}$ just suffices to determine $Q_{00}{}^{(7)}$. It is perhaps significant that an explicit appeal to the field equations is necessary before the radiation-reaction terms in the equations of motion can be made determinate: it emphasizes the essential nonlinear character of the theory.

Considering equation (63), we readily verify that no contributions of $O(c^{-7})$ to R_{00} are made by terms in the Ricci tensor which are quadratic in the first derivatives of g_{ij}—terms which are usually written as products of the Christoffel symbols. Precisely, the contributions of $O(c^{-7})$ to R_{00} arise from the terms

$$\underset{7}{R_{00}} = \underbrace{\tfrac{1}{2}g^{\mu\nu}\frac{\partial^2 g_{00}}{\partial x_\mu \partial x_\nu}}_{7} + \underbrace{g^{\mu\nu}\frac{\partial}{\partial x_0}\left(\frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x_0} - \frac{\partial g_{\mu 0}}{\partial x_\nu}\right)}_{7}; \tag{64}$$

and we find

$$\underset{7}{R_{00}} = -\tfrac{1}{2}\nabla^2 Q_{00}{}^{(7)} - \tfrac{1}{2}Q_{\mu\nu}{}^{(5)}\frac{\partial^2}{\partial x_\mu \partial x_\nu}\underset{2}{g_{00}} - \frac{\partial}{\partial t}\left(\frac{1}{2}\frac{\partial Q_{\mu\mu}{}^{(5)}}{\partial t} - \frac{\partial Q_{0\mu}{}^{(6)}}{\partial x_\mu}\right). \tag{65}$$

Remembering that

$$\underset{2}{g_{00}} = -2U\,, \tag{66}$$

and making use, also, of the relations (61) and (62), we can rewrite equation (65) in the form

$$\underset{7}{R_{00}} = -\tfrac{1}{2}\nabla^2 Q_{00}{}^{(7)} + Q_{\mu\nu}{}^{(5)}\frac{\partial^2 U}{\partial x_\mu \partial x_\nu} + \tfrac{1}{2}\frac{\partial^2 Q_{00}{}^{(5)}}{\partial t^2}\,. \tag{67}$$

Considering next the right-hand side of equation (63), we readily find that

$$\underbrace{T_{00} - \tfrac{1}{2}g_{00}T}_{3} = \tfrac{1}{2}\rho Q_{00}{}^{(5)}\,. \tag{68}$$

The equation governing $Q_{00}{}^{(7)}$ is, therefore,

$$\nabla^2 Q_{00}{}^{(7)} = 8\pi G\rho Q_{00}{}^{(5)} + 2Q_{\mu\nu}{}^{(5)}\frac{\partial^2 U}{\partial x_\mu \partial x_\nu} + \frac{\partial^2 Q_{00}{}^{(5)}}{\partial t^2}\,, \tag{69}$$

where it may be recalled that $Q_{00}{}^{(5)}$ and $Q_{\mu\nu}{}^{(5)}$ are, according to equations (43) and (45), functions of time only.

a) The Solution for $Q_{00}{}^{(7)}$

The part of the integral of equation (69) which derives from the first two terms on the right-hand side of equation (69) is readily written down. We clearly have

$$-2Q_{00}{}^{(5)}U - Q_{\mu\nu}{}^{(5)}\frac{\partial^2 \chi}{\partial x_\mu \partial x_\nu}\,. \tag{70}$$

It is less straightforward to write down the part of the integral which derives from the last term. The principal question here is one of uniqueness. It appears that the question can be resolved as follows.

First, we state the following readily verifiable lemma.

LEMMA 4: *If*

$$\Psi^{(n)}(x) = \int \psi(x')|x - x'|^n dx'\,, \tag{71}$$

where $\psi(x)$ is a good function,[7] *then*

$$\nabla^2\Psi^{(n)}(x) = n(n+1)\int\psi(x')|x - x'|^{n-2}dx' = n(n+1)\Psi^{(n-2)}(x)\,. \tag{72}$$

In writing the integrals corresponding to terms such as (cf. eq. [43])

$$\frac{\partial^2 Q_{00}{}^{(5)}}{\partial t^2} = \tfrac{1}{3}G\frac{\partial^5}{\partial t^5}\int\rho(x', t)|x - x'|^2 dx' + 2G\frac{\partial^3}{\partial t^3}\int\Theta_{\mu\mu}(x', t)dx' \tag{73}$$

in equation (69), we shall make use of the *converse of Lemma 4* in the operational form

$$(\nabla^2)^{-1}\Psi^{(n)}(x) = \frac{\Psi^{(n+2)}(x)}{(n+2)(n+3)}\,. \tag{74}$$

The justification for this method of inversion is that in this way the solutions are unambiguously and uniquely expressed in terms of the sources. But more importantly, the only terms (such as $\partial^2 Q_{00}{}^{(5)}/\partial t^2$ in eq. [69]) which require this procedure, when so inverted, lead precisely to those terms in the expansion of the solution,

$$-\frac{4G}{c^4}\int\frac{dx'}{|x - x'|}\Theta^{ik}(x', t - |x - x'|/c)\,, \tag{75}$$

[7] In the following sense defined by Lighthill (1958, p. 15): "a good function is one which is everywhere differentiable any number of times and such that it and all its derivatives are $O(|x|^{-N})$ as $|x| \to \infty$ for all N."

of equation (31) which appear in the *requisite order* as (cf. eq. [10] with T^{ik} replaced by Θ^{ik})

$$\frac{4G}{c^{n+4}}\frac{(-1)^{n+1}}{n!}\frac{\partial^n}{\partial t^n}\int \underset{0}{\Theta^{ik}}(x', t)\,|x - x'|^{n-1}dx' , \tag{76}$$

where $\underset{0}{\Theta^{ik}}$ is the *Newtonian expression* for the complex. Thus when we are considering $Q_{00}{}^{(7)}$ the requisite order is c^{-7}; and, in accordance with expression (75) and equation (42), the term in question is

$$\tfrac{1}{2}\left[\frac{4G}{5!c^7}\frac{\partial^5}{\partial t^5}\int \rho(x', t)\,|x - x'|^4dx' + \frac{4G}{3!c^7}\frac{\partial^3}{\partial t^3}\int \Theta^{\mu\mu}(x', t)\,|x - x'|^2dx'\right]; \tag{77}$$

whereas, inverting the expression on the right-hand side of equation (73) in accordance with equation (74), we have

$$\tfrac{1}{60}G\frac{\partial^5}{\partial t^5}\int \rho(x', t)\,|x - x'|^4dx' + \tfrac{1}{3}G\frac{\partial^3}{\partial t^3}\int \Theta_{\mu\mu}(x', t)\,|x - x'|^2dx' ; \tag{78}$$

and we observe that the two results agree.

Combining the results (70) and (78), we can now write the complete integral of equation (69); we have

$$\begin{aligned} Q_{00}{}^{(7)} = -2Q_{00}{}^{(5)}U - Q_{\mu\nu}{}^{(5)}\frac{\partial^2\chi}{\partial x_\mu \partial x_\nu} + \tfrac{1}{60}G\frac{\partial^5}{\partial t^5}\int \rho(x', t)\,|x - x'|^4dx' \\ + \tfrac{1}{3}G\frac{\partial^3}{\partial t^3}\int \Theta_{\mu\mu}(x', t)\,|x - x'|^2dx' . \end{aligned} \tag{79}$$

Since $Q_{\mu\nu}{}^{(5)}$ is traceless, we can in view of the relation (52) rewrite the solution (79) in the form

$$\begin{aligned} Q_{00}{}^{(7)} = -2Q_{00}{}^{(5)}U - Q_{\mu\nu}{}^{(5)}\mathfrak{B}_{\mu\nu} + \tfrac{1}{60}G\frac{\partial^5}{\partial t^5}\int \rho(x', t)\,|x - x'|^4dx' \\ + \tfrac{1}{3}G\frac{\partial^3}{\partial t^3}\int \Theta_{\mu\mu}(x', t)\,|x - x'|^2dx' . \end{aligned} \tag{80}$$

V. CHRISTOFFEL SYMBOLS AND RELATED QUANTITIES

With the metric coefficients $Q_{00}{}^{(7)}$, $Q_{0\alpha}{}^{(6)}$, $Q_{\alpha\beta}{}^{(5)}$, and $Q_{00}{}^{(5)}$ determined, we readily find that

$$\underset{5}{g^{00}} = -\frac{1}{c^5}Q_{00}{}^{(5)} , \quad \underset{5}{g^{\alpha\beta}} = -\frac{1}{c^5}Q_{\alpha\beta}{}^{(5)} , \quad \text{and} \quad \underbrace{\sqrt{-g}}_{5} = \frac{1}{2c^5}Q_{00}{}^{(5)} . \tag{81}$$

With this knowledge, we find on evaluating the various Christoffel symbols that the terms in them which follow those listed in Paper II, equations (47), are

$$\begin{aligned} &\underset{6}{\Gamma^0{}_{00}} = \frac{1}{2c^6}\frac{dQ_{00}{}^{(5)}}{dt} , \quad \underset{7}{\Gamma^0{}_{0\alpha}} = \frac{1}{2c^7}\left(\frac{\partial Q_{00}{}^{(7)}}{\partial x_\alpha} + 2Q_{00}{}^{(5)}\frac{\partial U}{\partial x_\alpha}\right) , \\ &\underset{6}{\Gamma^0{}_{\alpha\beta}} = \frac{1}{2c^6}\left(\frac{\partial Q_{0\alpha}{}^{(6)}}{\partial x_\beta} + \frac{\partial Q_{0\beta}{}^{(6)}}{\partial x_\alpha} - \frac{\partial Q_{\alpha\beta}{}^{(5)}}{\partial t}\right) , \quad \underset{5}{\Gamma^\alpha{}_{\beta\gamma}} = 0 , \\ &\underset{7}{\Gamma^\alpha{}_{00}} = \frac{1}{2c^7}\left(\frac{\partial Q_{00}{}^{(7)}}{\partial x_\alpha} - 2\frac{\partial Q_{0\alpha}{}^{(6)}}{\partial t} - 2Q_{\alpha\mu}{}^{(5)}\frac{\partial U}{\partial x_\mu}\right) , \quad \underset{6}{\Gamma^\alpha{}_{0\beta}} = -\frac{1}{2c^6}\frac{dQ_{\alpha\beta}{}^{(5)}}{dt} . \end{aligned} \tag{82}$$

Also,

$$\underset{6}{y_0} = \underbrace{\partial \log \sqrt{-g}/\partial x_0}_{6} = \frac{1}{2c^6}\frac{dQ_{00}{}^{(5)}}{dt} \quad \text{and} \quad \underset{5}{y_\alpha} = 0\,. \tag{83}$$

VI. THE EQUATIONS GOVERNING THE FLUID IN THE $2\frac{1}{2}$-POST-NEWTONIAN APPROXIMATION

First we observe that our present increased knowledge of the metric coefficients enables us to supplement the expressions for the components of the energy-momentum tensor given in Paper II, equations (49)–(51), by the additional terms

$$\underset{5}{u^0} = -\frac{1}{2c^5}Q_{00}{}^{(5)}\,, \qquad \underset{3}{T^{00}} = -\frac{1}{c^3}\rho Q_{00}{}^{(5)}\,, \qquad \underset{4}{T^{0\alpha}} = -\frac{1}{c^4}\rho Q_{00}{}^{(5)}v_\alpha\,,$$

and

$$\underset{5}{T^{\alpha\beta}} = \frac{1}{c^5}(-\rho v_\alpha v_\beta Q_{00}{}^{(5)} + pQ_{\alpha\beta}{}^{(5)})\,. \tag{84}$$

The relevant equations governing the motion of the fluid in the $2\frac{1}{2}$-post-Newtonian approximation can now be obtained by simply writing out the equation

$$T^{ij}{}_{;j} = 0 \tag{85}$$

to the required order with the aid of equations (82)–(84).

Considering first the 0-component of equation (85), we find that *there are no additional terms beyond those included in the second post-Newtonian approximation.* The reason for this circumstance is that the term of $O(c^{-5})$ in the conserved density, $\rho u^0\sqrt{-g}$, vanishes (cf. eqs. [81] and [84]):

$$\underbrace{\rho u^0\sqrt{-g}}_{5} = 0\,. \tag{86}$$

In other words, *no alteration in the baryon number, conserved in the second post-Newtonian approximation, is introduced in this higher approximation.* This fact is consistent with the *requirement* that the present "intermediate" approximation does not alter the expression for the density conserved in the *third* post-Newtonian approximation given in Paper II, equation (95) (which in turn is required in the determination of the energy conserved in the *second* post-Newtonian approximation).

Considering next the α-component of equation (85), we find

$$\frac{1}{c}\underset{4}{T^{\alpha j}}{}_{;j} = \frac{1}{c^5}\Big[-\rho\frac{d}{dt}(Q_{00}{}^{(5)}v_\alpha + Q_{\alpha\beta}{}^{(5)}v_\beta + Q_{0\alpha}{}^{(6)}) + \rho v_\beta\frac{\partial Q_{0\beta}{}^{(6)}}{\partial x_\alpha}$$
$$+\tfrac{1}{2}\rho v_\alpha\frac{dQ_{00}{}^{(5)}}{dt} + \rho Q_{00}{}^{(5)}\frac{\partial U}{\partial x_\alpha} + \tfrac{1}{2}\rho\frac{\partial Q_{00}{}^{(7)}}{\partial x_\alpha}\Big]\,. \tag{87}$$

By making use of the relations (62), we find that the foregoing equation can be brought to the somewhat simpler form

$$\frac{1}{c}\underset{4}{T^{\alpha j}}{}_{;j} = \frac{1}{c^5}\Big[-\rho Q_{00}{}^{(5)}\frac{dv_\alpha}{dt} - \tfrac{1}{2}\rho v_\alpha\frac{dQ_{00}{}^{(5)}}{dt} - \rho\frac{d}{dt}(Q_{\alpha\beta}{}^{(5)}v_\beta)$$
$$-\rho\frac{\partial Q_{0\alpha}{}^{(6)}}{\partial t} + \rho Q_{00}{}^{(5)}\frac{\partial U}{\partial x_\alpha} + \tfrac{1}{2}\rho\frac{\partial Q_{00}{}^{(7)}}{\partial x_\alpha}\Big]\,. \tag{88}$$

Finally, inserting in equation (88) the solution (80) for $Q_{00}{}^{(7)}$ and simplifying, we are left with

$$\frac{1}{c}\underset{4}{T}^{\alpha j}{}_{;j}=\frac{1}{c^5}\Big[-\rho Q_{00}{}^{(5)}\frac{dv_\alpha}{dt}-\tfrac{1}{2}\rho v_\alpha\frac{dQ_{00}{}^{(5)}}{dt}-\rho\frac{d}{dt}(Q_{\alpha\beta}{}^{(5)}v_\beta)-\tfrac{1}{2}\rho Q_{\mu\nu}{}^{(5)}\frac{\partial\mathfrak{B}_{\mu\nu}}{\partial x_\alpha}$$
$$-\rho\frac{\partial Q_{0\alpha}{}^{(6)}}{\partial t}+\tfrac{1}{30}\rho G\frac{\partial^5}{\partial t^5}\int\rho(x',t)|x-x'|^2(x_\alpha-x'_\alpha)dx' \tag{89}$$
$$+\tfrac{1}{3}\rho G\frac{\partial^3}{\partial t^3}\int\Theta_{\mu\mu}(x',t)(x_\alpha-x_\alpha')dx'\Big].$$

VII. THE COMPONENTS OF THE LANDAU-LIFSHITZ COMPLEX IN THE 2½-POST-NEWTONIAN APPROXIMATION

Expressions for the components of the Landau-Lifshitz complex in the second post-Newtonian approximation have been given in Paper II, equations (58), (72), and (90). The terms of one higher order that must be added to these expressions in the 2½-post-Newtonian approximation can now be evaluated with the aid of the Christoffel symbols listed in Paper II, equations (47), supplemented by equations (82) of the present paper. We find that

$$\underset{3}{\Theta}^{00}=0\quad\text{and}\quad\underset{4}{\Theta}^{0\alpha}=0\,, \tag{90}$$

while

$$\underset{5}{\Theta}^{\alpha\beta}=\frac{1}{c^5}[-\rho Q_{00}{}^{(5)}v_\alpha v_\beta+pQ_{\alpha\beta}{}^{(5)}+Q_{00}{}^{(5)}(\rho v_\alpha v_\beta+p\delta_{\alpha\beta}+t_{\alpha\beta})]$$
$$+\frac{1}{16\pi Gc^5}\Big\{-2Q_{\alpha\beta}{}^{(5)}\Big(\frac{\partial U}{\partial x_\mu}\Big)^2-8Q_{00}{}^{(5)}\frac{\partial U}{\partial x_\alpha}\frac{\partial U}{\partial x_\beta}-\frac{\partial U}{\partial t}\frac{dQ_{\alpha\beta}{}^{(5)}}{dt}$$
$$+\tfrac{1}{2}\Big(\frac{\partial P_\alpha}{\partial x_\beta}+\frac{\partial P_\beta}{\partial x_\alpha}\Big)\frac{dQ_{00}{}^{(5)}}{dt}+\frac{\partial P_\mu}{\partial x_\alpha}\frac{dQ_{\beta\mu}{}^{(5)}}{dt}+\frac{\partial P_\mu}{\partial x_\beta}\frac{dQ_{\alpha\mu}{}^{(5)}}{dt}$$
$$+4\frac{\partial U}{\partial x_\mu}\Big(Q_{\alpha\mu}{}^{(5)}\frac{\partial U}{\partial x_\beta}+Q_{\beta\mu}{}^{(5)}\frac{\partial U}{\partial x_\alpha}\Big) \tag{91}$$
$$+4\Big(\frac{\partial U}{\partial x_\alpha}\frac{\partial Q_{0\beta}{}^{(6)}}{\partial t}+\frac{\partial U}{\partial x_\beta}\frac{\partial Q_{0\alpha}{}^{(6)}}{\partial t}\Big)-2\Big(\frac{\partial U}{\partial x_\alpha}\frac{\partial Q_{00}{}^{(7)}}{\partial x_\beta}+\frac{\partial U}{\partial x_\beta}\frac{\partial Q_{00}{}^{(7)}}{\partial x_\alpha}\Big)$$
$$+\delta_{\alpha\beta}\Big[4Q_{00}{}^{(5)}\Big(\frac{\partial U}{\partial x_\mu}\Big)^2-2Q_{\mu\nu}{}^{(5)}\frac{\partial U}{\partial x_\mu}\frac{\partial U}{\partial x_\nu}+2\frac{\partial U}{\partial t}\frac{\partial Q_{0\mu}{}^{(6)}}{\partial x_\mu}$$
$$-4\frac{\partial U}{\partial x_\mu}\frac{\partial Q_{0\mu}{}^{(6)}}{\partial t}-\frac{\partial P_\mu}{\partial x_\nu}\frac{dQ_{\mu\nu}{}^{(5)}}{dt}+2\frac{\partial U}{\partial x_\mu}\frac{\partial Q_{00}{}^{(7)}}{\partial x_\mu}\Big]\Big\}.$$

Accordingly, *the* (0, 0)- *and the* (0, α)-*components of the Landau-Lifshitz complex retain, in the 2½-post-Newtonian approximation, the same values as in the second post-Newtonian approximation.* These facts are consistent with what we shall find in Part III, namely, that the terms (89) of $O(c^{-5})$ in the equation of motion governing the fluid principally contribute only to the dissipation of the energy and the angular momentum that are conserved in the second post-Newtonian approximation.[8]

Next by evaluating the divergence of the terms (91) (with respect to β), we find that we simply recover the terms (87). Therefore, in view of equations (90), we may write

$$\frac{1}{c}\underset{4}{\Theta}^{\alpha j}{}_{,j}=\underset{5}{\Theta}^{\alpha\beta}{}_{,\beta}=\frac{1}{c}\underset{4}{T}^{\alpha j}{}_{;j}\,. \tag{92}$$

[8] The statement is strictly correct only with respect to the angular momentum; a slight amplification is necessary with respect to the energy (cf. Part IV).

In other words, *the identity of the equations,*

$$\Theta^{\alpha j}{}_{,j} = 0 \quad \text{and} \quad T^{\alpha j}{}_{;j} = 0\,, \tag{93}$$

verified in the first and the second post-Newtonian approximations, continues to be maintained in the $2\frac{1}{2}$-post-Newtonian approximation.

PART III

THE DISSIPATION OF ENERGY AND ANGULAR MOMENTUM COMPARISON WITH THE LINEARIZED THEORY OF GRAVITATIONAL RADIATION

S. CHANDRASEKHAR

VIII. THE RATE OF DISSIPATION OF ENERGY

The equations governing the fluid motions in the $2\frac{1}{2}$-post-Newtonian approximation can be written as the sum of the second post-Newtonian terms, included in Paper II, equation (54), and the terms of $O(c^{-5})$, included in equation (89). Symbolically, we may write

$$\frac{1}{c}\underset{\text{2d post-N}}{T^{\alpha j}{}_{;j}} + \frac{1}{c}\underset{4}{T^{\alpha j}{}_{;j}} = 0\,. \tag{94}$$

Contracting this equation with v_α and integrating over the volume occupied by the fluid, we obtain

$$\frac{1}{c}\int_V v_\alpha \underset{\text{2d post-N}}{T^{\alpha j}{}_{;j}}\, dx + \frac{1}{c}\int_V v_\alpha \underset{4}{T^{\alpha j}{}_{;j}} dx = 0\,. \tag{95}$$

The first term on the left-hand side of this equation represents $d(\int \mathfrak{E}_{(2)}dx)/dt$, where $\mathfrak{E}_{(2)}$ (given by Paper II, eq. [110]) is the energy conserved in the second post-Newtonian approximation (cf. Chandrasekhar 1970). We can, therefore, write

$$\frac{d}{dt}\int_V \mathfrak{E}_{(2)}dx + \frac{1}{c}\int_V v_\alpha \underset{4}{T^{\alpha j}{}_{;j}} dx = 0\,. \tag{96}$$

We shall presently show that this equation predicts a rate of dissipation of energy by the system that is in accord with the predictions of the linearized theory of gravitational radiation.

First, we shall simplify the last two terms on the right-hand side of equation (89). By expanding $|x - x'|^2$ that occurs under the integral sign in the first of them and ignoring those terms which vanish by virtue of the conservation of mass and of linear momentum, we obtain

$$\begin{aligned}
&\frac{\partial^5}{\partial t^5}\int_V \rho(x', t)\,|x - x'|^2(x_\alpha - x'_\alpha)dx' \\
&= x_\alpha \frac{\partial^5}{\partial t^5}\int_V \rho(x', t)(|x|^2 + |x'|^2 - 2x_\mu x'_\mu)dx' \\
&\quad - \frac{\partial^5}{\partial t^5}\int_V \rho(x', t)x'_\alpha(|x|^2 + |x'|^2 - 2x_\mu x'_\mu)dx' \\
&= x_\alpha \frac{d^5}{dt^5}\int_V \rho(x', t)\,|x'|^2 dx' \\
&\quad - \frac{\partial^5}{\partial t^5}\int_V \rho(x', t)x'_\alpha(|x'|^2 - 2x_\mu x'_\mu)dx' \\
&= x_\alpha \frac{d^5 I_{\mu\mu}}{dt^5} - \frac{d^5 I_{\mu\mu\alpha}}{dt^5} + 2x_\mu \frac{d^5 I_{\mu\alpha}}{dt^5}\,,
\end{aligned} \tag{97}$$

where we have let

$$I_{\alpha\beta\gamma} = \int_V \rho x_\alpha x_\beta x_\gamma dx \tag{98}$$

denote the third-order moment of inertia. Next, by making use of Lemma 2, we can reduce the last term on the right-hand side of equation (89) in the manner

$$\begin{aligned}\frac{\partial^3}{\partial t^3} \int \Theta_{\mu\mu}(x', t)(x_\alpha - x'_\alpha)dx' &= x_\alpha \frac{d^3}{dt^3} \int \Theta_{\mu\mu} dx - \frac{d^3}{dt^3} \int \Theta_{\mu\mu} x_\alpha dx \\ &= \tfrac{1}{2} x_\alpha \frac{d^5 I_{\mu\mu}}{dt^5} - \frac{d^3}{dt^3} \int \Theta_{\mu\mu} x_\alpha dx .\end{aligned} \tag{99}$$

Also, by equation (59),

$$- \rho \frac{\partial Q_{0\alpha}{}^{(6)}}{\partial t} = \tfrac{2}{3} \rho G \frac{d^4}{dt^4} \int_V \rho v_\alpha |x|^2 dx - \tfrac{2}{3} \rho G x_\mu \frac{d^5 I_{\mu\alpha}}{dt^5} . \tag{100}$$

Substituting the results of these reductions in equation (89), we obtain

$$\begin{aligned}\frac{1}{c} \underset{4}{T^{\alpha j}}{}_{;j} = \frac{1}{c^5} \Bigg[&\underbrace{- \rho Q_{00}{}^{(5)} \frac{dv_\alpha}{dt} - \tfrac{1}{2} \rho v_\alpha \frac{dQ_{00}{}^{(5)}}{dt}}_{\text{(I)}} \underbrace{- \rho \frac{d}{dt} (v_\mu Q_{\mu\alpha}{}^{(5)})}_{\text{(II)}} \\ &\underbrace{- \tfrac{1}{2} \rho Q_{\mu\nu}{}^{(5)} \frac{\partial \mathfrak{B}_{\mu\nu}}{\partial x_\alpha}}_{\text{(III)}} + \underbrace{\tfrac{1}{5} \rho x_\alpha G \frac{d^5 I_{\mu\mu}}{dt^5} - \tfrac{3}{5} \rho x_\mu G \frac{d^5 I_{\mu\alpha}}{dt^5}}_{\text{(IV)}} \\ &\underbrace{- \tfrac{1}{30} \rho G \frac{d^5 I_{\mu\mu\alpha}}{dt^5} - \tfrac{1}{3} \rho G \frac{d^3}{dt^3} \int \Theta_{\mu\mu} x_\alpha dx + \tfrac{2}{3} \rho G \frac{d^4}{dt^4} \int \rho v_\alpha |x|^2 dx}_{\text{(V)}} \Bigg] .\end{aligned} \tag{101}$$

We shall now write down the results of contracting the different groups of terms in equation (101) by v_α and integrating over the volume occupied by the fluid. The first three groups of terms give

$$\text{(I)}: \ -Q_{00}{}^{(5)} \int_V \rho v_\alpha \frac{dv_\alpha}{dt} dx - \tfrac{1}{2} \frac{dQ_{00}{}^{(5)}}{dt} \int_V \rho |v|^2 dx = - \frac{d}{dt} (Q_{00}{}^{(5)} \mathfrak{T}_{\mu\mu}) , \tag{102}$$

$$\begin{aligned}\text{(II)}: \ &- Q_{\alpha\mu}{}^{(5)} \int_V \rho v_\alpha \frac{dv_\mu}{dt} dx - \frac{dQ_{\alpha\mu}{}^{(5)}}{dt} \int_V \rho v_\alpha v_\mu dx \\ &= -Q_{\alpha\mu}{}^{(5)} \frac{d\mathfrak{T}_{\alpha\mu}}{dt} - 2 \frac{dQ_{\alpha\mu}{}^{(5)}}{dt} \mathfrak{T}_{\alpha\mu} = -2 \frac{d}{dt} (Q_{\alpha\mu}{}^{(5)} \mathfrak{T}_{\alpha\mu}) + Q_{\alpha\mu}{}^{(5)} \frac{d\mathfrak{T}_{\alpha\mu}}{dt} ,\end{aligned} \tag{103}$$

and

$$\begin{aligned}\text{(III)}: \ - \tfrac{1}{2} Q_{\mu\nu}{}^{(5)} \int_V \rho v_\alpha \frac{\partial \mathfrak{B}_{\mu\nu}}{\partial x_\alpha} dx &= \tfrac{1}{2} Q_{\mu\nu}{}^{(5)} \int_V \frac{\partial}{\partial x_\alpha} (\rho v_\alpha) \mathfrak{B}_{\mu\nu} dx \\ &= -\tfrac{1}{2} Q_{\mu\nu}{}^{(5)} \int_V \frac{\partial \rho}{\partial t} \mathfrak{B}_{\mu\nu} dx = \tfrac{1}{2} Q_{\mu\nu}{}^{(5)} \frac{d\mathfrak{W}_{\mu\nu}}{dt} .\end{aligned} \tag{104}$$

These three groups of terms combine to give

$$\text{(I)} + \text{(II)} + \text{(III)}: \ - \frac{d}{dt} (Q_{00}{}^{(5)} \mathfrak{T}_{\mu\mu} + 2 Q_{\alpha\mu}{}^{(5)} \mathfrak{T}_{\alpha\mu}) + \tfrac{1}{2} Q_{\alpha\mu}{}^{(5)} \left(2 \frac{d\mathfrak{T}_{\alpha\mu}}{dt} + \frac{d\mathfrak{W}_{\alpha\mu}}{dt} \right), \tag{105}$$

or in view of the traceless character of $Q_{\alpha\mu}{}^{(5)}$ and the tensor virial theorem, we can write

$$(\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}) : -\frac{d}{dt}\left(Q_{00}{}^{(5)}\mathfrak{T}_{\mu\mu} + 2Q_{\alpha\mu}{}^{(5)}\mathfrak{T}_{\alpha\mu}\right) + \tfrac{1}{4}Q_{\alpha\mu}{}^{(5)}\frac{d^3 I_{\alpha\mu}}{dt^3}. \tag{106}$$

Finally, substituting for $Q_{\alpha\mu}{}^{(5)}$ its expression given in equation (60), we obtain

$$\begin{aligned}(\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}) : &-\frac{d}{dt}\left(Q_{00}{}^{(5)}\mathfrak{T}_{\mu\mu} + 2Q_{\alpha\mu}{}^{(5)}\mathfrak{T}_{\alpha\mu}\right)\\ &+ \tfrac{1}{6}G\left(3\frac{d^3 I_{\alpha\mu}}{dt^3}\frac{d^3 I_{\alpha\mu}}{dt^3} - \frac{d^3 I_{\sigma\sigma}}{dt^3}\frac{d^3 I_{\mu\mu}}{dt^3}\right).\end{aligned} \tag{107}$$

Returning to equation (101), we find that the contribution by the group of terms IV is

$$\begin{aligned}(\mathrm{IV}) : \tfrac{1}{5}G\frac{d^5 I_{\mu\mu}}{dt^5}\int_V \rho x_\alpha v_\alpha dx - \tfrac{3}{5}G\frac{d^5 I_{\mu\alpha}}{dt^5}\int_V \rho x_\alpha v_\mu dx\\ = -\tfrac{1}{10}G\left(3\frac{d^5 I_{\mu\alpha}}{dt^5}\frac{dI_{\mu\alpha}}{dt} - \frac{d^5 I_{\mu\mu}}{dt^5}\frac{dI_{\sigma\sigma}}{dt}\right)\\ = -\tfrac{1}{10}G\left(3\frac{d^3 I_{\mu\alpha}}{dt^3}\frac{d^3 I_{\mu\alpha}}{dt^3} - \frac{d^3 I_{\mu\mu}}{dt^3}\frac{d^3 I_{\sigma\sigma}}{dt^3}\right)\\ + \tfrac{1}{10}G\frac{d}{dt}\left(3\frac{d^3 I_{\mu\alpha}}{dt^3}\frac{d^2 I_{\mu\alpha}}{dt^2} - \frac{d^3 I_{\mu\mu}}{dt^3}\frac{d^2 I_{\sigma\sigma}}{dt^2} - 3\frac{d^4 I_{\mu\alpha}}{dt^4}\frac{dI_{\mu\alpha}}{dt} + \frac{d^4 I_{\mu\mu}}{dt^4}\frac{dI_{\sigma\sigma}}{dt}\right).\end{aligned} \tag{108}$$

The contribution by the last group of terms V can be directly written down and calls for no special comment.

Combining then the results of the foregoing reductions, we obtain

$$\begin{aligned}\frac{1}{c}\int_V T^{\alpha j}{}_{;j}{}_4 v_\alpha dx = \frac{1}{c^5}\Bigg[&-\frac{d}{dt}\left(Q_{00}{}^{(5)}\mathfrak{T}_{\mu\mu} + 2Q_{\alpha\mu}{}^{(5)}\mathfrak{T}_{\alpha\mu}\right) + \tfrac{1}{15}G\left(3\frac{d^3 I_{\mu\alpha}}{dt^3}\frac{d^3 I_{\mu\alpha}}{dt^3} - \frac{d^3 I_{\mu\mu}}{dt^3}\frac{d^3 I_{\sigma\sigma}}{dt^3}\right)\\ &+ \tfrac{1}{10}G\frac{d}{dt}\left(3\frac{d^3 I_{\alpha\mu}}{dt^3}\frac{d^2 I_{\alpha\mu}}{dt^2} - \frac{d^3 I_{\mu\mu}}{dt^3}\frac{d^2 I_{\sigma\sigma}}{dt^2} - 3\frac{d^4 I_{\mu\alpha}}{dt^4}\frac{dI_{\mu\alpha}}{dt} + \frac{d^4 I_{\mu\mu}}{dt^4}\frac{dI_{\sigma\sigma}}{dt}\right)\\ &+ G\left(\tfrac{2}{3}\frac{d^4}{dt^4}\int_V \rho v_\alpha |x|^2 dx - \tfrac{1}{30}\frac{d^5 I_{\mu\mu\alpha}}{dt^5} - \tfrac{1}{3}\frac{d^3}{dt^3}\int \Theta_{\mu\mu} x_\alpha dx\right)\int_V \rho v_\alpha dx\Bigg].\end{aligned} \tag{109}$$

In terms of the tensor,

$$D_{\alpha\beta} = 3I_{\alpha\beta} - \delta_{\alpha\beta} I_{\mu\mu}, \tag{110}$$

defining the quadrupole moment of the system, equation (109) can be written more compactly in the form

$$\begin{aligned}\frac{1}{c}\int_V T^{\alpha j}{}_{;j}{}_4 v_\alpha dx = \frac{1}{c^5}\Bigg[&-\frac{d}{dt}\left(Q_{00}{}^{(5)}\mathfrak{T}_{\mu\mu} + 2Q_{\alpha\mu}{}^{(5)}\mathfrak{T}_{\alpha\mu}\right) + \tfrac{1}{45}G\frac{d^3 D_{\alpha\beta}}{dt^3}\frac{d^3 D_{\alpha\beta}}{dt^3}\\ &+ \tfrac{1}{30}G\frac{d}{dt}\left(\frac{d^3 D_{\alpha\beta}}{dt^3}\frac{d^2 D_{\alpha\beta}}{dt^2} - \frac{d^4 D_{\alpha\beta}}{dt^4}\frac{dD_{\alpha\beta}}{dt}\right)\\ &+ G\left(\tfrac{2}{3}\frac{d^4}{dt^4}\int_V \rho v_\alpha |x|^2 dx - \tfrac{1}{30}\frac{d^5 I_{\mu\mu\alpha}}{dt^5} - \tfrac{1}{3}\frac{d^3}{dt^3}\int \Theta_{\mu\mu} x_\alpha dx\right)\int_V \rho v_\alpha dx\Bigg].\end{aligned} \tag{111}$$

Equation (111) can now be inserted in equation (96). We shall consider the result in a frame of reference in which the center of mass is at rest. In this frame, the terms in the last line of equation (111) vanish and we shall obtain

$$\frac{d}{dt}\int \mathfrak{E}_{(2)}dx - \frac{1}{c^5}\frac{d}{dt}(Q_{00}{}^{(5)}\mathfrak{T}_{\mu\mu} + 2Q_{\alpha\mu}{}^{(5)}\mathfrak{T}_{\alpha\mu}) + \frac{G}{30c^5}\frac{d}{dt}\left(\frac{d^3D_{\alpha\beta}}{dt^3}\frac{d^2D_{\alpha\beta}}{dt^2} - \frac{d^4D_{\alpha\beta}}{dt^4}\frac{dD_{\alpha\beta}}{dt}\right) = -\frac{G}{45c^5}\frac{d^3D_{\alpha\beta}}{dt^3}\frac{d^3D_{\alpha\beta}}{dt^3}. \tag{112}$$

In Part IV (§ XII) we shall show that we must associate with the 2½-post-Newtonian approximation the additional energy (cf. eq. [167] below)

$$\int \underset{5}{\mathfrak{E}}dx = \frac{1}{c^5}\left[-(Q_{00}{}^{(5)}\mathfrak{T}_{\mu\mu} + 2Q_{\alpha\mu}{}^{(5)}\mathfrak{T}_{\alpha\mu}) - \tfrac{1}{3}G\frac{d^4I_{\alpha\beta}}{dt^4}\frac{dI_{\alpha\beta}}{dt} + \tfrac{1}{18}G\frac{d^3D_{\alpha\beta}}{dt^3}\frac{d^2D_{\alpha\beta}}{dt^2} - \tfrac{1}{90}G\frac{d^4D_{\alpha\beta}}{dt^4}\frac{dD_{\alpha\beta}}{dt}\right]. \tag{113}$$

Including this energy together with $\mathfrak{E}_{(2)}$, we can rewrite equation (112) in the form

$$\frac{d}{dt}\int \mathfrak{E}_{(2.5)}dx = -\frac{G}{45c^5}\frac{d^3D_{\alpha\beta}}{dt^3}\frac{d^3D_{\alpha\beta}}{dt^3} + \frac{G}{45c^5}\frac{d^2}{dt^2}\left(\frac{d^3D_{\alpha\beta}}{dt^3}\frac{dD_{\alpha\beta}}{dt}\right) - \frac{G}{6c^5}\frac{d^2}{dt^2}\left(2\frac{d^3I_{\alpha\beta}}{dt^3}\frac{dI_{\alpha\beta}}{dt} - \frac{d^2I_{\alpha\beta}}{dt^2}\frac{d^2I_{\alpha\beta}}{dt^2}\right). \tag{114}$$

We observe that the first term on the right-hand side of equation (114) is *negative definite*. It therefore represents a *secular decrease of the integrated energy in the system.* In contrast to the first term, the two remaining terms (being total time derivatives) may be expected to vanish when averaged over a long enough interval of time. We may therefore write

$$\left\langle\frac{d}{dt}\int \mathfrak{E}_{(2.5)}dx\right\rangle = -\frac{G}{45c^5}\left\langle\left|\frac{d^3D_{\alpha\beta}}{dt^3}\right|^2\right\rangle. \tag{115}$$

This result is in exact agreement with the rate of emission of gravitational radiant energy predicted by the linearized theory of gravitational radiation.

IX. THE RATE OF DISSIPATION OF ANGULAR MOMENTUM

We have seen in § VII that there is no contribution of $O(c^{-4})$ to $\Theta^{\alpha 0}$. Accordingly, we may write the equation satisfied by the α-component of the Landau-Lifshitz complex in the form

$$\frac{1}{c}\frac{\partial\Theta^{0\alpha}{}_{(2)}}{\partial t} + \frac{\partial\Theta^{\alpha\beta}{}_{(2)}}{\partial x_\beta} + \frac{\partial}{\partial x_\beta}\underset{5}{\Theta^{\alpha\beta}} = 0, \tag{116}$$

where $\Theta^{0\alpha}{}_{(2)}$ and $\Theta^{\alpha\beta}{}_{(2)}$ are the contributions to the complex in the second post-Newtonian approximation. (Explicit expressions for these quantities are given in Paper II, eqs. [72] and [90].) In view of the relation (92) established in § VII, we may write instead

$$\frac{1}{c}\frac{\partial\Theta^{0\alpha}{}_{(2)}}{\partial t} + \frac{\partial\Theta^{\alpha\beta}{}_{(2)}}{\partial x_\beta} + \frac{1}{c}\underset{4}{T^{\alpha j}{}_{;j}} = 0, \tag{117}$$

where the last term on the left-hand side is again given by equation (101).

Now applying to equation (117) the same procedure that one follows in obtaining the angular-momentum integral from the equation satisfied by the Landau-Lifshitz complex, we shall obtain

$$\frac{\partial}{\partial t}\int L_{\gamma(2)}dx = \frac{1}{c}\int_V (x_\beta \underset{4}{T^{\alpha j}}_{;j} - x_\alpha \underset{4}{T^{\beta j}}_{;j})dx\,, \tag{118}$$

where $L_{\gamma(2)}(= x_\alpha\Theta^{\beta 0}{}_{(2)} - x_\beta\Theta^{\alpha 0}{}_{(2)})$ is the angular momentum that is conserved in the second post-Newtonian approximation.

The results of multiplying the first four groups of terms in equation (101) by x_β and integrating over the volume occupied by the fluid are

$$(\mathrm{I}):\ \tfrac{1}{2}Q_{00}{}^{(5)}(2\mathfrak{T}_{\alpha\beta} - \mathfrak{W}_{\alpha\beta} - \mathfrak{P}\delta_{\alpha\beta}) - \tfrac{1}{2}\frac{d}{dt}\Big(Q_{00}{}^{(5)}\int_V \rho v_\alpha x_\beta dx\Big)\,, \tag{119}$$

$$(\mathrm{II}):\ 2Q_{\alpha\mu}{}^{(5)}\mathfrak{T}_{\mu\beta} - \frac{d}{dt}\Big(Q_{\alpha\mu}{}^{(5)}\int_V \rho v_\mu x_\beta dx\Big)\,, \tag{120}$$

$$(\mathrm{III}):\ Q_{\alpha\mu}{}^{(5)}\mathfrak{W}_{\mu\beta} + \tfrac{3}{4}Q_{\mu\nu}{}^{(5)}\mathfrak{W}_{\mu\nu\alpha\beta}\,, \tag{121}$$

and

$$(\mathrm{IV}):\ \tfrac{1}{5}GI_{\alpha\beta}\frac{d^5 I_{\sigma\sigma}}{dt^5} - \tfrac{3}{5}GI_{\mu\beta}\frac{d^5 I_{\alpha\mu}}{dt^5}\,, \tag{122}$$

where

$$\mathfrak{W}_{\mu\nu\alpha\beta} = G\int_V\int_V \rho(x)\rho(x')\frac{(x_\mu - x'_\mu)(x_\nu - x'_\nu)(x_\alpha - x'_\alpha)(x_\beta - x'_\beta)}{|x - x'|^5}dxdx' \tag{123}$$

is a completely symmetric (Cartesian) tensor.

The last group of terms (V) in equation (101) does not contribute to the required integral in the center-of-mass frame. And we shall ignore these terms on the understanding that the evaluation is carried out in this frame.

Combining the results (119)–(122) and antisymmetrizing the sum with respect to α and β in accordance with equation (118), we find after some further reductions

$$\begin{aligned}\frac{d}{dt}\int L_{\gamma(2)}dx = {}& \frac{2G}{5c^5}\Big(\frac{d^2 I_{\beta\mu}}{dt^2}\frac{d^3 I_{\alpha\mu}}{dt^3} - \frac{d^2 I_{\alpha\mu}}{dt^2}\frac{d^3 I_{\beta\mu}}{dt^3}\Big)\\ &+ \frac{3G}{5c^5}\frac{d}{dt}\Big(\frac{d^4 I_{\beta\mu}}{dt^4}I_{\alpha\mu} - \frac{d^4 I_{\alpha\mu}}{dt^4}I_{\beta\mu} - \frac{d^3 I_{\beta\mu}}{dt^3}\frac{dI_{\alpha\mu}}{dt} + \frac{d^3 I_{\alpha\mu}}{dt^3}\frac{dI_{\beta\mu}}{dt}\Big)\\ &- \frac{1}{c^5}\frac{d}{dt}\Big(-\tfrac{1}{2}Q_{00}{}^{(5)}L_{\gamma(0)} + Q_{\alpha\mu}{}^{(5)}\int_V \rho v_\mu x_\beta dx - Q_{\beta\mu}{}^{(5)}\int_V \rho v_\mu x_\alpha dx\Big)\,.\end{aligned} \tag{124}$$

On averaging equation (124) over a long enough interval of time, we may expect the terms in the second and the third lines of equation (124) to vanish; and we shall be left with

$$\Big\langle\frac{d}{dt}\int L_{\gamma(2)}dx\Big\rangle = \frac{2G}{5c^5}\Big\langle\frac{d^2 I_{\beta\mu}}{dt^2}\frac{d^3 I_{\alpha\mu}}{dt^3} - \frac{d^2 I_{\alpha\mu}}{dt^2}\frac{d^3 I_{\beta\mu}}{dt^3}\Big\rangle\,. \tag{125}$$

This last result for the rate of dissipation of the angular momentum, which is conserved in the second post-Newtonian approximation, is again in agreement with the predictions of the linearized theory of gravitational radiation.

PART IV

THE ENERGY ASSOCIATED WITH THE 2½-POST-NEWTONIAN APPROXIMATION

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X. THE EXPRESSION FOR $\underset{5}{\mathfrak{E}}$

To obtain the energy in the 2½-post-Newtonian approximation, we must evaluate Θ^{00} and $c^2\rho u^0\sqrt{-g}$ to $O(c^{-5})$ (i.e., in the 3½-post-Newtonian approximation) in order that their difference, the energy $\mathfrak{E}$, may be known to $O(c^{-5})$, appropriate for the 2½-post-Newtonian approximation. As we shall presently see, the evaluation of Θ^{00} and $c^2\rho u^0\sqrt{-g}$ to $O(c^{-5})$ requires a knowledge of $g_{\alpha\beta}$ to $O(c^{-7})$.

Letting, then,

$$\underset{7}{g_{\alpha\beta}} = \frac{1}{c^7} Q_{\alpha\beta}{}^{(7)} , \tag{126}$$

we find

$$- \underset{7}{g} = \frac{1}{c^7}(Q_{00}{}^{(7)} - Q_{\mu\mu}{}^{(7)} + 6UQ_{00}{}^{(5)}) ,$$

$$\underbrace{\sqrt{-g}}_{7} = \frac{1}{2c^7}(Q_{00}{}^{(7)} - Q_{\mu\mu}{}^{(7)} + 4UQ_{00}{}^{(5)}) , \tag{127}$$

$$\underbrace{\log\sqrt{-g}}_{7} = \frac{1}{2c^7}(Q_{00}{}^{(7)} - Q_{\mu\mu}{}^{(7)} + 2UQ_{00}{}^{(5)}) ,$$

and

$$\begin{aligned} \underset{7}{u^0} &= -\frac{1}{2c^7}[Q_{00}{}^{(7)} + 2Q_{0\mu}{}^{(6)}v_\mu + Q_{\mu\nu}{}^{(5)}v_\mu v_\nu + \tfrac{3}{2}(v^2 + 2U)Q_{00}{}^{(5)}] , \\ \underset{7}{[u^0]^2} &= -\frac{1}{c^7}[Q_{00}{}^{(7)} + 2Q_{0\mu}{}^{(6)}v_\mu + Q_{\mu\nu}{}^{(5)}v_\mu v_\nu + 2(v^2 + 2U)Q_{00}{}^{(5)}] . \end{aligned} \tag{128}$$

With the aid of these results we find

$$\underbrace{c^2\rho u^0\sqrt{-g}}_{5} = -\frac{1}{2c^5}\rho(Q_{\mu\mu}{}^{(7)} + 2Q_{0\mu}{}^{(6)}v_\mu + Q_{\mu\nu}{}^{(5)}v_\mu v_\nu + Q_{00}{}^{(5)}v^2) . \tag{129}$$

We also find that

$$\underset{5}{T^{00}} = -\frac{1}{c^5}\rho[Q_{00}{}^{(7)} + 2Q_{0\mu}{}^{(6)}v_\mu + Q_{\mu\nu}{}^{(5)}v_\mu v_\nu + Q_{00}{}^{(5)}(2v^2 + 4U + \Pi)] \tag{130}$$

and

$$\underbrace{-gT^{00}}_{5} = -\frac{1}{c^5}\rho(Q_{\mu\mu}{}^{(7)} + 2Q_{0\mu}{}^{(6)}v_\mu + Q_{\mu\nu}{}^{(5)}v_\mu v_\nu + Q_{00}{}^{(5)}v^2) . \tag{131}$$

Turning next to the evaluation of the terms of $O(c^{-5})$ in t^{00}—the (0, 0)-component of the pseudo-tensor of Landau and Lifshitz—we find that we need the terms in $\Gamma^\alpha{}_{\beta\gamma}$ and

$y_\alpha(=\partial \log \sqrt{-g}/\partial x_\alpha)$ beyond those listed in equations (82) and (83). We find that these terms are given by

$$\underset{7}{\Gamma^\alpha{}_{\beta\gamma}} = \frac{1}{c^7}\left[-\tfrac{1}{2}\left(\frac{\partial Q_{\alpha\beta}{}^{(7)}}{\partial x_\gamma} + \frac{\partial Q_{\alpha\gamma}{}^{(7)}}{\partial x_\beta} - \frac{\partial Q_{\beta\gamma}{}^{(7)}}{\partial x_\alpha}\right) + Q_{\alpha\beta}{}^{(5)}\frac{\partial U}{\partial x_\gamma} + Q_{\alpha\gamma}{}^{(5)}\frac{\partial U}{\partial x_\beta} - \delta_{\beta\gamma}Q_{\alpha\mu}{}^{(5)}\frac{\partial U}{\partial x_\mu}\right] \tag{132}$$

and

$$\underset{7}{y_\alpha} = \frac{1}{2c^7}\left[\frac{\partial}{\partial x_\alpha}\left(Q_{00}{}^{(7)} - Q_{\mu\mu}{}^{(7)}\right) + 2Q_{00}{}^{(5)}\frac{\partial U}{\partial x_\alpha}\right]. \tag{133}$$

And evaluating t^{00} to $O(c^{-5})$, we find

$$\underset{5}{t^{00}} = \frac{1}{16\pi Gc^5}\left[14Q_{00}{}^{(5)}\left(\frac{\partial U}{\partial x_\mu}\right)^2 + 2\frac{\partial U}{\partial x_\mu}\left(3\frac{\partial Q_{\sigma\sigma}{}^{(7)}}{\partial x_\mu} - 2\frac{\partial Q_{\mu\nu}{}^{(7)}}{\partial x_\nu} - 3Q_{\mu\nu}{}^{(5)}\frac{\partial U}{\partial x_\nu}\right) + \frac{\partial U}{\partial t}\frac{\partial Q_{00}{}^{(5)}}{\partial t} + \frac{\partial P_\mu}{\partial x_\nu}\left(\frac{\partial Q_{0\mu}{}^{(6)}}{\partial x_\nu} + \frac{\partial Q_{0\nu}{}^{(6)}}{\partial x_\mu} - \frac{\partial Q_{\mu\nu}{}^{(5)}}{\partial t}\right)\right]. \tag{134}$$

Remembering that

$$\underbrace{-gt^{00}}_{5} = \underset{5}{t^{00}} + \frac{1}{c^5}Q_{00}{}^{(5)}\underset{0}{t^{00}}, \tag{135}$$

where (cf. Paper III, eq. [46])

$$\underset{0}{t^{00}} = -\frac{7}{8\pi G}\left(\frac{\partial U}{\partial x_\mu}\right)^2, \tag{136}$$

we can now combine equations (129), (131), and (134)–(136) to give

$$\begin{aligned}\underset{5}{\mathfrak{E}} &= \underset{5}{\Theta^{00}} - \underbrace{c^2\rho u^0\sqrt{-g}}_{5} = \underbrace{-g(T^{00}+t^{00})}_{5} - \underbrace{c^2\rho u^0\sqrt{-g}}_{5}\\ &= -\frac{1}{2c^5}\rho\left(Q_{\mu\mu}{}^{(7)} + 2Q_{0\mu}{}^{(6)}v_\mu + Q_{\mu\nu}{}^{(5)}v_\mu v_\nu + Q_{00}{}^{(5)}v^2\right)\\ &\quad + \frac{1}{16\pi Gc^5}\left[2\frac{\partial U}{\partial x_\mu}\left(3\frac{\partial Q_{\sigma\sigma}{}^{(7)}}{\partial x_\mu} - 2\frac{\partial Q_{\mu\nu}{}^{(7)}}{\partial x_\nu} - 3Q_{\mu\nu}{}^{(5)}\frac{\partial U}{\partial x_\nu}\right)\right.\\ &\qquad \left. + \frac{\partial U}{\partial t}\frac{\partial Q_{00}{}^{(5)}}{\partial t} + \frac{\partial P_\mu}{\partial x_\nu}\left(\frac{\partial Q_{0\mu}{}^{(6)}}{\partial x_\nu} + \frac{\partial Q_{0\nu}{}^{(6)}}{\partial x_\mu} - \frac{\partial Q_{\mu\nu}{}^{(5)}}{\partial t}\right)\right].\end{aligned} \tag{137}$$

The integral of this last expression over the whole of the three-dimensional space—we shall verify that the integral converges in the center-of-mass frame—will give the contribution of the $2\frac{1}{2}$-post-Newtonian approximation to the "conserved" energy (conserved, that is, in the absence of the effects of the radiation-reaction terms in the equation of motion).

We observe that in the expression (137) for $\underset{5}{\mathfrak{E}}$ the $3\frac{1}{2}$-post-Newtonian term $Q_{\mu\nu}{}^{(7)}$ occurs; it must accordingly be determined. For this last purpose we shall derive in § XI below the equation which governs this term.

XI. THE EQUATION DETERMINING $Q_{\alpha\beta}{}^{(7)}$

The equation governing $Q_{\alpha\beta}{}^{(7)}$ is readily obtained by considering

$$\underset{7}{R_{\alpha\beta}} = -\frac{8\pi G}{c^4}\underbrace{(T_{\alpha\beta} - \tfrac{1}{2}g_{\alpha\beta}T)}_{3} . \tag{138}$$

We find

$$\nabla^2 Q_{\alpha\beta}{}^{(7)} - \frac{\partial}{\partial x_\alpha}\left(\frac{\partial Q_{\mu\beta}{}^{(7)}}{\partial x_\mu} - \frac{1}{2}\frac{\partial Q_{\sigma\sigma}{}^{(7)}}{\partial x_\beta}\right) - \frac{\partial}{\partial x_\beta}\left(\frac{\partial Q_{\mu\alpha}{}^{(7)}}{\partial x_\mu} - \frac{1}{2}\frac{\partial Q_{\sigma\sigma}{}^{(7)}}{\partial x_\alpha}\right) = S_{\alpha\beta}{}^{(7)} , \tag{139}$$

where

$$\begin{aligned} S_{\alpha\beta}{}^{(7)} = & -8\pi G\rho Q_{\alpha\beta}{}^{(5)} + 2Q_{00}{}^{(5)}\frac{\partial^2 U}{\partial x_\alpha \partial x_\beta} + \frac{\partial^2 Q_{00}{}^{(7)}}{\partial x_\alpha \partial x_\beta} \\ & + 2\delta_{\alpha\beta}Q_{\mu\nu}{}^{(5)}\frac{\partial^2 U}{\partial x_\mu \partial x_\nu} - 2Q_{\mu\alpha}{}^{(5)}\frac{\partial^2 U}{\partial x_\beta \partial x_\mu} - 2Q_{\mu\beta}{}^{(5)}\frac{\partial^2 U}{\partial x_\alpha \partial x_\mu} \\ & + \frac{\partial}{\partial t}\left(\frac{\partial Q_{\alpha\beta}{}^{(5)}}{\partial t} - \frac{\partial Q_{0\alpha}{}^{(6)}}{\partial x_\beta} - \frac{\partial Q_{0\beta}{}^{(6)}}{\partial x_\alpha}\right). \end{aligned} \tag{140}$$

As was to be expected, there is an integrability condition for the solvability of equation (139), namely (cf. Paper II, § III*a*, eq. [19]),

$$\frac{\partial}{\partial x_\alpha}(S_{\alpha\beta}{}^{(7)} - \tfrac{1}{2}\delta_{\alpha\beta}S_{\sigma\sigma}{}^{(7)}) = 0 . \tag{141}$$

It can be verified that this integrability condition is indeed satisfied by virtue of equation (69) governing $Q_{00}{}^{(7)}$.

With the integrability condition (141) satisfied, the general solution of equation (139) involves an arbitrary vector function W_α (cf. Paper II, §§ III*a* and III*d*). Thus, if $Q_{\alpha\beta}{}^{(7)}$ is a solution, then so is

$$Q_{\alpha\beta}{}^{(7)} + \frac{\partial W_\alpha}{\partial x_\beta} + \frac{\partial W_\beta}{\partial x_\alpha} . \tag{142}$$

To be specific, we shall select the solution which satisfies the coordinate condition

$$\frac{\partial Q_{\alpha\beta}{}^{(7)}}{\partial x_\beta} - \frac{1}{2}\frac{\partial Q_{\sigma\sigma}{}^{(7)}}{\partial x_\alpha} = 0 . \tag{143}$$

In this "gauge," the equation satisfied by $Q_{\alpha\beta}{}^{(7)}$ is

$$\nabla^2 Q_{\alpha\beta}{}^{(7)} = S_{\alpha\beta}{}^{(7)} . \tag{144}$$

For the determination of $\underset{5}{\mathfrak{E}}$ in § XII below, we shall be particularly interested in the solution for $Q_{\mu\mu}{}^{(7)}$. By contracting equation (140) and making use of equation (69) satisfied by $Q_{00}{}^{(7)}$, we find

$$S_{\sigma\sigma}{}^{(7)} = 4Q_{\mu\nu}{}^{(5)}\frac{\partial^2 U}{\partial x_\mu \partial x_\nu} . \tag{145}$$

Therefore, in the gauge (143), the equation satisfied by $Q_{\mu\mu}{}^{(7)}$ is

$$\nabla^2 Q_{\mu\mu}{}^{(7)} = 4Q_{\mu\nu}{}^{(5)}\frac{\partial^2 U}{\partial x_\mu \partial x_\nu} . \tag{146}$$

Since $Q_{\mu\nu}^{(5)}$ is independent of the spatial coordinates, the required solution of equation (146) is

$$Q_{\mu\mu}^{(7)} = -2Q_{\mu\nu}^{(5)} \frac{\partial^2 \chi}{\partial x_\mu \partial x_\nu}\,; \tag{147}$$

or, in view of the traceless character of $Q_{\mu\nu}^{(5)}$ and the relation (52), we can write

$$Q_{\mu\mu}^{(7)} = -2Q_{\mu\nu}^{(5)} \mathfrak{B}_{\mu\nu}\,. \tag{148}$$

XII. THE ENERGY IN THE $2\frac{1}{2}$-POST-NEWTONIAN APPROXIMATION

Before we insert in equation (137) the solution for $Q_{\mu\nu}^{(7)}$ derived in the gauge (143), it is important to verify that the integral of $\underset{5}{\mathfrak{E}}$ over the whole of space (which is the only quantity of interest) is independent of the choice of gauge. This independence on the choice of gauge is necessary since the integral in question pertains to the $2\frac{1}{2}$-post-Newtonian approximation and as such should not depend on the choice of gauge relevant in the $3\frac{1}{2}$-post-Newtonian approximation.

To show the required independence on gauge, consider the terms in equation (137) which are dependent on $Q_{\alpha\beta}^{(7)}$. These are (apart from the factor c^{-5})

$$-\tfrac{1}{2}\rho Q_{\mu\mu}^{(7)} + \frac{1}{8\pi G}\frac{\partial U}{\partial x_\mu}\left(3\frac{\partial Q_{\sigma\sigma}^{(7)}}{\partial x_\mu} - 2\frac{\partial Q_{\mu\nu}^{(7)}}{\partial x_\nu}\right). \tag{149}$$

If in these terms we replace $Q_{\mu\nu}^{(7)}$ (defined, say, in some particular gauge) by the general solution (142), we shall find the additional terms

$$-\rho\frac{\partial W_\mu}{\partial x_\mu} + \frac{1}{4\pi G}\frac{\partial U}{\partial x_\mu}\left(2\frac{\partial^2 W_\nu}{\partial x_\mu \partial x_\nu} - \nabla^2 W_\mu\right). \tag{150}$$

It can be readily verified that *these additional terms are zero*, *modulo divergence*, if

$$\frac{\partial U}{\partial x_\mu}\frac{\partial W_\sigma}{\partial x_\sigma} \quad \text{and} \quad U\frac{\partial^2 W_\sigma}{\partial x_\mu \partial x_\sigma} \tag{151}$$

vanish at infinity more rapidly than r^{-2}.

In view, then, of the invariance of the terms (149) to the choice of gauge, we may, without loss of generality, consider them in the special gauge (143). In this gauge the terms reduce to

$$\begin{aligned}-\tfrac{1}{2}\rho Q_{\mu\mu}^{(7)} + \frac{1}{4\pi G}\frac{\partial U}{\partial x_\mu}\frac{\partial Q_{\sigma\sigma}^{(7)}}{\partial x_\mu} &\equiv -\tfrac{1}{2}\rho Q_{\mu\mu}^{(7)} - \frac{Q_{\mu\mu}^{(7)}}{4\pi G}\nabla^2 U \text{ (mod div)} \\ &\equiv +\tfrac{1}{2}\rho Q_{\mu\mu}^{(7)} = -\rho Q_{\mu\nu}^{(5)}\mathfrak{B}_{\mu\nu}\,,\end{aligned} \tag{152}$$

where in the last step of the reductions we have substituted for $Q_{\mu\mu}^{(7)}$ its solution (148).

Also, it can be verified that

$$\begin{aligned}-\frac{6}{16\pi G}Q_{\mu\nu}^{(5)}\frac{\partial U}{\partial x_\mu}\frac{\partial U}{\partial x_\nu} &\equiv -\frac{3}{16\pi G}Q_{\mu\nu}^{(5)}U\frac{\partial^2}{\partial x_\mu \partial x_\nu}\nabla^2\chi \text{ (mod div)} \\ &\equiv \tfrac{3}{4}\rho Q_{\mu\nu}^{(5)}\frac{\partial^2\chi}{\partial x_\mu \partial x_\nu} = \tfrac{3}{4}\rho Q_{\mu\nu}^{(5)}\mathfrak{B}_{\mu\nu}\,.\end{aligned} \tag{153}$$

Now replacing the terms (149), in the expression (137) for $\underset{5}{\mathfrak{E}}$, by their equivalent (152) and substituting also the result of the reductions (153), we obtain

$$\underset{5}{\mathfrak{G}} \equiv -\frac{1}{2c^5}\rho(2Q_{0\mu}{}^{(6)}v_\mu + Q_{\mu\nu}{}^{(5)}v_\mu v_\nu + \tfrac{1}{2}Q_{\mu\nu}{}^{(5)}\mathfrak{B}_{\mu\nu} + Q_{00}{}^{(5)}v^2)$$
$$+\frac{1}{16\pi Gc^5}\left[\frac{\partial U}{\partial t}\frac{\partial Q_{00}{}^{(5)}}{\partial t} + \frac{\partial P_\nu}{\partial x_\mu}\left(\frac{\partial Q_{0\mu}{}^{(6)}}{\partial x_\nu} + \frac{\partial Q_{0\nu}{}^{(6)}}{\partial x_\mu} - \frac{\partial Q_{\mu\nu}{}^{(5)}}{\partial t}\right)\right]. \tag{154}$$

The terms in brackets in equation (154) can be simplified by substituting for $Q_{00}{}^{(5)}$, $Q_{0\mu}{}^{(6)}$, and $Q_{\mu\nu}{}^{(5)}$ their values given in equations (58)–(60) and making further use of the relation

$$\frac{\partial P_\mu}{\partial x_\mu} = -3\frac{\partial U}{\partial t}. \tag{155}$$

We thus finally obtain

$$\underset{5}{\mathfrak{G}} = -\frac{1}{2c^5}\rho(2Q_{0\mu}{}^{(6)}v_\mu + Q_{\mu\nu}{}^{(5)}v_\mu v_\nu + \tfrac{1}{2}Q_{\mu\nu}{}^{(5)}\mathfrak{B}_{\mu\nu} + Q_{00}{}^{(5)}v^2) - \frac{1}{72\pi c^5}\frac{\partial P_\nu}{\partial x_\mu}\frac{d^4 D_{\mu\nu}}{dt^4}, \tag{156}$$

where it may be recalled that

$$P_\nu = 4U_\nu - \tfrac{1}{2}\frac{\partial^2\chi}{\partial t\partial x_\nu} = 4G\int_V \frac{\rho(x', t)v_\nu(x', t)}{|x - x'|}dx' - \tfrac{1}{2}\frac{\partial^2\chi}{\partial t\partial x_\nu}. \tag{157}$$

a) *The Evaluation of* $\int \underset{5}{\mathfrak{G}} dx$

We shall now evaluate the integral of $\underset{5}{\mathfrak{G}}$ over the whole of the three-dimensional space in a frame of reference in which the center of mass is at rest—it will appear that only in this frame does the integral converge.

The evaluation of the integral of the terms in $\underset{5}{\mathfrak{G}}$ which occur with the factor ρ is straightforward if appropriate use is made of the relations familiar in the theory of the tensor virial theorem. We find that they combine to give

$$\frac{1}{c^5}\left[-(Q_{00}{}^{(5)}\mathfrak{T}_{\mu\mu} + 2Q_{\mu\nu}{}^{(5)}\mathfrak{T}_{\mu\nu}) + \tfrac{1}{18}G\frac{d^2 D_{\mu\nu}}{dt^2}\frac{d^3 D_{\mu\nu}}{dt^3} - \tfrac{1}{3}G\frac{d^4 I_{\mu\nu}}{dt^4}\frac{dI_{\mu\nu}}{dt}\right]. \tag{158}$$

The evaluation of the integral over all space of the term in $\partial P_\nu/\partial x_\mu$ in $\underset{5}{\mathfrak{G}}$ is less straightforward since its existence will have to be established at the same time. On this last account, we shall first evaluate the integral over a large spherical volume of radius r (bounded by the surface S, say). We shall then let both r and S tend to infinity. The limit we have to consider is then

$$-\frac{1}{72\pi c^5}\frac{d^4 D_{\mu\nu}}{dt^4}\int_{S\to\infty} P_\nu dS_\mu = -\frac{1}{72\pi c^5}\frac{d^4 D_{\mu\nu}}{dt^4}\int_{S\to\infty}\left(4U_\nu - \tfrac{1}{2}\frac{\partial^2\chi}{\partial t\partial x_\nu}\right)dS_\mu. \tag{159}$$

Consider first the contribution to the surface integral (159) by U_ν. Since U_ν is the Newtonian potential derived from the "source" ρv_ν, it is clear that in the center-of-mass frame, in which the integral of this source vanishes, the asymptotic behavior of U_ν as $r \to \infty$ is

$$U_\nu \to G\frac{x_\sigma}{r^3}\int_V \rho(x', t)v_\nu(x', t)x'_\sigma dx' = G\frac{l_\sigma}{r^2}\int_V \rho v_\nu x_\sigma dx, \tag{160}$$

where l denotes the direction cosine of the outward radial direction. Therefore,

$$\int_{S\to\infty} U_\nu dS_\mu = 4\pi G\langle l_\sigma l_\mu\rangle\int_V \rho v_\nu x_\sigma dx = \tfrac{4}{3}\pi G\int_V \rho v_\nu x_\mu dx. \tag{161}$$

To evaluate the contribution to the surface integral (159) by the term in χ, we first observe that in the center-of-mass frame the asymptotic behavior of $\partial\chi/\partial t$ as $r \rightarrow \infty$, as deduced from the integral representation

$$\chi = -G \int_V \rho(x', t) |x - x'| dx' , \tag{162}$$

is

$$\frac{\partial \chi}{\partial t} \rightarrow -\tfrac{1}{2} G \frac{dI_{\sigma\sigma}}{dt} \frac{1}{r} + \tfrac{1}{2} G \frac{dI_{\alpha\beta}}{dt} \frac{x_\alpha x_\beta}{r^3} . \tag{163}$$

From this last formula it follows that

$$\frac{\partial^2 \chi}{\partial t \partial x_\nu} \rightarrow \frac{G}{r^2} \left(\tfrac{1}{2} \frac{dI_{\sigma\sigma}}{dt} l_\nu - \tfrac{3}{2} \frac{dI_{\alpha\beta}}{dt} l_\alpha l_\beta l_\nu + \frac{dI_{\alpha\nu}}{dt} l_\alpha \right) . \tag{164}$$

Therefore,

$$\begin{aligned} \int_{S\rightarrow\infty} \frac{\partial^2 \chi}{\partial t \partial x_\nu} dS_\mu &= 4\pi G \left(\tfrac{1}{2} \frac{dI_{\sigma\sigma}}{dt} \langle l_\nu l_\mu \rangle - \tfrac{3}{2} \frac{dI_{\alpha\beta}}{dt} \langle l_\alpha l_\beta l_\nu l_\mu \rangle + \frac{dI_{\alpha\mu}}{dt} \langle l_\alpha l_\mu \rangle \right) \\ &= \frac{4\pi G}{15} \left(\delta_{\mu\nu} \frac{dI_{\sigma\sigma}}{dt} + 2 \frac{dI_{\mu\nu}}{dt} \right) . \end{aligned} \tag{165}$$

Inserting the results (161) and (165) in equation (159) and remembering that $D_{\mu\nu}$ is traceless, we obtain

$$-\frac{1}{72\pi c^5} \frac{d^4 D_{\mu\nu}}{dt^4} \int_{S\rightarrow\infty} P_\nu dS_\mu = -\frac{G}{30c^5} \frac{d^4 D_{\mu\nu}}{dt^4} \frac{dI_{\mu\nu}}{dt} = -\frac{G}{90c^5} \frac{d^4 D_{\mu\nu}}{dt^4} \frac{dD_{\mu\nu}}{dt} . \tag{166}$$

Combining the results (158) and (166), we now have

$$\begin{aligned} \int_5 \mathfrak{E} dx = \frac{1}{c^5} \Big[&-(Q_{00}{}^{(5)}\mathfrak{T}_{\mu\mu} + 2Q_{\mu\nu}{}^{(5)}\mathfrak{T}_{\mu\nu}) - \tfrac{1}{3} G \frac{d^4 I_{\mu\nu}}{dt^4} \frac{dI_{\mu\nu}}{dt} \\ &+ \tfrac{1}{18} G \frac{d^2 D_{\mu\nu}}{dt^2} \frac{d^3 D_{\mu\nu}}{dt^3} - \tfrac{1}{90} G \frac{d^4 D_{\mu\nu}}{dt^4} \frac{dD_{\mu\nu}}{dt} \Big] . \end{aligned} \tag{167}$$

An alternative form of this expression is

$$\begin{aligned} \int_5 \mathfrak{E} dx = &-\frac{1}{c^5} (Q_{00}{}^{(5)}\mathfrak{T}_{\mu\mu} + 2Q_{\mu\nu}{}^{(5)}\mathfrak{T}_{\mu\nu}) + \frac{G}{90c^5} \frac{d}{dt} \left(3 \frac{d^2 D_{\mu\nu}}{dt^2} \frac{d^2 D_{\mu\nu}}{dt^2} - \frac{d^3 D_{\mu\nu}}{dt^3} \frac{dD_{\mu\nu}}{dt} \right) \\ &+ \frac{G}{6c^5} \frac{d}{dt} \left(\frac{d^2 I_{\mu\nu}}{dt^2} \frac{d^2 I_{\mu\nu}}{dt^2} - 2 \frac{d^3 I_{\mu\nu}}{dt^3} \frac{dI_{\mu\nu}}{dt} \right) . \end{aligned} \tag{168}$$

It is of interest to observe that, if equation (168) is averaged over a long enough interval of time, the terms in the last two lines of the equation may be expected to vanish and we shall be left with

$$\langle \int_5 \mathfrak{E} dx \rangle = -\frac{1}{c^5} \langle Q_{00}{}^{(5)}\mathfrak{T}_{\mu\mu} + 2Q_{\mu\nu}{}^{(5)}\mathfrak{T}_{\mu\nu} \rangle . \tag{169}$$

XIII. CONCLUDING REMARKS

Papers I, II, and III, together with the present one, have carried the solution of Einstein's field equations for a perfect fluid far enough to incorporate, in an explicit set of equations governing the motion of the fluid, the first radiative corrections. It is worth noting that at no stage in the development of the theory was it necessary to make any assumptions beyond those already implicit in the initial choice of the *form* of the energy-momentum tensor and the postulation of the conservation of the rest-mass energy (or, equivalently, the conservation of the baryon number). The entire development manifests once again the marvelous logical simplicity and the inner self-consistency of the general theory of relativity.

One of us (S. C.) is grateful to Professor G. Wentzel and Dr. Kip S. Thorne for many fruitful discussions on topics related to this paper; in particular, Dr. Thorne's alternative ideas on the subject have greatly influenced this paper.

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Notes added in proof by S. C.:

1. Professor Andrzej Trautman, to whom I sent a preprint version of this paper, wrote on October 1, 1969:

"During the last years of his life, Leopold Infeld worked, together with my wife, on the problem of radiation and its connection with that of motion. Among other results, they obtained the correct expression (i.e. in agreement with yours) for the lowest order radiative terms in the metric corresponding to a system of point particles. This is contained in a paper by L. Infeld and R. Trautman about to appear in the Annals of Physics. Your results are certainly more general than theirs."

The papers to which Professor Trautman refers have since appeared: *Annals of Physics*, **55**, 561–575 and 576–586, 1969.

2. Professor Philip C. Peters has similarly drawn my attention to his paper on "Gravitational radiation and the motion of two point masses" (*Phys. Rev.*, **136**, 1224, 1964) in which he has derived expressions (in the context of the two-body problem) for the rates of dissipation of energy and angular momentum in agreement with the linearized theory of gravitational radiation. I regret that I was not aware of Professor Peters's paper.

It will, however, be clear that the object of the present series of papers on post-Newtonian approximations is to have a complete set of post-Newtonian equations and conservation laws derived explicitly to the requisite orders; it does not appear that this completeness has been attempted before.

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