Two elementary proofs of the Wigner theorem on symmetry in quantum mechanics

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In quantum theory, symmetry has to be defined necessarily in terms of the family of unit rays, the state space. The theorem of Wigner asserts that a symmetry so defined at the level of rays can always be lifted into a linear unitary or an antilinear antiunitary operator acting on the underlying Hilbert space. We present two proofs of this theorem which are both elementary and economical. Central to our proofs is the recognition that a given Wigner symmetry can, by post-multiplication by a unitary symmetry, be taken into either the identity or complex conjugation. Our analysis involves a judicious interplay between the effect a given Wigner symmetry has on certain two-dimensional subspaces and the effect it has on the entire Hilbert space.

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I. INTRODUCTION

The Wigner unitary-antiunitary theorem on the representation of symmetry operations in quantum mechanics is an important result belonging to the mathematical foundations of the subject. The dominant role of unitary group representations in quantum physics can be traced to this theorem which states that any (invertible) map of the space of pure states (unit rays) of a quantum system onto itself preserving transition probabilities is induced by either a linear unitary transformation or an antilinear antiunitary transformation at the level of vectors in the complex Hilbert space pertaining to the system. That is, invariance of all transition probabilities demands that all probability amplitudes be either preserved or complex conjugated uniformly.

While the theorem was originally proved by Wigner in 1931 [1], over the decades many authors have presented new proofs, extensions, etc.[2]-[28]. Prominent among these is a proof by Bargmann in 1964 [6] which is extremely elegant and, in a sense, elementary. We also mention some insightful remarks on this subject by Wick in 1966 [7]. Some of these authors cited above view this theorem as a consequence of the fundamental theorem of projective geometry.

The purpose of this paper is to present two elementary

proofs of the Wigner theorem which we believe have some attractive features. We already know that composition of symmetries in the Wigner sense results in new symmetries; and that unitary transformations on Hilbert space do induce Wigner symmetries on the ray space. It is also clear that complex conjugation (in any chosen orthonormal basis) at the Hilbert space level induces a Wigner symmetry on the ray space. Our strategy, which is largely conditioned by our experience in classical wave optics, then in analysing a given Wigner symmetry is to take it to a canonical form by composition with a unitary symmetry naturally suggested by the given Wigner symmetry, and then to examine the resulting (simpler) Wigner symmetry step by step until it becomes completely transparent that the Wigner symmetry in its canonical form is either the identity map or complex conjugation.

In this process of demonstrating that the group of all Wigner symmetries is the union of just two cosets with respect to the unitary group, as we shall see, there is a judicious interplay of 'local' and 'global' aspects, the former involving two-dimensional subspaces of Hilbert space and the latter involving general vectors not restricted to any subspace.

The material of this paper is organised as follows. Section II serves the dual purpose of introducing our notation and making a precise statement of the problem. Certain two-dimensional subspaces of the Hilbert space and their associated Poincaré spheres are defined. Our first proof of the Wigner theorem is presented in Section III in a sequence of six elementary steps. Some comments on the proof are presented in Section IV. A second proof, based on induction in the dimension N of the Hilbert

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space under consideration, is given in Section V, and we conclude in Section VI with further remarks.

II. NOTATIONAL PRELIMINARIES, STATEMENT OF THE PROBLEM

Let \mathcal{H} be the N-dimensional Hilbert space pertaining to some quantum system, with N finite or infinite. Vectors and the inner product are denoted as usual by $|\psi\rangle, |\phi\rangle, \cdots, \langle\phi|\psi\rangle$. The unit sphere \mathcal{B} consists of all vectors of unit length,

$$\mathcal{B} = \{ |\psi\rangle \in \mathcal{H} \mid \langle \psi | \psi \rangle = 1 \} \subset \mathcal{H}. \tag{2.1}$$

For finite N, \mathcal{B} is a manifold of real odd dimension (2N-1). Rays are (equivalence) classes of vectors differing by phases, of the form $\{e^{i\alpha} | \psi\rangle, 0 \leq \alpha < 2\pi\}$. Physical pure states correspond one–to–one to normalised or unit rays (i.e., to one-dimensional projections or density matrices). So we define

$$\mathcal{R} = \{ \rho(|\psi\rangle) = |\psi\rangle\langle\psi| \mid |\psi\rangle \in \mathcal{B} \}, \qquad (2.2)$$

and call it the ray space. For finite N, \mathcal{R} has real even dimension 2(N-1). Neither \mathcal{B} nor \mathcal{R} is a vector space. There is a natural projection $\pi : \mathcal{B} \to \mathcal{R}$ given by

$$\pi: |\psi\rangle \in \mathcal{B} \to \pi(|\psi\rangle) = \rho(|\psi\rangle) \in \mathcal{R}.$$
 (2.3)

Given two rays, we have a 'scalar product'

$$\operatorname{Tr}(\rho(|\psi_1\rangle)\rho(|\psi_2\rangle)) = |\langle \psi_1|\psi_2\rangle|^2 > 0, \qquad (2.4)$$

which has the standard quantum mechanical interpretation as transition probability.

In our first proof of Wigner's theorem, a family of two-dimensional subspaces of \mathcal{H} plays an important role, so we define it now. Let $\{|n\rangle\}$, $n=1,2,\cdots,N$, be an orthonormal basis (ONB) for \mathcal{H} . With respect to this basis, for each pair (j,k) with j < k we define a two-dimensional linear subspace $\mathcal{H}_{jk} \subset \mathcal{H}$:

$$\mathcal{H}_{jk} = \{ \alpha | j \rangle + \beta | k \rangle \mid \alpha, \beta \in \mathcal{C} \} \subset \mathcal{H},$$

$$j, k = 1, 2, \dots, N, \quad j < k. \tag{2.5}$$

The intersection of \mathcal{B} and \mathcal{H}_{jk} is the set of unit vectors in \mathcal{H}_{jk} :

$$\mathcal{B}_{jk} = \mathcal{B} \cap \mathcal{H}_{jk}$$

= $\{ \alpha | j \rangle + \beta | k \rangle | |\alpha|^2 + |\beta|^2 = 1 \} \subset \mathcal{B}. (2.6)$

Upon projection this maps onto a subset $\mathcal{R}_{jk} \subset \mathcal{R}$ which can conveniently be parametrised using spherical polar angles on \mathcal{S}^2 :

$$\mathcal{R}_{jk} = \pi(\mathcal{B}_{jk}) = \{ \rho(|\theta; \phi\rangle_{jk}) \} \subset \mathcal{R},$$

$$|\theta; \phi\rangle_{jk} = \cos\frac{\theta}{2} |j\rangle + \sin\frac{\theta}{2} e^{i\phi} |k\rangle \in \mathcal{B}_{jk},$$

$$0 < \theta < \pi, \ 0 < \phi < 2\pi.$$
(2.7)

Thus each \mathcal{R}_{jk} , j < k, has the form of a Poincaré sphere. A Wigner symmetry (hereafter WS) is a one-to-one, onto (hence invertible) map $\Omega : \mathcal{R} \to \mathcal{R}$ which preserves the 'inner product' (2.4):

$$\Omega : \rho(|\psi\rangle) \in \mathcal{R} \to \Omega(\rho(|\psi\rangle)) \in \mathcal{R},$$

$$\operatorname{Tr}(\Omega(\rho(|\psi_1\rangle))\Omega(\rho(|\psi_2\rangle))) = |\langle \psi_1 | \psi_2 \rangle|^2. (2.8)$$

With any such WS it is convenient to associate a map $\tilde{\Omega}: \mathcal{B} \to \mathcal{R}$ by composing Ω and π using (2.3):

$$\tilde{\Omega} = \Omega \circ \pi : \mathcal{B} \to \mathcal{R} : \ \tilde{\Omega}(|\psi\rangle) = \Omega(\rho(|\psi\rangle)), \ |\psi\rangle \in \mathcal{B}.$$
(2.9)

Then the condition (2.8) on Ω appears slightly simpler:

$$\operatorname{Tr}(\tilde{\Omega}(|\psi_1\rangle)\tilde{\Omega}(|\psi_2\rangle)) = |\langle \psi_1 | \psi_2 \rangle|^2. \tag{2.10}$$

We hereafter refer to this (or equivalently (2.8)) as the symmetry condition or SC. Clearly the composition $\Omega_1 \circ \Omega_2$ of two WS's is another WS.

Every unitary transformation U on \mathcal{H} leads to an associated WS \mathcal{U} by conjugation of density matrices:

$$U: \mathcal{H} \to \mathcal{H}: ||U|\psi\rangle || = |||\psi\rangle || \Rightarrow$$

$$U: \mathcal{R} \to \mathcal{R}: \mathcal{U}(\rho(|\psi\rangle)) = \rho(U|\psi\rangle)$$

$$= U\rho(|\psi\rangle)U^{-1}. (2.11)$$

Therefore given a WS Ω and a unitary operator U on \mathcal{H} , by composition in either order we get new WS's:

$$\Omega' = \mathcal{U} \circ \Omega, \quad \tilde{\Omega}' = \mathcal{U} \circ \tilde{\Omega};
\Omega'' = \Omega \circ \mathcal{U}, \quad \tilde{\Omega}'' = \Omega \circ \tilde{\mathcal{U}}.$$
(2.12)

We will use only the former in our considerations.

We can now state the problem: Given a WS Ω : $\mathcal{R} \to \mathcal{R}$, can we find or construct a one-to-one onto map ω : $\mathcal{B} \to \mathcal{B}$ such that

$$\tilde{\Omega} = \Omega \circ \pi = \pi \circ \omega \tag{2.13}$$

corresponding to the diagram

$$\begin{array}{cccc}
 & \omega? \\
 & \xrightarrow{} & \mathcal{B} \\
 & \pi & \downarrow & \downarrow & \pi & ? \\
 & \mathcal{R} & \xrightarrow{} & \mathcal{R}
\end{array} (2.14)$$

If such ω exists, we ask if it can be extended in a natural way from \mathcal{B} to \mathcal{H} , and in that case how it acts on a general vector $|\psi\rangle \in \mathcal{H}$.

Wigner's Theorem states that this is always possible, and that ω extended to \mathcal{H} is either a linear unitary operator U, so $\Omega = \mathcal{U}$; or an antilinear antiunitary operator, namely $\omega = KU$ where K is complex conjugation in some orthonormal basis $\{|n\rangle\}$ and U is unitary. In the latter case we can also express ω as the product U'K with U' = KUK also linear unitary.

It is useful to make two remarks before we present our proofs of Wigner's theorem:

Let us denote by $\{\mathcal{U}\}$ the set of symmetries induced by unitary transformations U on the Hilbert space, and by $\{\mathcal{UK}\}$ the set of symmetries induced by antilinear antiunitary transformations UK on the Hilbert space. The union $\{\mathcal{U}\}\cup\{\mathcal{UK}\}$ thus constitutes a set of obvious symmetries in the sense of Wigner. Wigner's theorem is the assertion that there are no more symmetries beyond these obvious ones.

Maps which are positive but not completely positive are fundamental to quantum information theory. Our second remark is concerning the implication of Wigner's theorem in this context: $\{\mathcal{U}\}$ is the set of all completely positive maps which map the family of pure states onto itself. Since complex conjugation $\{\mathcal{K}\}$ and matrix transposition $\{\mathcal{T}\}$ are equivalent maps at the level of density operators we have: $\{\mathcal{UT}\}$ is the set of all positive but not completely positive maps which map the family of pure states onto itself. That is, any other positive but not completely positive map will take some pure states into mixed states.

III. FIRST PROOF OF WIGNER'S THEOREM

Let a WS Ω with its associated map $\tilde{\Omega}: \mathcal{B} \to \mathcal{R}$ be given. It is clear from eqs. (2.8), (2.9) that for any $|\psi\rangle \in \mathcal{B}$ we have

$$\tilde{\Omega}(|\psi\rangle) \equiv \Omega(\pi(|\psi\rangle)) = \pi(|\psi'\rangle), \tag{3.1}$$

where $|\psi'\rangle \in \mathcal{B}$ is determined upto a phase. We use this fact repeatedly in the following. The proof is made up of six steps, each quite elementary. We now present them in sequence.

Step 1: Choose some (any) ONB $\{|n\rangle\}$ for \mathcal{H} . From the SC (2.10) for pairs of basis vectors and eq. (3.1) it follows that

$$\tilde{\Omega}(|n\rangle) = \pi(|n;\Omega\rangle), \ n = 1, 2, \dots, N,$$
 (3.2)

where each $|n;\Omega\rangle$ is determined upto a phase and the collection $\{|n;\Omega\rangle\}$ also is an ONB for \mathcal{H} . Make some (any) choices for the vectors $|n;\Omega\rangle$ and define a unitary transformation U on \mathcal{H} by

$$U|n;\Omega\rangle = |n\rangle, \ n = 1, 2, \cdots, N.$$
 (3.3)

We now define a WS Ω' by

$$\Omega' = \mathcal{U} \circ \Omega, \quad \tilde{\Omega}' = \mathcal{U} \circ \tilde{\Omega},$$
 (3.4)

and reduce the analysis of Ω to that of Ω' . This WS has a simple action on the vectors $\{|n\rangle\}$, namely,

$$\tilde{\Omega}'(|n\rangle) = \pi(|n\rangle), \ n = 1, 2, \cdots, N, \tag{3.5}$$

and by the SC (2.10) for $|n\rangle$ and general $|\psi\rangle \in \mathcal{B}$ we have

$$\tilde{\Omega}'(\sum_{n=1}^{N} c_n |n\rangle) = \pi(\sum_{n=1}^{N} c'_n |n\rangle), |c'_n| = |c_n|.$$
 (3.6)

To repeat, the symmetry $\tilde{\Omega}'$ leaves invariant the standard set of orthonormal rays $\{\pi(|n\rangle)\}$, and this can be arranged for any Wigner symmetry Ω .

We hasten to add that $\Omega' = \mathcal{U} \circ \Omega$ is not yet the intended canonical form for Ω . The reason for this is the fact that the unitary operator U could have been postmultiplied by any diagonal unitary operator, still leaving invariant the standard set of orthonormal rays. The canonical form for Ω will indeed get fixed once we exercise (and exhaust) this freedom in Step 3.

Step 2: Next we limit ourselves to vectors in \mathcal{B}_{jk} for a chosen j < k, and study the action of $\tilde{\Omega}'$ on such vectors. It is in fact adequate to look at the action on a (latitude) circle of vectors $|\theta_0;\phi\rangle_{jk}$, eq. (2.7), for fixed $\theta_0 \in (0,\pi)$ and varying $\phi \in [0,2\pi)$. From eq. (3.6) we see that

$$\tilde{\Omega}'(|\theta_0;\phi\rangle_{jk}) = \pi(|\theta_0;\phi'\rangle_{jk}),\tag{3.7}$$

with θ_0, j, k unchanged and ϕ' dependent on ϕ (and possibly also on θ_0, j, k) in an invertible manner. Since

$$|j_{k}\langle\theta_{0};\phi_{1}|\theta_{0};\phi_{2}\rangle_{jk}|^{2} = 2\cos^{2}\frac{\theta_{0}}{2}\sin^{2}\frac{\theta_{0}}{2}\cos(\phi_{1}-\phi_{2}) + \cos^{4}\frac{\theta_{0}}{2}+\sin^{4}\frac{\theta_{0}}{2}, \quad (3.8)$$

use of the SC (2.10) for this pair of vectors on the latitude circle θ_0 shows that ϕ_1, ϕ_2 are carried by (3.7) into ϕ_1', ϕ_2' such that

$$\cos(\phi_1' - \phi_2') = \cos(\phi_1 - \phi_2). \tag{3.9}$$

It follows that the change $\phi \to \phi'$ in eq. (3.7) caused by $\tilde{\Omega}'$ action is of the form

$$\phi' = \phi_{jk} + \epsilon_{jk}\phi, \ \phi_{jk} \in [0, 2\pi), \ \epsilon = \pm 1.$$
 (3.10)

(This and similar later equations are understood to be valid mod 2π .) Thus the action (3.7) reads

$$\tilde{\Omega}'(|\theta_0;\phi\rangle_{jk}) = \pi(|\theta_0;\phi_{jk} + \epsilon_{jk}\phi\rangle_{jk}). \tag{3.11}$$

For fixed θ_0, j, k this means that $\tilde{\Omega}'$ acts on ϕ via some element of the group O(2): ϕ_{jk} denotes an SO(2) element, while ϵ_{jk} determines whether we have a proper or improper rotation.

Step 3: We now use the freedom, noted at the end of Step 1, to multiply each vector $|n\rangle$ in the ONB $\{|n\rangle\}$ by an independent phase factor, preserving the structure of the results obtained upto this point. Thus we define a (diagonal) unitary transformation U' on \mathcal{H} by

$$U'|n\rangle = e^{-i\phi_n}|n\rangle, \ n = 1, 2, \dots, N,$$
 (3.12)

and pass from the WS Ω' to the (final) WS Ω'' by

$$\Omega'' = \mathcal{U}' \circ \Omega', \quad \tilde{\Omega}'' = \mathcal{U}' \circ \tilde{\Omega}'.$$
 (3.13)

This helps us simplify the phases or SO(2) angles ϕ_{jk} in eq. (3.11) to some extent by suitable choices of the ϕ_n . We have :

$$\tilde{\Omega}''(|\theta_0;\phi\rangle_{jk}) = \tilde{\mathcal{U}}'(|\theta_0;\phi_{jk} + \epsilon_{jk}\phi\rangle_{jk})
= \pi(|\theta_0;\phi_{jk} + \phi_j - \phi_k + \epsilon_{jk}\phi\rangle_{jk}).$$
(3.14)

Remembering that $k \geq 2$, we choose

$$\phi_1 = 0, \ \phi_n = \phi_{1n} \quad \text{for } n \ge 2.$$
 (3.15)

Then we have

$$\tilde{\Omega}''(|\theta_0;\phi\rangle_{jk}) = \pi(|\theta_0;\phi'_{jk} + \epsilon_{jk}\phi\rangle_{jk}),$$

$$\phi'_{jk} = \phi_{jk} + \phi_j - \phi_k,$$

$$\phi'_{1k} = 0, \quad k \ge 2.$$
(3.16)

Compared to eq. (3.11), the sign factors ϵ_{jk} are unchanged, while the SO(2) angles ϕ_{jk} have been simplified to ϕ'_{jk} with $\phi'_{1k} = 0$.

Step 4: Since Ω'' is related to Ω' by the diagonal unitary transformation U', eq. (3.11), it is clear that the structure of eq. (3.6) is retained for $\tilde{\Omega}''$:

$$\tilde{\Omega}''(\sum_{n=1}^{N} c_n |n\rangle) = \pi(\sum_{n=1}^{N} c_n'' |n\rangle), \quad |c_n''| = |c_n|. \quad (3.17)$$

Now we show that in the action (3.16) by $\tilde{\Omega}''$, not only is $\phi'_{1k} = 0$ but in fact $\phi'_{jk} = 0$ for all j < k. This is a very important consequence of the SC (2.10). We choose a single special 'real' (normalized) vector $\mathbf{r}^{(0)} = (r_1^{(0)}, r_2^{(0)}, r_3^{((0)}, \cdots)^T$, and define $|\mathbf{r}^{(0)}\rangle \in \mathcal{B}$ with the following form:

$$|\mathbf{r}^{(0)}\rangle = \sum_{n=1}^{N} r_n^{(0)} |n\rangle, \quad r_n^{(0)} \text{ real and } \neq 0.$$
 (3.18)

Under $\tilde{\Omega}''$ action we have, from eq. (3.17),

$$\tilde{\Omega}''(|\mathbf{r}^{(0)}\rangle) = \pi(\sum_{n=1}^{N} r_n^{(0)} e^{i\eta_n} |n\rangle),$$
 (3.19)

for some phases η_n . We now invoke the SC (2.10) for the pair of vectors $|\theta_0;\phi\rangle_{jk}$, $|\boldsymbol{r}^{(0)}\rangle$, any j < k, under $\tilde{\Omega}''$ action to get:

$$|r_j^{(0)} e^{i\eta_j} \cos \frac{\theta_0}{2} + r_k e^{i\eta_k} \sin \frac{\theta_0}{2} e^{-i(\phi'_{jk} + \epsilon_{jk}\phi)}|^2$$
$$= |r_j^{(0)} \cos \frac{\theta_0}{2} + r_k^{(0)} \sin \frac{\theta_0}{2} e^{-i\phi}|^2.$$

That is,

$$\cos(\epsilon_{jk}\phi + \phi'_{jk} + \eta_j - \eta_k) = \cos\phi, \ 0 \le \phi < 2\pi. \ (3.20)$$

This implies

$$\phi'_{ik} = \eta_k - \eta_j, \ j < k. \tag{3.21}$$

For j = 1, from eq.(3.16) we have $\eta_k = \eta_1$ independent of k. Putting this back into eq. (3.21) gives $\phi'_{jk} = 0$ for all j < k. Thus the actions (3.16), (3.20) simplify to

$$\tilde{\Omega}''(|\theta_0;\phi\rangle_{jk}) = \pi(|\theta_0;\epsilon_{jk}\phi\rangle_{jk}),
\tilde{\Omega}''(|\mathbf{r}^{(0)}\rangle) = \pi(|\mathbf{r}^{(0)}\rangle).$$
(3.22)

For each pair j < k, only a sign factor ϵ_{jk} remains. We will soon prove that this factor cannot depend on j, k.

Step 5: Let us next invoke the SC (2.10) under Ω'' for the pair of vectors $|c\rangle$, $|\theta_0;\phi\rangle_{jk}$ where $|c\rangle$ is the general normalised linear combination occurring on the left hand sides of eqs. (3.6), (3.17). On the basis of the results (3.17), (3.22) we have:

$$|c_{j}''\cos\frac{\theta_{0}}{2} + c_{k}''\sin\frac{\theta_{0}}{2} e^{-i\epsilon_{jk}\phi}|^{2}$$

$$= |c_{j}\cos\frac{\theta_{0}}{2} + c_{k}\sin\frac{\theta_{0}}{2} e^{-i\phi}|^{2},$$
i.e., $c_{j}''c_{k}''^{*}e^{i\epsilon_{jk}\phi} + c_{j}''^{*}c_{k}''e^{-i\epsilon_{jk}\phi}$

$$= c_{j}c_{k}^{*}e^{i\phi} + c_{j}^{*}c_{k}e^{-i\phi}, \quad \phi \in [0, 2\pi).$$
(3.23)

Therefore the transition $\{c_n\} \to \{c_n''\}$ must follow

$$c_{j}''c_{k}''^{*} = c_{j}c_{k}^{*} \text{ if } \epsilon_{jk} = +1,$$

 $c_{j}''c_{k}''^{*} = c_{j}^{*}c_{k} \text{ if } \epsilon_{jk} = -1.$ (3.24)

Step 6: This is the final step in the proof. We show that consistency demands that the choice of ϵ_{jk} cannot depend on the pair j, k; it must be uniformly +1 or uniformly -1 for all pairs. Choose any j, k, ℓ with $j < k < \ell$. For any vector $|c\rangle \in \mathcal{B}$, we have the elementary result

$$c_j c_k^* c_k c_l^* (c_j c_\ell^*)^* = |c_j c_k c_\ell|^2 = \text{real} \ge 0.$$
 (3.25)

Therefore if $|c\rangle$ is taken by $\tilde{\Omega}''$ action to $|c''\rangle$ as in eq. (3.17) we must necessarily have

$$c_j''c_k''^* \cdot c_k''c_\ell''^* \cdot (c_j''c_\ell''^*)^* = \text{real} \ge 0.$$
 (3.26)

Depending on the values of ϵ_{jk} , $\epsilon_{k\ell}$, $\epsilon_{j\ell}$, by eq. (3.24) this requires that, whatever $\{c_n\}$ may be

$$(c_j c_k^* \text{ or } c_j^* c_k) \cdot (c_k c_\ell^* \text{ or } c_k^* c_\ell) \cdot (c_j c_l^* \text{ or } c_j^* c_\ell)^* = \text{real} \ge 0.$$
(3.27)

In each factor we have the first expression for $\epsilon=1$, the second for $\epsilon=-1$. It is now immediate that if $\epsilon_{jk}=\epsilon_{k\ell}=\epsilon_{j\ell}=+1$, or if $\epsilon_{jk}=\epsilon_{k\ell}=\epsilon_{j\ell}=-1$, this condition is obeyed. But in every other case, the left hand side of

eq. (3.27) is an expression which is in general complex. Therefore we have the final result:

$$\tilde{\Omega}''(\sum_{n=1}^{N} c_n |n\rangle) = \pi(\sum_{n=1}^{N} c_n |n\rangle) \text{ if all } \epsilon_{jk} = +1,$$

$$= \pi(\sum_{n=1}^{N} c_n^* |n\rangle) \text{ if all } \epsilon_{jk} = -1. (3.28)$$

These are the only consistent possibilities. For action on vectors in \mathcal{B} , the originally given WS Ω is thus either the product $\mathcal{U}^{-1} \circ \mathcal{U}'^{-1}$ of unitary WS's or the product $\mathcal{U}^{-1} \circ \mathcal{U}'^{-1} \circ \mathcal{K}$, where \mathcal{K} is the WS corresponding to complex conjugation in the ONB $\{|n\rangle\}$. These immediately extend from \mathcal{B} to \mathcal{H} in the natural ways, thus establishing Wigner's Theorem.

IV. EXTENSIONS OF WS ACTIONS ON SPECIAL VECTORS

In the course of the proof of Wigner's theorem given in the preceding Section, important roles were played by the specially chosen vectors $|\theta_0;\phi\rangle_{jk} \in \mathcal{B}_{jk} \subset \mathcal{H}_{jk}$ in each two-dimensional subspace defined by eqs. (2.5), (2.6); and the 'real' vector $|\mathbf{r}^{(0)}\rangle$ of eq. (3.18). We emphasize that only one (latitude) circle of vectors in \mathcal{B}_{jk} , and a single 'real' vector $|\mathbf{r}^{(0)}\rangle$, were actually used in the proof. However it is easy to show that, at corresponding steps of the proof, Steps 2 and 4, we can obtain more information about the actions of $\tilde{\Omega}'$, $\tilde{\Omega}''$ respectively.

Consider two values θ_0, θ'_0 of the polar angle over S^2 and let eq. (3.11) in the two cases read

$$\tilde{\Omega}'(|\theta_0;\phi\rangle_{jk}) = \pi(|\theta_0;\phi_{jk}+\epsilon_{jk}\phi\rangle_{jk}),
\tilde{\Omega}'(|\theta'_0;\phi'\rangle_{jk}) = \pi(|\theta'_0;\phi'_{jk}+\epsilon'_{jk}\phi'\rangle_{jk}).$$
(4.1)

The SC (2.10) for the two vectors on the left here is:

$$|\cos\frac{\theta_0}{2}\cos\frac{\theta_0'}{2} + \sin\frac{\theta_0}{2}\sin\frac{\theta_0'}{2} e^{i(\epsilon_{jk}\phi + \phi_{jk} - \epsilon_{jk}'\phi' - \phi_{jk}')}|^2$$
$$= |\cos\frac{\theta_0}{2}\cos\frac{\theta_0'}{2} + \sin\frac{\theta_0}{2}\sin\frac{\theta_0'}{2} e^{i(\phi - \phi')}|^2.$$

That is,

i.e.,
$$\cos(\phi - \phi') = \cos(\epsilon_{jk}\phi - \epsilon'_{jk}\phi' + \phi_{jk} - \phi'_{jk}),$$

$$0 \le \phi, \ \phi' < 2\pi. \quad (4.2)$$

This immediately implies

$$\epsilon'_{jk} = \epsilon_{jk}, \ \phi'_{jk} = \phi_{jk},$$
 (4.3)

so in fact we have in place of eq. (3.11):

$$\tilde{\Omega}'(|\theta;\phi\rangle_{jk}) = \pi(|\theta;\phi_{jk} + \epsilon_{jk}\phi\rangle_{jk}),$$

$$0 < \theta < \pi, \quad 0 < \phi < 2\pi. \quad (4.4)$$

with ϕ_{jk} , ϵ_{jk} constant over S^2 .

Next, in connection with the result at Step 4, we easily see that we have the wider result

$$|\mathbf{r}\rangle = \sum_{n=1}^{N} r_n |n\rangle \in \mathcal{B}, \ r_n \text{ real, and } r_1 \neq 0$$

$$\Rightarrow \tilde{\Omega}''(|\mathbf{r}\rangle) = \pi(|\mathbf{r}\rangle)$$
(4.5)

for all 'real' vectors of this form.

V. SECOND PROOF OF WIGNER'S THEOREM

The crucial step in the proof of the Wigner theorem given in Section III is the first one – the passage from Ω to Ω' in eq. (3.4), resulting in eqs. (3.5), (3.6). That this is so is clearly brought out by the second proof we now present, based on induction in the dimension of the Hilbert space.

For a two-dimensional quantum system, as is so well known and used earlier, the space \mathcal{R} is the Poincaré sphere \mathcal{S}^2 . As in eq. (2.7), each pure state density matrix corresponds one—to—one to a point on this sphere, and we have:

$$\hat{\boldsymbol{n}} \in \mathcal{S}^2 \to \rho(\hat{\boldsymbol{n}}) = \frac{1}{2} (1 + \hat{\boldsymbol{n}} \cdot \boldsymbol{\sigma}),$$

$$\operatorname{Tr}(\rho(\hat{\boldsymbol{n}}_1)\rho(\hat{\boldsymbol{n}}_2)) = \frac{1}{2} (1 + \hat{\boldsymbol{n}}_1 \cdot \hat{\boldsymbol{n}}_2). \tag{5.1}$$

(Orthogonal states correspond to antipodal points). Thus a WS in this case is a one–to–one onto map $\Omega: \mathcal{S}^2 \to \mathcal{S}^2$ preserving angles between pairs of points:

$$\Omega(\hat{\boldsymbol{n}}_1) \cdot \Omega(\hat{\boldsymbol{n}}_2) = \hat{\boldsymbol{n}}_1 \cdot \hat{\boldsymbol{n}}_2. \tag{5.2}$$

As is very well known, such maps are either proper rotations belonging to SO(3), induced by two dimensional unitary transformations of U(2) on the underlying two-dimensional Hilbert space $\mathcal{H}^{(2)}$; or they are improper rotations in O(3), involving in addition complex conjugation on $\mathcal{H}^{(2)}$ (mirror reflection on \mathcal{S}^2). This is Wigner's theorem in this case.

For general finite dimension we use induction. We assume the theorem is true in N dimensions, then prove it for (N+1) dimensions. Let $\mathcal{H}^{(N+1)}$ be an (N+1) dimensional Hilbert space, and Ω a WS for the corresponding quantum system. Choose any ONB $\{|j\rangle, j=1,2,\cdots,N+1\}$ for $\mathcal{H}^{(N+1)}$. As in the initial steps of Section III, pass from Ω to another WS Ω' by composition with a suitable unitary symmetry. Then as in eqs. (3.5), (3.6) we have:

$$\tilde{\Omega}'(|j\rangle) = \pi(|j\rangle), \quad j = 1, 2, \dots, N+1;
\tilde{\Omega}'(\sum_{j=1}^{N+1} c_j|j\rangle) = \pi(\sum_{j=1}^{N+1} c'_j|j\rangle), \quad |c_j| = |c'_j|. (5.3)$$

Let us identify the N-dimensional subspace $\mathcal{H}^{(N)} \subset \mathcal{H}^{(N+1)}$ as the subspace spanned by the first N basis vectors:

$$\mathcal{H}^{(N)} = \text{Sp}\{|j\rangle, j = 1, 2, \dots, N\} \subset \mathcal{H}^{(N+1)}.$$
 (5.4)

By eq.(5.3), Ω' is a WS for $\mathcal{H}^{(N)}$. By our assumption, may be after composition with a diagonal unitary symmetry which we suppress for simplicity, we have either

$$\tilde{\Omega}'(\sum_{j=1}^{N} c_j |j\rangle) = \pi(\sum_{j=1}^{N} c_j |j\rangle), \quad \forall \ \boldsymbol{c} \equiv (c_1, \cdots, c_N)^T,$$

or

$$\tilde{\Omega}'(\sum_{j=1}^{N} c_j |j\rangle) = \pi(\sum_{j=1}^{N} c_j^* |j\rangle), \quad \forall \ \boldsymbol{c}.$$
 (5.5)

Suppose the first choice holds. To deal with $\mathcal{H}^{(N+1)}$, a general vector here is

$$|c\rangle + c_{N+1}|N+1\rangle \in \mathcal{H}^{(N+1)},$$

 $|c\rangle = \sum_{j=1}^{N} c_j|j\rangle \in \mathcal{H}^{(N)}.$ (5.6)

Let

$$\tilde{\Omega}'(|c\rangle + c_{N+1}|N+1\rangle) = \pi(|c'\rangle + c'_{N+1}|N+1\rangle), |c'_{j}| = |c_{j}|, \quad j = 1, 2, \dots, N+1. \quad (5.7)$$

The SC (2.10) for a general $|\mathbf{c}^{(1)}\rangle \in \mathcal{H}^{(N)}$ and the vector (5.6) in $\mathcal{H}^{(N+1)}$ yields

$$|\sum_{j=1}^{N} c_{j}^{(1)*} c_{j}'|^{2} = |\sum_{j=1}^{N} c_{j}^{(1)*} c_{j}|^{2}, \quad \forall \mathbf{c}^{(1)}$$

$$\Rightarrow \mathbf{c}^{(1)\dagger} \mathbf{c}' \mathbf{c}'^{\dagger} \mathbf{c}^{(1)} = \mathbf{c}^{(1)\dagger} \mathbf{c} \mathbf{c}^{\dagger} \mathbf{c}^{(1)}, \quad \forall \mathbf{c}^{(1)}$$

$$\Rightarrow \mathbf{c}' = (\text{phase}) \mathbf{c}. \tag{5.8}$$

Then eq. (5.7) simplifies to

$$\tilde{\Omega}'(|\mathbf{c}\rangle + c_{N+1}|N+1\rangle) = \pi(|\mathbf{c}\rangle + c_{N+1}''|N+1\rangle), |c_{N+1}''| = |c_{N+1}|.$$
 (5.9)

Next the SC for two vectors in $\mathcal{H}^{(N+1)}$ yields:

$$\begin{aligned} |\boldsymbol{c}^{(1)}\rangle + c_{N+1}^{(1)}|N+1\rangle, \quad |\boldsymbol{c}^{(2)}\rangle + c_{N+1}^{(2)}|N+1\rangle &\in \mathcal{H}^{(N+1)}, \\ |\boldsymbol{c}^{(1)\dagger}\boldsymbol{c}^{(2)} + c_{N+1}^{(1)\prime\prime\ast}c_{N+1}^{(2)\prime\prime}|^2 &= |\boldsymbol{c}^{(1)\dagger}\boldsymbol{c}^{(2)} + c_{N+1}^{(1)\ast}c_{N+1}^{(2)}|^2, \\ &\forall \boldsymbol{c}^{(1)}, \boldsymbol{c}^{(2)}. \end{aligned}$$

That is,

$$c_{N+1}^{(1)"*}c_{N+1}^{(2)"} = c_{N+1}^{(1)*}c_{N+1}^{(2)}. (5.10)$$

So

$$c_{N+1}'' = e^{i\phi}c_{N+1},$$

$$\tilde{\Omega}'(|\mathbf{c}\rangle + c_{N+1}|N+1\rangle) = \pi(|\mathbf{c}\rangle + e^{i\phi}c_{N+1}|N+1\rangle),$$
(5.11)

the phase ϕ being independent of $|c\rangle$. Then by a diagonal unitary phase transformation we can pass to a WS $\tilde{\Omega}''$ which acts trivially on $\mathcal{H}^{(N+1)}$. This proves, in this case, that validity of Wigner's theorem in N dimensions implies its validity in (N+1) dimensions.

For the second option in (5.5), a similar argument holds. Validity of Winger's theorem for N=2 is evident as noted in the opening paragraphs of this Section. Hence, by induction, proof of Wigner' theorem is complete for all $N=2, 3, \cdots$.

We conclude this Section with some additional remarks on the infinite-dimensional case. For a WS Ω on a Hilbert space \mathcal{H} of infinite dimension, choose an ONB $\{|n\rangle, n=1, 2, \cdots\}$, denote by ψ a column vector $(\psi_1, \psi_2, \psi_3, \cdots)^T$, and pass to Ω' such that

$$\tilde{\Omega}'(|n\rangle) = \pi(|n\rangle), \quad n = 1, 2, \cdots,
\tilde{\Omega}'(|\psi\rangle) = \sum_{n=1}^{\infty} \psi_n |n\rangle) = \pi(|\psi'\rangle) = \sum_{n=1}^{\infty} \psi'_n |n\rangle);
|\psi'_n| = |\psi_n|.$$
(5.12)

This ONB gives a sequence of subspaces $\mathcal{H}^{(2)} \subset \mathcal{H}^{(3)} \cdots \subset \mathcal{H}^{(N)} \subset \mathcal{H}^{(N+1)} \cdots$. So as in the above we can pass from the WS Ω' to an Ω'' whose action on $\mathcal{H}^{(N)}$ is either trivial for all finite N or is complex conjugation for all finite N. Now the SC for $|c\rangle \in \mathcal{H}^{(N)}$, $|\psi\rangle \in \mathcal{H}$ in the trivial case is

$$\tilde{\Omega}''(|\mathbf{c}\rangle \text{ or } |\psi\rangle) = \pi(|\mathbf{c}\rangle \text{ or } |\psi'\rangle),$$

$$|\psi'_n| = |\psi_n|, \ \forall n;$$

$$\mathbf{c}^{\dagger}\psi'\psi'^{\dagger}\mathbf{c} = \mathbf{c}^{\dagger}\psi\psi^{\dagger}\mathbf{c}.$$
(5.13)

While $\psi \psi^{\dagger}$ is in general infinite dimensional, (5.13) holds for all c for all finite N, hence

$$\psi' = (\text{phase})\,\psi,\tag{5.14}$$

and the triviality of $\tilde{\Omega}''$ action is established on \mathcal{H} . A similar argument holds when complex conjugation on all $\mathcal{H}^{(N)}$ is needed.

VI. CONCLUDING REMARKS

We hope to have provided new elementary proofs of Wigner's unitary-antiunitary theorem on the representation of symmetry operations in quantum mechanics which are both elementary and economical on the one hand, and effectively combine 'local' and 'global' aspects on the other. By the latter we mean that while for the most part we deal with the action of a Wigner Symmetry on vectors in certain two-dimensional subspaces of the full Hilbert space \mathcal{H} , at all stages there is a clear and evolving understanding of the action on general vectors in \mathcal{H} . This may well be contrasted with the well known and extremely elegant proof by Bargmann; there,

at every stage, the arguments work with just two or three vectors, so to that extent, the global picture seems missing

In connection with Steps 3 and 4 of the proof in Section III, the following comment may be made. At the start of Step 3, there is a 'matrix' of phases ϕ_{jk} still to be examined; and the diagonal unitary transformation U' involves a much smaller number of phases ϕ_n . Therefore it is understandable that by Step 3, which does not use the SC (2.10), it is only possible to transform the phases ϕ_{1k} to zero. Step 4, however, by use of the SC (2.10) is able to show that after the action of U', all the phases ϕ_{jk} have been transformed to zero. This seems to be a particular feature of the present first proof, bringing out the power of the SC (2.10) in a direct fashion.

The second proof of Section V is different in spirit from the first proof, though both rest on the idea of passing from Ω to Ω' . This is reminiscent of 'passing to the rest frame' or to a 'local inertial frame' in relativistic problems. After this first step, however, the two proofs are structured differently, though both are quite elementary.

Finally, it is clear that the set of all Wigner symmetries of a system of Hilbert space dimension N forms a group. This group is not twice as large as U(N) or SU(N). The centre Z_N of SU(N) has N elements, all of which are multiples of identity; and these elements leave every ray invariant. Thus the relevant group is not SU(N), but the quotient $SU(N)/Z_N$; and the adjoint representation of SU(N) is indeed a faithful representation of this quotient group. [It is again in view of this nontrivial centre that U(N) is not $SU(N) \times U(1)$ but the quotient $(SU(N) \times U(N))$ $U(1)/Z_N$. It is well known, and already noted at the beginning of Section V, that in the case N=2 the group of Wigner symmetries is the union of two copies of SO(3). The point being made is that this feature is true for all N, with $SO(3) = SU(2)/Z_2$ replaced by $SU(N)/Z_N$. Acknowledgement: The authors would like to thank P.P. Divakaran for many insightful discussions and, in particular, for drawing the authors' attention to the work of Wick[7]. V. Srinivasan would like to acknowledge an Emeritus Fellowship of the University Grants Commission which made this work possible.

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