Canonical Partition Functions for Parastatistical Systems of any order

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Abstract

A general formula for the canonical partition function for a system obeying any statistics based on the permutation group is derived. The formula expresses the canonical partition function in terms of sums of Schur functions. The only hitherto known result due to Suranyi [Phys. Rev. Lett. **65**, 2329 (1990)] for parasystems of order two is shown to arise as a special case of our general formula. Our results also yield all the relevant information about the structure of the Fock spaces for parasystems.

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There exist two approaches to parastatistics. The first is the field theoretic approach[1−3] based on a generalization of the creation-annihilation operator algebra for bosons and fermions. This is the way in which parastatistics was first introduced by Green.[1] In the second approach pioneered by Messiah and Greenberg[4] and further investigated by Hartle Stolt and Taylor,[5] parastatistics arises in the quantum mechanical description of an assembly of N-identical particles. Here the permutation group S_N plays a central role. Of the two, for calculational purposes, the first seems to have found greater favour with the workers in this field.[6,7] Thus, for instance, using this approach Suranyi[7] has given the canonical partition function for a parabose and parafermi gas of order $p = 2$. The calculation involves a clever use of the simplifications which occur in the para algebra when the order of the parastatistics is two. In this letter we show that the quantum mechanical approach to parastatistics when combined with the machinery of symmetric functions[8] yields a powerful method which enables one to answer, with great facility, all questions pertaining to statistics based on the permutation group. In particular, we give the canonical partition function for a parabose or parafermi system of an arbitrary order.

To set the notation we begin with a brief summary of some familiar results. This is also necessitated by the fact that it is an in-depth appreciation and examination of what is all too familiar along with the intution gained in the process which leads us, in one stroke to the desired results. Consider a Hilbert space $\mathcal{H}$ built by an N-fold tensor product of a Hilbert space H of dim M. We shall assume that $M \geq N$. Let $1, 2, 3, \cdots, M$ denote the basis vectors of H. The M^N basis vectors of $\mathcal{H}$ correspond to each term in the product

$$\frac{(1+2+\cdots+M)(1+2+\cdots+M)\cdots(1+2+\cdots+M)}{N \text{ factors}} \quad . \tag{1}$$

One may consider two decompositions of this set of M^N states.

1. *Decomposition based on occupation numbers*:

Here one groups together states which have the same number of 1's, 2's $\cdots$ etc., regardless of their location in the product. Each such group is characterized by a set of occupation numbers which give the number of times $1, 2, \cdots M$ occur in the states in that group. All relevant aspects of this decomposition are encapsulated in the following decomposition of the symmetric function $(x_1 + \cdots + x_M)^N$,

$$Z_N^{inf}(x_1, \cdots, x_M) \equiv (x_1 + x_2 \cdots + x_M)^N = \sum_{\substack{\lambda \\ |\lambda|=N}} \frac{N!}{\lambda_1! \cdots \lambda_M!} \, m_\lambda(x_1, \cdots, x_M) \quad . \tag{2}$$

Here $\lambda \equiv (\lambda_1, \lambda_2, \cdots, \lambda_M)$, $\lambda_1 \geq \lambda_2 \geq \lambda_3 \cdots \geq \lambda_M$ is a partition of N (indicated by $|\lambda| = N$) and $m_\lambda(x_1, \cdots, x_M)$ denotes the monomial symmetric function[8] corresponding to the partition λ

$$m_\lambda(x_1, \cdots, x_M) = \sum x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_M^{\lambda_M} \quad . \tag{3}$$

The sum on the R.H.S. of (3) is over all distinct permutations of $(\lambda_1, \cdots, \lambda_M)$. Setting $x_1 = x_2 = x_M = 1$ in (2) we obtain

$$M^N = \sum_\lambda \frac{N!}{\lambda_1 \ \cdots \lambda_M!} \, m_\lambda(1, \cdots, 1) \quad , \tag{4}$$

which tells us that each partition $\lambda = (\lambda_1, \cdots, \lambda_M)$ corresponds to $m_\lambda(1, \cdots, 1)$ sets of occupation numbers obtained by distinct permutations of λ_i's and each such set contains $N!/\lambda_1! \cdots \lambda_M!$ states. The number $m_\lambda(1, \cdots, 1)$ is given by

$$m_\lambda(1, 1, \cdots, 1) = \frac{M!}{m_1! m_2! \cdots} \quad , \tag{5}$$

where m_i's denote the number of times λ_i's occur in the given partition λ.

Note also that with the identification

$$x_1 = e^{-\beta\epsilon_1} \quad , \quad x_2 = e^{-\beta\epsilon_2}, \cdots, x_M = e^{-\beta\epsilon_M} \quad , \tag{6}$$

where $\epsilon_1, \epsilon_2 \cdots, \epsilon_M$ are taken to denote the "single particle energies" corresponding to the states $1, \cdots, M$, the symmetric function $Z_N^{inf}(x_1, \cdots, x_M)$ represents the partition function of the system under consideration. This fact is made more explicitly by rewriting (2) as

$$Z_N^{inf}(x_1, \cdots, x_M) = (x_1 + \cdots + x_M)^N = \sum_{\substack{n_i \\ \Sigma n_i = N}} \frac{N!}{n_1! n_2! \cdots n_M!} \, x_1^{n_1} x_2^{n_2} \cdots x_M^{n_M} \quad . \tag{7}$$

(The symbol Z_N was introduced with a view to emphasizing this point. The superscript "inf" indicates that we are dealing with the infinite statistics[9])

1. *Decomposition based on the permutation group*:

In this decomposition we regard the M^N states as the carrier space for an M^N dimensional representation of the permutation group S_N. This reducible representation can be decomposed into the irreducible representations of S_N which, as is well known, are in one to one correspondence with the partitions of N. In this case, the relation analogous to (2) which summarises all relevant features of this decomposition is

$$Z_N^{inf}(x_1, \cdots, x_M) \equiv (x_1 + \cdots + x_M)^N = \sum_{\substack{\lambda \\ |\lambda|=N}} n(\lambda) S_\lambda(x_1, \cdots, x_M) \quad , \tag{8}$$

where $S_\lambda(x_1, \cdots, x_M)$ are the Schur functions[8]

$$S_\lambda(x_1, \cdots, x_M) = \frac{\det(x_i^{\lambda_j + M - j})}{\det(x_i^{M-j})} \quad ; \quad 1 \le i, j \le M \quad , \tag{9}$$

and $n(\lambda)$ denotes the dimension of the irreducible representation λ of S_N. Setting $x_1 = x_2 = \cdots = x_M = 1$ in (8) we get

$$M^N = \sum_{\substack{\lambda \\ |\lambda|=N}} n(\lambda) S_\lambda(1, \cdots, 1) \quad , \tag{10}$$

which tells us that $S_\lambda(1, \cdots, 1)$ is the number of times the irreducible representation λ occurs in this decomposition.

So far we had been dealing with $\mathcal{H}$. Following refs. 4 and 5 we now construct out of it a generalized ray space $\mathcal{H}_{phy}$ by

- (a) admitting only those operators on $\mathcal{H}$ which are permutation symmetric,
- (b) identifying those states in $\mathcal{H}$ which have the same expectation values for all permutation symmetric operators.

These assumptions, together with the Schur's Lemma, imply that all states belonging to an irreducible representation λ of S_N count as one state of $\mathcal{H}_{phy}$. This immediately tells us that the symmetric function of degree N appropriate to $\mathcal{H}_{phy}$ is simply obtained by setting $n(\lambda) = 1$ in (8).

$$Z_N^{HST}(x_1,\cdots,x_M)=\sum_{\substack{\lambda\\|\lambda|=N}} S_\lambda(x_1,\cdots,x_M) \quad . \tag{11}$$

This is the key result of this work. (Here we use the superscripts HST to denote Hartle Stolt and Taylor in honour of their contributions to parastatistics). This symmetric function captures all aspects of $\mathcal{H}_{phy}$ much the same way as $(x_1+\cdots+x_M)^N$ does for $\mathcal{H}$.

So far no restrictions have been put on λ — the sum on the R.H.S. of (11) is over all partitions of N. We shall refer to this statistics as HST statistics. Parabose case of order p arises when we retrict the sum in (11) to only those partitions of N whose length $l(\lambda)$ (the number of the non-zero λ_i's) is less than equal to $\le p$. In terms of Young tableaux, this amounts to retaining only those irreducible representations pf S_N in which the number of boxes in the first column is $\le p$. The appropriate symmetric function for this case is

$$Z_N^{P.B}(x_1,\cdots,x_M;p)=\sum_{\substack{\lambda\ |\lambda|=N\\ l(\lambda)\le p}} S_\lambda(x_1,\cdots,x_M) \quad . \tag{12}$$

Similarly parafermi case of order p arises when we restrict λ in (11) to those partitions whose conjugate partition λ' is of length $\le p$. In terms of Young tableaux this implies retaining only those irreducible representations of S_N in which the number of boxes in the first row is $\le p$. The symmetric function appropriate to this case is

$$Z_N^{PF}(x_1,\cdots,x_M;p)=\sum_{\substack{\lambda\\|\lambda|=N\\ l(\lambda')\le p}} S_\lambda(x_1,\cdots,x_M) \quad . \tag{13}$$

Likewise, for the (p,q) statistics the corresponding symmetric function $Z_N^{(p,q)}(x_1,\cdots,x_M)$ is obtained by restricting the sum in (11) to those partitions for which $l(\lambda)\le p$ and $l(\lambda')\le q$. It may be noted that if $p,q\ge N$, all these cases reduce to HST.

The symmetric functions above contain all the relevant information about the appropriate $\mathcal{H}_{phy}$. For instance the dimension of $\mathcal{H}_{phys}$ is obtained by setting $x_1=x_2=\cdots x_M=1$. The appropriate canonical partition function is obtained by setting $x_1=e^{-\beta\epsilon_1}$, $x_2=e^{-\beta\epsilon_2},\cdots,x_M=e^{-\beta\epsilon_M}$. Some formulae which prove to be extremely useful in carrying out the sums in (12) and (13) with restrictions on the lengths of the partitions are as follows.[8]

$$S_\lambda(x_1,\cdots,x_M)=\det(h_{\lambda_i}-i+j) \;\; ; \;\; 1\le i,j\le l(\lambda) \;\; , \tag{14}$$

$$S_\lambda(x_1,\cdots,x_M)=\det(e_{\lambda'_i}-i+j) \;\; ; \;\; 1\le i,j\le l(\lambda') \;\; . \tag{15}$$

Here the complete symmetric functions $h_r(x_1,\cdots,x_M)$ and the elementary symmetric functions $e_r(x_1,\cdots,x_M)$ are defined as follows

$$h_r(x_1,\cdots,x_M)=\sum_{\substack{\lambda\\|\lambda|=r}} m_\lambda(x_1,\cdots,x_M) \;\; , \tag{16}$$

$$e_r(x_1,\cdots,x_M)=\sum_{i_1<i_2<\cdots<i_r} x_{i_1}x_{i_2}\cdots x_{i_r} \;\; . \tag{17}$$

Using these formulae one can express the appropriate symmetric functions in terms of either h's or e's. As an illustration, let us consider the Bose case. Here, since $l(\lambda)\le 1$, we have only one term on the R.H.S. of (13) corresponding to $\lambda=(N,0,0\cdots,0)$. Using (14) we obtain

$$Z_N^B(x_1,\cdots,x_M)=h_N(x_1,\cdots,x_M) \;\; . \tag{18}$$

Similarly, for the Fermi case, one has

$$Z_N^F(x_1,\cdots,x_M)=e_N(x_1,\cdots,x_M) \;\; . \tag{19}$$

Consider parabose of order 2. Using (12) and (14) we obtain

$$Z_N^{PB}(x_1,\cdots,x_M;2)=h_N+\sum_{\substack{\lambda_1+\lambda_2=N\\ \lambda_1\ge\lambda_2}} \det\begin{pmatrix} h_{\lambda_1} & h_{\lambda_1+1} \\ h_{\lambda_2-1} & h_{\lambda_2}\end{pmatrix} \;\; , \tag{20}$$

which on simplification leads to

$$Z_N^{PB}(x_1,\cdots,x_M;2) \begin{array}{l} =h_P^2(x_1,\cdots,x_M) \quad \text{if} \quad N=2P \;\; , \\ \\ =h_{P+1}(x_1,\cdots,x_M)h_P(x_1,\cdots,x_M) \quad \text{if} \quad N=2P+1 \;\; . \end{array} \tag{21}$$

The result for parafermi of order 2 is obtained by replacing h's by e's. Thus we obtain the the results due to Suranyi which arise as a special case of (12) and (13). One can carry out similar calculations for any order p.

For the HST case a number of interesting formulae can be derived from the following result for the Schur functions[8]

$$\sum_N \sum_{\substack{\lambda \\ |\lambda|=N}} S_\lambda(x_1, \cdots, x_M) = \prod_i \frac{1}{(1-x_i)} \prod_{i<j} \frac{1}{(1-x_i x_j)} \quad . \tag{22}$$

The R.H.S. of (22), with appropriate identifications of the x_i's, can be seen to be the grand canonical partition function for the HST statistics. Further, setting $x_1 = \cdots = x_M = t$ and reading off the coefficient of t^N in the resulting expression on the R.H.S. one obtains

$$\begin{aligned} \dim(\mathcal{H}_{phys}) &= \sum_{\substack{\lambda \\ |\lambda|=N}} S_\lambda(1, \cdots, 1) \\ &= \sum_{S=0}^{[N/2]} \begin{pmatrix} M+N-2S-1 \\ N-2S \end{pmatrix} \begin{pmatrix} M(M-1)/2+S-1 \\ S \end{pmatrix} \quad . \end{aligned} \tag{23}$$

This formula is interesting in its own right as it gives the number of irreducible representations of S_N which occur in the decomposition of the M^N dimensional reducible representation of S_N discussed above.

Finally, having derived the symmetric function appropriate to $\mathcal{H}_{phy}$ (with or without restrictions on the irreducible representations of the permutation group) we can immediately obtain all the relevant information regarding its occupation number decomposition. All that needs to be done is to expand the appropriate symmetric functions in terms of the monomial symmetric functions $m_\lambda(x_1, \cdots, x_M)$. Now since $S_\lambda(x_1, \cdots, x_M)$ and $m_\lambda(x_1 \cdots x_M)$; $|\lambda| = N$, serve as bases for symmetric polynomials in $x_1, \cdots, x_M$ of degree N, we can expand one in terms of the other

$$S_\chi(x_1, \cdots, x_M) = \sum_{\substack{\lambda \\ |\lambda|=N}} K_{\chi\lambda} m_\lambda(x_1, \cdots, x_M) \quad , \tag{24}$$

where $K_{x\lambda}$ are Kostka-Foulkes numbers.[10] Substituting this in (11) we obtain

$$Z_N^{HST}(x_1, \cdots, x_M) = \sum_\lambda (\sum_\chi K_{\chi\lambda}) m_\lambda(x_1, \cdots, x_M) \quad . \tag{25}$$

(For parabose or parafermi or (p, q) case one has to put appropriate restrictions on the partitions χ in (26)). The coefficients $\sum_\chi K_{\chi\lambda}$ immediately give us the number of states corresponding to the occupation numbers $\lambda = (\lambda_1, \cdots, \lambda_M)$ or any distinct permutation thereof.

To conclude, by combining the approach propounded in refs. 4 and 5, with the theory of symmetric functions, we have been able to obtain partition functions for all statistics based on the permutation group. Detailed analyses of the thermodynamic properties derivable from these results would be published elsewhere.

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