Predictions for the unitarity triangle angles in a new parametrization

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Abstract

A new approach to the parametrization of the CKM matrix, V, is considered in which V is written as a linear combination of the unit matrix I and a non-diagonal matrix U which causes intergenerational-mixing, that is $V = \cos\theta\ I + i\sin\theta\ U$. Such a V depends on 3 real parameters including the parameter θ. It is interesting that a value of $\theta = \pi/4$ is required to fit the available data on the CKM-matrix including CP-violation. Predictions of this fit for the angles α, β and γ for the unitarity triangle corresponding to $V_{11}V_{13}^* + V_{21}V_{23}^* + V_{31}V_{33}^* = 0$, are given. For $\theta=\pi/4$, we obtain $\alpha = 88.46°$, $\beta = 45.046°$ and $\gamma = 46.5°$. These values are just about in agreement, within errors, with the present data. It is very interesting that the unitarity triangle is expected to be approximately a right-angled, isosceles triangle. Our prediction $\sin 2\beta = 1$ is in excellent agreement with the value $0.99 \pm 0.15 \pm 0.05$ reported by the Belle collaboration at the Lepton-Photon 2001 meeting.

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1 Introduction

After the first explicit parametrization for three generations [1], many different parametrizations have been suggested [2, 3] for the Cabibbo-Kobayashi-Maskawa (CKM) matrix. Even today we do not have a deep understanding of the observed mixing of the quark flavors in the standard model.

Recently, a new approach to the parametrization of of the CKM-matrix, V, was suggested motivated by the question whether V consists of two parts. That is, a trivial part (taken to be the unit matrix I) for which the physical (or quark mass-eigenstate) basis and the the gauge basis are the same and a non-trivial part (represented by a non- diagonal diagonal matrix U) for which the two bases are different and which causes quark flavor mixing. This possibility was explored in some detail in reference 4 where the CKM-matrix was taken to be given by the linear combination

$$V(\theta) = \cos\theta \ I + i\sin\theta \ U. \tag{1}$$

The value of θ determines the relative importance of the two parts. It also determines the magnitude of CP-violation in this approach. It was shown [4] that the value of θ near $\pi/4$ gave a good fit to the data for the CKM-matrix given in year 1998 by the Particle Data Group [3]. For $0 < \theta < \pi/2$, it is clear that for V to be unitary, U (independent of θ) has to be hermitian and unitary. For three generations, mathematically, such a 3×3 matrix U can depend on at most 4 real parameters, namely, two moduli and two phases. In section 2, we give the explicit form of U and V. It is shown that using the freedom of re-phasing transformations on V, one can eliminate the two phases, so that, in effect, U contains only 2 real parameters. Thus, V depends on 3 parameters, one less than in the usual parametrizations.

In section 3, $V(\theta)$ is confronted with the data [5] given in the year 2000 by the Particle Data Group. This update of the fits is necessary as the recent data differs from the earlier data of 1998. A satisfactory fit to the latest data is obtained and points strongly to a value of θ equal to $\pi/4$.

In section 4, the predictions for the the angles of the triangle implied by the unitarity constraint

$$V_{11}V_{13}^* + V_{21}V_{23}^* + V_{31}V_{33}^* = 0, \tag{2}$$

are considered. This can be written as

$$z_1 + z_2 + z_3 = 0, \tag{3}$$

where the complex numbers $z_i = V_{i1}V_{i3}^*$; $i = 1,\ 2,\ 3$. In standard notation, the three angles of the triangle in terms of z_i are $\alpha = arg(-z_3/z_1),\ \beta =$

$arg(-z_2/z_3)$, and $\gamma = arg(-z_1/z_2)$. Numerical results for these angles in our parametrization are compared with the available data.

Finally, section 5, contains a brief summary with some speculative concluding remarks.

2   Parameters and form of U and V

To determine the general form of U, one starts with a general hermitian matrix and then requires it to be unitary. For the case of three generations, it was shown earlier [4] that such a matrix can be parametrized in terms of three complex numbers a, b and c which satisfy two constraints, one on their moduli and the other on their phases ϕ_a, ϕ_b and ϕ_c. Explicitly,

$$U = I - 2 \begin{pmatrix} |a|^2 + |b|^2 & b^*c & a^*c^* \\ bc^* & |a|^2 + |c|^2 & ab^* \\ ac & a^*b & |b|^2 + |c|^2 \end{pmatrix}, \tag{4}$$

with the constraints

$$|a|^2 + |b|^2 + |c|^2 = 1, \tag{5}$$

$$\phi_a - \phi_b + \phi_c = \pi/2. \tag{6}$$

Mathematically, this is the most general U which is hermitian and unitary. It depends on four real parameters, two moduli and two phases. However, this U is a part of the CKM-matrix V for which one has the freedom to make re-phasing transformations without affecting its physical predictions. This freedom allows one to eliminate the phases in U making it a real matrix depending on only two real positive parameters, the two moduli. In our ansatz, V and its physical predictions depend on three real parameters, the angle θ and the two moduli in U. This fact was not realised in reference 4.

We consider the explicit phase transformation which eliminates the phases. Let $P(\lambda)$ denote the diagonal phase matrix, diag $(e^{i\lambda_1}, \ e^{i\lambda_2}, 1)$. Then,

$$V'(\theta) = P(\lambda)V(\theta)P^*(\lambda) = \cos\theta \ I + i\sin\theta \ U', \tag{7}$$

where the real matrix

$$U' = I - 2 \begin{pmatrix} |a|^2 + |b|^2 & |bc| & -|ac| \\ |bc| & |a|^2 + |c|^2 & |ab| \\ -|ac| & |ab| & |b|^2 + |c|^2 \end{pmatrix}, \tag{8}$$

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is obtained by choosing the phases $\lambda_1 = \phi_b - \pi/2$ and $\lambda_2 = \phi_c - \pi/2$, while the phase constraint fixes ϕ_a. It is interesting to note that the particular choice, $\phi_a = \phi_b = \phi_c = \pi/2$ (which respects the phase constraint) in Eq.(6) will also give Eq.(8). This was mistakenly referred to as a special case earlier [4] as the above points were not realised there. The important point is that the physical predictions are independent of the actual value of the individual phases of a, b and c as long as they satisfy the phase constraint. Since, under a phase transformation $|V_{ij}| = |V'_{ij}|$, it does not matter whether V or V' is used to confront the data. In either case, θ and the two moduli will determine the matrix elements of the CKM-matrix. Furthermore, the Jarlskog invariant [6], J, which gives CP-violation, is the same for V or V', namely

$$J(V(\theta)) = Im(V_{11}V_{22}V_{12}^*V_{21}^*) = \cos\theta|V_{12}V_{13}V_{23}| = 8\cos\theta\sin^3\theta|abc|^2. \quad (9)$$

Note that J is independent of the phases of a, b and c.

To confront $V(\theta)$ with experiment we need to specify θ. A simple and appealing choice is $\theta = \pi/4$ which gives equal weight to the two parts in V. In this case,

$$V(\pi/4) = \frac{1}{\sqrt{2}}(I + iU). \quad (10)$$

In this case two experimental inputs e.g. $|V_{12}|$ and $|V_{23}|$ are enough to determine all the the other $|V_{ij}|$. However, with 3 inputs,$|V_{12}|$, $|V_{23}|$ and $|V_{13}|$ from the data will determine all the $|V_{ij}|$ and also the value of $\sin\theta$. The numerical results for the 2 input ($\theta = \pi/4$) and the 3 input cases are given in the next section.

3 Fits to the recent data.

The experimentally determined CKM-matrix,V_{EX}, given by the Particle Data Group [5]

$$V_{EX} = \begin{pmatrix} 0.9742 - 0.9757 & 0.219 - 0.226 & 0.002 - 0.005 \\ 0.219 - 0.225 & 0.9734 - 0.9749 & 0.037 - 0.043 \\ 0.004 - 0.014 & 0.035 - 0.043 & 0.9990 - 0.9993 \end{pmatrix}. \quad (11)$$

The entries correspond to ranges for the moduli of the matrix elements. Since U is hermitian, this implies that $|V_{ij}|=|V_{ji}|$ in our approach. It is clear that $|V_{12}|=|V_{21}|$ and $|V_{23}| = |V_{32}|$ are satisfied for practically the whole range while the equality $|V_{13}| = |V_{31}|$ is merely suggested by the data. Since $|V_{13}|$ and $|V_{31}|$ are the most difficult to measure experimentally, it is possible they may

turn out to be equal. Note that for any unitary matrix V, unitarity requires that either $|V_{ij}| = |V_{ji}|$ for all three pairs or for none of them. To fit the data we convert the range for each modulus into a central value with errors, for example, $|V_{11}| = 0.97495 \pm 0.00075$. For each pair of off-diagonal elements a common value was obtained by averaging the central values. For example, the common value $|V_{13}| = |V_{31}| = 0.00625 \pm 0.00325$ is obtained by taking the average of the central values with errors of $|V_{13}| = 0.0035 \pm 0.0015$ and $|V_{31}| = 0.009 \pm 0.005$. Similar procedure was used for the other two pairs of off-diagonal matrix elements.

a) Two parameter fit. We take the experimentally well determined $|V_{12}|$ and $|V_{23}|$ as inputs. Given these, for general θ, one has

$$|a| = |V_{23}|/(2\sin\theta|b|), \tag{12}$$

$$|c| = |V_{12}|/(2\sin\theta|b|). \tag{13}$$

The constraint, Eq.(5), gives a quadratic for $|b|^2$ with the solutions,

$$|b|^2 = \frac{1}{2}\left[1 \pm \sqrt{1 - (|V_{12}|^2 + |V_{23}|^2)\csc^2\theta}\right]. \tag{14}$$

Clearly, we need the positive solution since $|V_{12}| > |V_{23}| > |V_{13}|$. Further, for $|b|^2$ to be real, θ has to be greater than some minimum value. For the numerical input values $|V_{12}| = 0.22225$ and $|V_{23}| = 0.0395$, we need $\theta \geq 13.046°$. For $\theta = \pi/4$, Eqs.(12-14) yield

$$|a| = 0.028303, \qquad |b| = 0.986832, \qquad |c| = 0.159251. \tag{15}$$

The calculated values of $|V_{ij}|$ for $V(\pi/4)$ are given in Table I. These are to be compared with the central values of the experimental $|V_{ij}|$ given in the first column. The agreement is fairly good.

The values of J for V_{EX} and $V(\pi/4)$ are also given in Table I. The value of $J(V_{EX})$ was calculated from the formula [7]

$$J^2 = |V_{11}V_{22}V_{12}^*V_{21}^*|^2 - \frac{1}{4}\left[1 - |V_{11}|^2 - |V_{22}|^2 - |V_{12}|^2 - |V_{21}|^2 + |V_{11}V_{22}|^2 + |V_{12}V_{21}|^2\right]^2, \tag{16}$$

using the central values of $|V_{ij}|$; $i,j = 1,2$ since these four are the best measured. The value of $J(V(\pi/4))$ was calculated using Eq.(9) and is about a factor 2 smaller than $J(V_{EX})$. The two values are in reasonable agreement considering the slight differences in the values of $|V_{ii}|$; $i = 1,2$ in the two cases and also because of the strong numerical cancellation between the two

terms on the rhs of Eq.(16).

b)Three input fit. We now consider a three input fit which determines θ directly from the data. This can be done using the equation

$$2 \sin \theta = \frac{|V_{12} V_{23}|}{|V_{13}|} + \frac{|V_{12} V_{13}|}{|V_{23}|} + \frac{|V_{13} V_{23}|}{|V_{12}|}. \tag{17}$$

With the numerical inputs $|V_{12}| = |V_{21}| = 0.22225$, $|V_{23}| = |V_{32}| = 0.0395$ and $|V_{13}| = |V_{31}| = 0.00625$, we obtain $\sin \theta =0.720448$ so that θ equal to $46.09°$. This is remarkably close to our earlier choice of $\theta = \pi/4$. The values obtained for the moduli are

$$|a| = 0.027765, \qquad |b| = 0.987331, \qquad |c| = 0.156223. \tag{18}$$

The calculated values of $|V_{ii}|$; $i = 1, 2, 3$ and J are also given in Table I. These are practically same as the values for the two input case. The reason is that the value obtained for $|V_{13}|$ for $\theta = \pi/4$ is very close to the input value used in (17). The more recent data [5] suggests more strongly than the earlier data [3] that the value of $\theta = \pi/4$. In other words, the diagonal part (I) and the non-diagonal part (U) in the CKM-matrix have equal weight.

4   Predictions for the angles α, β and γ

To determine the angles of the unitarity triangle given by Eq.(2), it is convenient to denote the complex number

$$-\frac{z_1}{z_2} = \rho + i\eta, \tag{19}$$

so that, using Eq.(3),

$$-\frac{z_3}{z_2} = (1 - \rho) - i\eta. \tag{20}$$

The notation used here for the real and imaginary parts has been chosen so that it coincides with that used by Wolfenstein [2] in his approximate parametrization of the CKM-matrix. This notation like that for the angles has become standard. However, the use of ρ and η in Eqs.(19-20) is purely a matter of notation and the formulae such as Eqs.(21-22) below are valid for any exact parametrization of the CKM-matrix. From the definitions of the angles in section 1 and Eqs.(19-20) it follows that

$$\sin \alpha = \frac{\sin \beta}{\sqrt{\rho^2 + \eta^2}} = \frac{\sin \gamma}{\sqrt{(1 - \rho)^2 + \eta^2}}, \tag{21}$$

and

$$\tan\gamma = \eta/\rho. \tag{22}$$

Thus, the knowledge of ρ and η is sufficient to determine the three angles of the unitarity triangle. In general, ρ and η are independent parameters. However, in our parametrization, one expects a relation between them [8] since V depends on 3 parameters only. Using the explicit form of V in terms of a, b and c one obtains

$$\rho = (1 - 2|c|^2)/2|b|^2 = -\frac{1}{2|b|^2}\frac{|V_{12}|^2 - |V_{23}|^2}{|V_{12}|^2 + |V_{23}|^2} + \frac{|V_{12}|^2}{|V_{12}|^2 + |V_{23}|^2}, \tag{23}$$

and

$$\eta = \cot\theta/2|b|^2, \tag{24}$$

where $|b|^2$ is the positive solution given in Eq.(14). The second equality in Eq.(23) requires the use of Eqs.(12), (13) and Eq.(5). The relation between ρ and η is not simple. To derive it one notes that Eq.(24) together with Eq.(14) can be used to solve for $|b|^2$ in terms of η and $s = |V_{12}|^2 + |V_{23}|^2$. The result is

$$\frac{s}{2|b|^2} = 1 - \sqrt{1 - s - \eta^2 s^2}. \tag{25}$$

Substitution of this in Eq.(23) gives ρ in terms of η, $|V_{12}|$ and $|V_{23}|$. Numerical values of $|V_{13}|$, ρ, and η for some representative values of θ are given in Table II. For input values $|V_{12}|$=0.22225 and $|V_{23}|$=0.0395, as θ increases from its minimum value of 13.046 to 90 degrees, ρ increases from 0.03062 to 0.49386 and η decreases from 4.31567 to 0. Algebraic expressions for the limits on ρ and η can be easily derived. Note, the variation with θ of η is stronger than that of ρ. The latter increases from about 0.47328 to 0.49386, while former decreases from 0.91532 to 0 as θ goes from $\pi/6$ to $\pi/2$. For $\theta = \pi/4$, we obtain

$$\eta = 0.5134 \qquad \rho = 0.4874. \tag{26}$$

Using these values in Eqs.(21-22) gives

$$\alpha = 88.46°, \beta = 45.046°, \gamma = 46.5°. \tag{27}$$

So, the unitarity triangle is predicted to be approximately a right-angled isosceles triangle. The near isosceles nature of the triangle follows from the

fact that in our parametrization $|V_{13}| = |V_{31}|$ and experimentally $|V_{11}|$ and $|V_{33}|$ are nearly equal. Since ρ and $1 - \rho$ are approximately equal over a wide of θ (approximately from 30 to 90 degrees), there is a simple mnemonic for the variation of the angles α, β and γ with θ. A change in $\theta \to \theta + \delta$ implies approximately that $\beta \to \beta - \delta$, $\gamma \to \gamma - \delta$ while $\alpha \to \alpha + 2\delta$ to satisfy $\alpha + \beta + \gamma = \pi$. For example, for $\theta = 30°$ (that is, $\delta = -15°$) one obtains from direct calculations $\alpha = 57.26°$, $\beta = 60.08°$ and $\gamma = 62.66°$, an approximate equilateral triangle!

A direct way to obtain the angles is to use Eq.(21) in the form

$$\frac{\sin \alpha}{|z_2|} = \frac{\sin \beta}{|z_1|} = \frac{\sin \gamma}{|z_3|} \equiv \lambda > 0, \tag{28}$$

where λ is a positive real number. Since $\sin \alpha = \sin(\beta + \gamma)$, this gives

$$4|z_1 z_2 z_3|^2 \; \lambda^2 = (|z_1|^2 + |z_2|^2 + |z_3|^2)^2 - 2(|z_1|^4 + |z_2|^4 + |z_3|^4). \tag{29}$$

Since $|z_1| = |V_{11}||V_{13}|$ etc. are known one can determine λ and hence the angles. The results are tabulated in Table III. Using the values of $|V_{ij}|$ for the three cases in Table I one obtains nearly the same values, for λ and the angles. Note that the values of the angles, for the 2-input case when $\theta = \pi/4$, are very close to those in Eq.(27) as they should be.

The values of the angles in Eq.(27) imply that $\sin 2\alpha = 0.054$ and $\sin 2\beta = 1$. The former and the value of γ in Eq.(27) are compatible with a recent theoretical analysis [9], but which obtains $0.49 < \sin 2\beta < 0.94$. The data has a large variation. Our result for $\sin 2\beta$ is compatible within errors with the values $0.79^{+0.41}_{-0.44}$ and $0.84^{+0.82}_{-1.04} \pm 0.16$ obtained by the CDF and ALEPH collaborations [10]. The values [11] quoted by BaBar $(0.34 \pm 0.20 \pm 0.05)$ and Belle $(0.58^{+0.32+0.09}_{-0.34-0.10})$ are lower, especially that of BaBar. However, the values reported at the Lepton-Photon 2001 [12, 13] are much larger. Babar [12] and Belle [13] give the values $0.59 \pm 0.14 \pm 0.05$ and $0.99 \pm 0.14 \pm 0.06$ respectively. The latter agrees with our prediction. Experiments with high statistics are in progress at both Belle and BaBar and hopefully more definitive values for the angles will be available in a year or so.

5   Summary and concluding remarks

The ansatz $V(\theta) = \cos \theta \; I + i \sin \theta \; U$, considered here, was motivated by the question whether the CKM-matrix is a linear combination of a trivial part (I) and a non-trivial part (U). The matrix U depends on 2 real

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parameters and so that the CKM-matrix V depends on 3 real parameters including θ which plays a double role. It determines the relative importance of the generation mixing term (U) relative to the generation diagonal term (I). It also determines the magnitude of CP-violation. The fits to the presently available data [5] for the CKM-matrix require that $\theta = \pi/4$, that is $V(\pi/4) = (I + iU)/\sqrt{2}$, implying equal importance of the two parts. This fit predicts that the unitarity triangle is approximately a right-angled isosceles triangle (see section 4). These results are compatible with present data wthin allowed errors. Experiments, in the offing, will soon decide the validity of the approach used here for parametrizing the CKM-matrix.

The ansatz considered here has been proposed [14] recently for the parametrization of the neutrino mixing matrix V_ν, the subscript 'ν' will denote the corresponding quantities for the neutrino sector. Remarkably, one finds that $V_\nu = (I + iU_\nu)/\sqrt{2}$ can explain the neutrino data with U_ν depending on only one small parameter. In this case, the atmospheric neutrino data [15], which requires maximal ν_μ and ν_τ mixing, forces the value of θ_ν to be equal to $\pi/4$. It is extremely interesting that in both the lepton and quark sectors the relative weight of the diagonal part and the non-diagonal pieces in the mixing matrix are the same! Even though V and V_ν are very different, one may speculate that an underlying quark-lepton symmetry seems to be suggested, in our approach, by the equality $\theta = \theta_\nu = \pi/4$ and furthermore, this equality may emerge naturally in a grand unification model.

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QUANTITY	EXPERIMENT	THEORY	
$|V_{12}| = |V_{21}|$	0.22225 ±0.00325	INPUT	INPUT
$|V_{23}| = |V_{32}|$	0.0395 ±0.0035	INPUT	INPUT
$|V_{13}| = |V_{31}|$	0.00625±0.00325	0.006374369	INPUT
$|V_{11}|$	0.97495±0.00075	0.97496887	0.97496968
$|V_{22}|$	0.97415±0.00075	0.97418925	0.97418925
$|V_{33}|$	0.99915±0.00015	0.99919924	0.99920002
$J\times10^5$	6.7	3.96	3.8

Table I Numerical values of the moduli of the matrix elements of the CKM-matrix V. Experimental quantities are average values obtained from V_{EX} in Eq.(11). The errors reflect the range of the values of $|V_{ij}|$ as explained in the text. Column 3 gives the results for the 2 input fit with $\theta = \pi/4$. Column 4 gives the results for the 3 input fit (see section 3).

θ	$|V_{13}|$	ρ	η
$\theta_{min} = 13.046°$	0.0388906	0.03062	4.31567
30°	0.00927858	0.473282	0.915321
40°	0.00705337	0.484561	0.615477
45°	0.00637437	0.48739	0.513432
50°	0.0058601	0.489343	0.429076
60°	0.00515763	0.491745	0.293752
90°	0.00444683	0.493863	0.0

Table II Numerical values of $|V_{13}|$, ρ and η as a function of θ. While ρ and η are calculated using Eqs. (24), (25), $|V_{13}|$ is calculated using $|V_{13}| = (|V_{12}||V_{23}|/s)(\sin\theta - \sqrt{\sin^2\theta - s})$.

QUANTITY	EXPERIMENT	THEORY	
		2-inputs	3-inputs
λ	113.9017	113.8689	113.9012
α	89.317	88.46	89.29
β	43.952	45.046	43.953
γ	45.358	46.49	45.342

Table III Values of the angles obtained directly from the general relation between the sides and angles satisfied by any triangle given in Eq.(28).