Wigner distributions for non Abelian finite groups of odd order

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Abstract

Wigner distributions for quantum mechanical systems whose configuration space is a finite group of odd order are defined so that they correctly reproduce the marginals and have desirable transformation properties under left and right translations. While for the Abelian case we recover known results, though from a different perspective, for the non Abelian case our results appear to be new.

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The notion of a Wigner distribution, introduced about seventy years ago in the context of quantum mechanical systems with $\mathcal{R}^n$ as the configuration space [1], has played a dominant role in various areas in physics both as a bridge between classical and quantum mechanics as well as a useful computational tool [2]. In the last twenty years there have been several attempts to extend the definition of the Wigner distribution beyond the Cartesian case $\mathcal{R}^n$ [3]- [12]. In a recent work [13] in this general direction, a notion of a Wigner distribution was developed from first principles for the case when the configuration space is a compact Lie group G. A detailed analysis of the quantum kinematics of such systems revealed that the transition from $\mathcal{R}^n$ to an arbitrary compact Lie Group G entails several extensions or modifications of the familiar Wigner distribution formalism in the Cartesian case. Some novel features of this formalism are listed below:

- The role of the continuous momenta in the Cartesian case is played by the discrete labels JMN of the unitary irreducible representation matrices $\mathcal{D}^J_{MN}(g)$ of the group G. Here J labels the irreducible representations of G and MN the rows and the columns.

- If one insists on the recovery of the marginal distributions and correct transformation properties under left and right translations, the Wigner distribution associated with a normalised wavefunction $\psi(g)$ on the compact Lie group G no longer turns out to be just a function $W(g; JMN)$ of the 'position coordinates' g and the 'momentum coordinates' JMN alone but rather a more elaborate object $W(g; JMN\ M'N')$:

$$W(g; JMN\ M'N') = N_J \int_G dg' \int_G dg''\ \delta\left(g^{-1}s(g', g'')\right) \mathcal{D}^J_{MN}(g')\ \psi(g')^*\ \mathcal{D}^J_{M'N'}(g'')^*\ \psi(g''). \quad (1)$$

  Here N_J is the dimension of the unitary irreducible representation J and dg is the normalised translation invariant volume element on G.

  In the case of Abelian groups, simplifications occur owing to the fact that all irreducible representations are one dimensional making the labels $M\ N\ M'\ N'$ redundant with the result that the Wigner distribution $W(g; JMN\ M'N')$ can simply be written as $W(g, J)$, a function of just the coordinates 'g' and the momenta 'J' as in the cases of $\mathcal{R}^n$ and $\mathcal{S}^1$.

- A key ingredient in the construction of the Wigner function above is the notion of the 'mid point' $s(g,\ g')$ of two group elements g and g'. (Properties of $s(g,\ g')$ are given in detail later). In the compact Lie group case, this object can be computed using the geodesics on the manifold of G.

- The form of the Wigner distribution in (1) has a wider range of applicability than just compact Lie groups as was demonstrated in [13] by recovering the known results for the non compact case $\mathcal{R}^n$ and $\mathcal{S}^1$ [14]. In these cases, owing to the Abelian nature of the groups involved, though the geodesic construction is not available for computing $s(g,\ g')$, it does turn out to be possible to find $s(g,\ g')$ satisfying the desired conditions and hence the known Wigner distributions appropriate to these cases.

The purpose of the present work is to show that the structure of the Wigner distribution given above, suitably adapted to finite groups, leads to a satisfactory definition of Wigner distributions for all finite groups of odd order both Abelian and non Abelian.

Let $\mathcal{H}$ denote the Hilbert space of complex valued functions on a finite group G of order $|G|$ and let $\psi(g),\ g \in G$ denote a normalised 'position-space' wavefunction:

$$||\psi||^2 \equiv \sum_{g\in G} |\psi(g)|^2 = 1. \tag{2}$$

Let $\mathcal{D}^J_{MN}(g)$ denote the unitary irreducible representation matrices of G with J labelling the irreducible representation and MN the rows and columns respectively. These matrices satisfy the following representation, orthogonality and completeness properties:

$$\sum_{M'} \mathcal{D}^J_{MM'}(g')\mathcal{D}^J_{M'N}(g) = \mathcal{D}^J_{MN}(g'g), \tag{3}$$

$$\frac{1}{|G|}\sum_{g\in G} \mathcal{D}^{J'}_{M'N'}(g)^*\mathcal{D}^J_{MN}(g) = \delta_{J'J}\delta_{M'M}\delta_{N'N}/N_J, \tag{4}$$

$$\frac{1}{|G|}\sum_{JMN} N_J\mathcal{D}^J_{MN}(g)\mathcal{D}^J_{MN}(g')^* = \delta_{g,g'}. \tag{5}$$

With the help of the matrices $\mathcal{D}^J_{MN}(g)$ we define the 'Fourier transform' ψ_{JMN} of $\psi(g)$:

$$\psi_{JMN} = \sqrt{\frac{N_J}{|G|}}\sum_{g\in G} \mathcal{D}^J_{MN}(g)^*\psi(g),$$

$$\sum_{JMN} |\psi_{JMN}|^2 = \sum_{g\in G} |\psi(g)|^2, \tag{6}$$

yielding the 'momentum-space' wavefunction ψ_{JMN}.

Following [13], we associate with $\psi(g)$ the Wigner distribution $W(g; JMN\ M'N')$ as follows:

$$W(g; JMN\ M'N') = \frac{N_J}{|G|}\sum_{g'\in G}\sum_{g''\in G} \delta_{g,s(g',g'')}$$
$$\mathcal{D}^J_{MN}(g')\ \psi(g')^*\ \mathcal{D}^J_{M'N'}(g'')^*\ \psi(g''). \tag{7}$$

This involves a group element $s(g', g'') \in G$, the 'mid-point' of the group elements g and g', which is required to satisfy the following conditions:

$$\begin{aligned} g', g'' \ \in\ G \rightarrow s(g', g'') \ &\in\ G, \\ s(g', g'') \ &=\ s(g'', g'), \\ s(g', g') \ &=\ g', \\ s\left(g_1g'g_2^{-1}, g_1g''g_2^{-1}\right) \ &=\ g_1\ s(g', g'')g_2^{-1}. \end{aligned} \tag{8}$$

By virtue of (8) and $(3-5)$ one finds that the Wigner distribution $W(g; JMN\ M'N')$ corresponding to any $\psi(g) \in \mathcal{H}$ possesses the following properties:

- *Hermiticity*:

$$W(g;JMN\ M'N')^* = W(g;JM'N'\ MN). \tag{9}$$

- *Marginals*

$$\begin{aligned} \sum_{g\in G} W(g;JMN\ M'N') &= \psi_{JM'N'}\psi_{JMN}{}^*, \\ \sum_{JMN} W(g;JMN\ MN) &= |\psi(g)|^2. \end{aligned} \tag{10}$$

- *Transformation under left translations*

$$\begin{aligned} &\psi'(g) = \psi(g_1^{-1}g) \rightarrow \\ &W'(g;JMN\ M'N') = \sum_{M_1M_1'} \mathcal{D}^J_{MM_1}(g_1)\mathcal{D}^J_{M'M_1'}(g_1)^* W\left(g_1^{-1}g;JM_1N\ M_1'N'\right). \end{aligned} \tag{11}$$

- *Transformation under right translations*

$$\begin{aligned} &\psi''(g) = \psi(gg_2) \rightarrow \\ &W''(g;JMN\ M'N') = \sum_{N_1N_1'} W\left(gg_2;JMN_1M'N_1'\right)\mathcal{D}^J_{N_1N}\left(g_2^{-1}\right)\mathcal{D}^J_{N_1'N'}\left(g_2^{-1}\right)^*. \end{aligned} \tag{12}$$

- *Traciality* If $\hat{\rho}_1$ and $\hat{\rho}_2$ denote two density operators and $W_1(g;JMN\ \ M'N')$ and $W_2(g;JMN\ M'N')$ the corresponding Wigner distributions, then

$$\sum_{JMM'} \frac{|G|}{N_J} \sum_{g\in G} \tilde{W}_1(g;JMM')\tilde{W}_2(g;JM'M) = \mathrm{Tr}(\hat{\rho}_1\hat{\rho}_2), \tag{13}$$

  where the auxiliary function $\tilde{W}(g;JMM')$ appearing in this equation is obtained from $W(g;JMN\ M'N')$ by setting $N=N'$ and summing over N:

$$\tilde{W}(g;JMM') = \sum_N W(g;JMN\ M'N). \tag{14}$$

  Since any density operator $\hat{\rho}_1$ is fully determined by the traces of its products with all other density operators $\hat{\rho}_2$, we can see that even the simpler function $\tilde{W}(g;JMM')$ fully characterises $\hat{\rho}$ leading to the conclusion that $W(g;JMN\ M'N')$ captures information contained in $\hat{\rho}$ in an overcomplete manner. This overcompleteness, however, disappears in the Abelian case.

Note that any choice of a function $s(g',g'')$ obeying conditions (8) leads to an acceptable definition of a Wigner distribution for quantum mechanics on a group G.

The covariance conditions in the last line in (8) help us simplify the problem of finding $s(g,g')$ to the choice of a suitable function $s_0(g)$, the 'square root' of the group element g, obeying the following conditions that ensure (8):

$$\begin{aligned} s(e,g) &= s_0(g), \\ s(g',g'') &= g'\ s_0(g'^{-1}g'') : \\ s_0(e) &= e; \\ s_0(g^{-1}) &= g^{-1}s_0(g); \\ s_0(g'gg'^{-1}) &= g'\ s_0(g)g'^{-1}. \end{aligned} \tag{15}$$

A consequence of these conditions on $s_0(g)$ is that

$$s_0\ (g)\ g = g\ s_0\ (g). \tag{16}$$

Thus the problem of setting up a Wigner distribution for any finite group reduces to constructing a function $s_0(g)$ for each $g \in G$ satisfying (15). This is easily accomplished if $|G|$ is odd. It is well known that for any finite group G of odd order the map $g \to g^2$ is one to one and onto. In other words for every group element $g_i \in G$ there is a unique $g_k \in G$ such that $g_i = g_k^2$. This being the case we can take $s_0(g_i) = g_k$. (One way of arriving at this result is to look at the cycles generated by individual group elements. On the one hand they are like one parameter subgroups or geodesics through the identity in the Lie group case; on the other hand each g obeys $g^{2m-1} = e$ or $g^{2m} = g$ for a least positive m giving g^m for the square root of g) It is easily verified that such a choice satisfies all the properties required of $s_0(g)$. This in turn enables us to construct $s(g,g')$ for all pairs $g, g' \in G$ satisfying (8) required for setting up Wigner distributions for all finite groups of odd order.

In the case of Abelian groups, as noted earlier, simplifications occur owing to the fact that all irreducible representations are one dimensional. Thus, for the cyclic group $\mathcal{Z}_N = \{0, 1, \cdots, N-1\}$ with N odd, the formalism presented above, with appropriate notational changes, leads to

$$W(k,J) = \frac{1}{N}\sum_{l=0}^{N-1}\sum_{m=0}^{N-1}\delta(k,s(l,m))\psi^*(l)\mathcal{D}^J(l)\psi(m)\mathcal{D}^J(m)^* \quad ;k,\ J = 0,1,2,\cdots,N-1, \tag{17}$$

where

$$\mathcal{D}^J(k) = \omega^{kJ};\ \omega = \exp(2\pi i/N). \tag{18}$$

The mid point $s(l,m)$ of two group elements l, m is n where $2n = l+m \pmod N$ and hence we can rewrite (17) as

$$W(k,J) = \frac{1}{N}\sum_{l=0}^{N-1}\psi^*(l)\psi(2k-l)\ \omega^{2J(l-k)}. \tag{19}$$

It is gratifying to see this result, extensively discussed in the literature [10] [11], come out from the general result in (7).

Next we turn to non Abelian finite groups. In particular we consider, in some detail, the smallest such group of odd order, a group of order 21 consisting of a semi-direct product $G = \mathcal{Z}_7 \times \mathcal{Z}_3$ of $\mathcal{Z}_7$ and $\mathcal{Z}_3$. We can display G as arising out of products of powers of two primitive generating elements a and b obeying the algebraic relations

$$a^7\ =\ b^3\ =\ e, \qquad b\ a = a^2\ b, \tag{20}$$

where e is the identity element of G. Thus a and b generate $\mathcal{Z}_7$ and $\mathcal{Z}_3$ respectively. From (20) we derive :

$$\begin{aligned} b^j\ a^\lambda &= a^{2^j\lambda}\ b^j, \quad j,\ \lambda \geq 0; \\ b\ a\ b^{-1} &= a^2, \qquad b^{-1}\ a\ b = a^4, \\ a\ b\ a^{-1} &= a^6\ b, \qquad a^{-1}\ b\ a = a\ b. \end{aligned} \tag{21}$$

We write the elements of G as ordered products of a power of a followed by a power of b, and denote them by the corresponding pair of non negative integers :

$$\begin{aligned} (\lambda,\ j) &= a^\lambda\ b^j, \quad 0\ \leq\ \lambda\ \leq\ 6,\ 0\ \leq\ j\ \leq\ 2, \\ (0,0) &= e. \end{aligned} \tag{22}$$

The composition law then follows from (21),

$$(\lambda,\ j)(\mu,\ k) = (\lambda + 2^j\mu \bmod 7,\ j+k \bmod 3). \tag{23}$$

The elements $\{(\lambda,\ 0)\},\ \{(0,\ j)\}$ constitute the subgroups $\mathcal{Z}_7,\ \mathcal{Z}_3$ respectively. The semidirect product structure is evident, with $\mathcal{Z}_7$ being the invariant subgroup.

Repeated use of (21) shows that the equivalence (or conjugacy) classes are five in number. Here we can use the fact that the exponent of b is unchanged upon conjugation :

$$g \in G \quad : \quad g(\lambda,\ j)g^{-1} = (\lambda',\ j). \tag{24}$$

The classes are

$$\begin{aligned} \mathcal{C}_1 &= \{(0,0) = e\}; \\ \mathcal{C}_2 &= \{(\lambda\ ,0) = a^\lambda,\ \lambda = 1,\ 2,\ 4\}; \\ \mathcal{C}_3 &= \{(\lambda,\ 0) = a^\lambda,\ \lambda = 3,\ 5,\ 6\}; \\ \mathcal{C}_4 &= \{(\lambda,\ 1) = a^\lambda\ b,\ 0\ \leq\ \lambda\ \leq\ 6\}; \\ \mathcal{C}_5 &= \{(\lambda,\ 2) = a^\lambda\ b^2,\ 0\ \leq\ \lambda\ \leq\ 6\}. \end{aligned} \tag{25}$$

There are therefore five inequivalent irreducible (unitary) representations of G. Since the squares of their dimensions must add up to 21, we see that there are three distinct one dimensional representations, and two distinct three-dimensional ones. We label them by $J = 1,\ 2,\ 3,\ 4,\ 5$ with dimensions $N_1 = N_2 = N_3 = 1,\ N_4 = N_5 = 3$.

The one dimensional representations obtain when the irreducible representations of $\mathcal{Z}_3$ are simply lifted to G and the element a is realised trivially:

$$\begin{aligned} J = 1,\ 2,\ 3\ :\quad a \rightarrow 1, \qquad b \rightarrow \omega^{J-1}, \quad \omega = \exp(2\pi i/3) \\ \mathcal{D}^J_{11}((\lambda,\ j))\ =\ \omega^{j(J-1)}. \end{aligned} \tag{26}$$

In the three dimensional representations we may assume that the matrix representing a is diagonal, with the eigenvalues being selected seventh roots of unity. We also have from (25) the matrices representing $a,\ a^2$ and a^4 (similarly $a^3,\ a^5$ and a^6) are similarity transforms of one another. One then easily arrives at the construction for, say, J=4:

$$J=4\ :\quad a\to\begin{pmatrix}\omega' & 0 & 0\\ 0 & \omega'^4 & 0\\ 0 & 0 & \omega'^2\end{pmatrix},\quad b\to\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix},$$

$$\omega'=\exp(2\pi i/7). \tag{27}$$

The primitive algebraic relations (20) are obeyed, so we do have a representation of G. Irreducibility follows from Schur's lemma : since a is diagonal non degenerate, any matrix commuting with it must be diagonal. If it also commutes with the permutation matrix b, it must be a multiple of the identity. To calculate the elements of the representation matrices we write (27), using $\omega'^2=\omega'^9$, as

$$\begin{aligned} a_{MN} &= \delta_{MN}\ (\omega')^{M^2},\\ b_{MN} &= \delta_{[M+2],N}\ ,\quad M,N=1,\ 2,\ 3\ \ , \end{aligned} \tag{28}$$

where $[M+2]$ indicates value mod 3, in the range $1,\ 2,\ 3$. For powers of a and b we then have :

$$\begin{aligned} (a^\lambda)_{MN} &= \delta_{MN}\ (\omega')^{\lambda M^2},\\ (b^j)_{MN} &= \delta_{[M+2j],N}\ , \end{aligned} \tag{29}$$

leading to

$$\begin{aligned} \mathcal{D}^4_{MN}((\lambda,\ j)) &= (a^\lambda\ b^j)_{MN}\\ &= (\omega')^{\lambda M^2}\ \delta_{[M+2j],N}\ \ . \end{aligned} \tag{30}$$

The other three dimensional irreducible representation $J=5$ is the complex conjugate of $J=4$.

$$J=5\ :\quad a\to\begin{pmatrix}\omega'^6 & 0 & 0\\ 0 & \omega'^3 & 0\\ 0 & 0 & \omega'^5\end{pmatrix},\quad b\to\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix};$$

$$(a^\lambda)_{MN}=\delta_{MN}\ (\omega')^{-\lambda M^2},$$

$$(b^j)_{MN}=\delta_{[M+2j],N}\ \ ;$$

$$\mathcal{D}^5_{MN}((\lambda,\ j))=(\omega')^{-\lambda M^2}\ \delta_{[M+2j],N}\ \ . \tag{31}$$

We next need expressions for the 'square root' elements $s_0(g)$ and the 'mid point' elements $s(g',g'')$ for general $g,\ g',\ g''\in G$. For the former, from the composition rule (23) we see that the square of the element $(\mu,\ k)$ is given by

$$\begin{aligned} &(\mu,\ k)^2=(\lambda,\ j)\ ,\\ &\qquad \lambda=(1+2^k)\mu \bmod 7,\quad j=2k \bmod 3. \end{aligned} \tag{32}$$

It is now indeed possible to solve for μ and k uniquely in terms of λ and j. By straight forward enumeration we find :

$$s_0((\lambda,\ j)) = ((4 + 2^{2+j} - 2^{2-j})\lambda \bmod 7,\ 2j \bmod 3),$$
$$0\ \leq \lambda\ \leq\ 6,\ \ 0\ \leq j\ \leq\ 2. \tag{33}$$

As for the mid point element $s(g', g'')$ we have from (15) :

$$s((\lambda,\ j),\ (\mu,\ k)) = (\lambda,\ j)\ s_0((\lambda,\ j)^{-1}(\mu,\ k)). \tag{34}$$

Since again from (23) we have

$$(\lambda,\ j)^{-1} = (-2^{3-j}\lambda \bmod 7,\ 2j \bmod 3),$$
$$0\ \leq \lambda\ \leq\ 6,\ \ 0\ \leq j\ \leq\ 2, \tag{35}$$

the computation of $s(g', g'')$ is complete.

Armed with the knowledge of the mid point of two group elements and the matrices of the irreducible representations of this group, we can now compute the Wigner distribution for any state $\psi(g)$ belonging to the Hilbert space of complex valued functions on this non-Abelian group. Thus, for instance, for $\psi(g)$ given by

$$\psi(g) = c_1\delta_{g,(3,1)} + c_2\delta_{g,(2,2)};\quad |c_1|^2 + |c_2|^2 = 1, \tag{36}$$

the Wigner distribution turns out to be

$$W(g; JMN\ M'N') = |c_1|^2 W_1(g; JMN\ M'N') + |c_2|^2 W_2(g; JMN\ M'N')$$
$$+ W_{\rm int}(g; JMN\ M'N'), \tag{37}$$

where

$$W_1(g; JMN\ M'N') = \frac{N_J}{21}\delta_{g,(3,1)} D^J_{MN}((3,1)) D^J_{M'N'}((3,1))^*, \tag{38}$$
$$W_2(g; JMN\ M'N') = \frac{N_J}{21}\delta_{g,(2,2)} D^J_{MN}((2,2)) D^J_{M'N'}((2,2))^*, \tag{39}$$

and

$$W_{\rm int}(g; JMN\ M'N') = \frac{N_J}{21}\delta_{g,(0,0)}[c_2^* c_1 D^J_{MN}((2,2)) D^J_{M'N'}((3,1))^*$$
$$+ c_1^* c_2\ D^J_{MN}((3,1)) D^J_{M'N'}((2,2))^*]. \tag{40}$$

The first two terms in (37) contain the Wigner functions of wave functions $\psi_1(g) = \delta_{g,(3,1)}$ and $\psi_2(g) = \delta_{g,(2,2)}$. The third term contains the 'interference' effects arising from the fact that the state ψ is a superposition of ψ_1 and ψ_2. Owing to the specific structure of ψ_1 and ψ_2, while $W_1(g; JMN\ M'N')$ and $W_2(g; JMN\ M'N')$ have supports at $g = (3,1)$ and $g = (2,2)$ respectively, $W_{\rm int}(g; JMN\ M'N')$ has support at $g = (0,0)$, the mid point of $(3,1)$ and $(2,2)$. The values of $W_1(g; JMN\ M'N'), W_2(g; JMN\ M'N')$ and $W_{\rm int}(g; JMN\ M'N')$ can be explicitly computed using (26), (30) and (31). Thus, for instance,

$$W_{\rm int}(g; 412\ 13) = \delta_{g,(0,0)} \times \frac{1}{7} c_2^* c_1 \exp\{-i\frac{8\pi}{7}\}. \tag{41}$$

To conclude, Fourier transforms on non Abelian finite groups find a wide variety of applications in areas such as signal processing, cryptology and quantum computation [15], [16] and we hope that the Wigner distribution formalism for all finite groups of odd order developed here may provide an alternative useful framework for examining some of these problems.

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