Bull. Indian Assoc. Cultiv. Sci. **5** 1–8 (1918)

The theory of the cyclical vibrations of a bowed string

C V RAMAN

Introduction

Ordinarily, the motion of every part of a stringed instrument excited by bowing is a periodic vibration capable of being represented by a simple Fourier series. The motion of the string and that of the bridge, belly and air, are all strictly periodic. Recently, in investigating a phenomenon the 'cello known as the "wolf-note", which is observed when the G-string or D-string is bowed at a pitch equal or nearly equal to that of the strongest resonance of the instrument, G W White* found that the motion of the belly was of large amplitude and, instead of being periodic, was cyclical in character. White considered the phenomenon to be due to the beats which arise when the pitch impressed on the system approaches the natural pitch of the resonator. This view, though it seems a plausible one at first sight, is open to objection, and on careful examination is found to be inconsistent with the observed facts. Perhaps the most effective criticism of the suggestion put forward by White is to be found in the fact first noticed by me that cyclical changes in the vibration of the belly can be obtained by bowing the G-string or D-string of a 'cello, rather near the bridge and with a suitable pressure, at *any* pitch lying between, say, 210 and 370 vibrations per second. (The pitch of the wolf-note is about 176 vibrations per second, and there is another fairly strong point of resonance at 360 vibrations per second). This fact alone is sufficient to dispose of any suggestion that the cyclical changes require for their production any close approximation between the natural frequency of the resonator and the impressed frequency of vibration.

From fundamental dynamical principles, it is evident that, if in any particular case the motion of the string is strictly periodic, the maintained vibration of every other part of the instrument must also be strictly periodic. It may be inferred, therefore, that when the motion of the belly is not simply periodic but is of a cyclical character, the vibrations of the string must also be of a cyclical character, and that the changes in the latter must precede (not follow) the corresponding changes in the former. This indication of theory can easily be verified experimentally, as shown in the communication by me published in *Nature*

*G W White, *Proc. Combridge Philos. Soc.*, June 1915.

(*London*).* As mentioned above, such cyclical changes occur not only at the wolf-note pitch, but also, under suitable conditions, at other frequencies of vibration, and any theory of the phenomenon must be capable of explaining this fact.

Modus operandi of the bow

It is in the mode of action of the bow that we must look for an explanation of the cyclical changes in the character of vibration of the string that occur under certain conditions. The system on which the bow acts consists of a stretched cord, one end of which is practically fixed, and the other end of which passes over the bridge, and is therefore capable of yielding under the periodic transverse and longitudinal components of the tension. Since the free periods of the string are modified by the yielding of the bridge, the problem of finding the motion maintained by the bow is by no means a simple one. As a preliminary to the discussion of the cyclical cases, we shall first briefly consider the general theory of the much more simple cases in which the vibration is of a periodic character.

The dissipation of energy through the communication of vibrations to the atmosphere as sound-waves is evidently the central feature of the problem. The maintenance of a steady, stable form of vibration of the string is only possible if there is an exact balance between these energy losses and the energy drawn from the bow, and if this balance is also not liable to be upset by any slight alteration in the pressure of the bow. If we assume that the pressure with which the bow is applied is sufficiently large, such energy-balance is only possible when the bowed point is carried forward by the bow with its own velocity for a considerable fraction of the period, and during the other part or parts of the period of vibration, slides down with a velocity which is not necessarily uniform or constant. For, if the relative velocity at the bowed point did not thus become zero during a considerable part of the period, the maintaining forces (due to the difference in the friction acting on the bowed point in the forward and backward motions) would be far in excess of those required to maintain the motion; whereas, by the relative velocity actually becoming zero, the frictional force in such stages falls below the maximum statical value and can thus automatically adjust itself so as to secure the requisite balance of energy supply and loss. Mathematically expressed, the relation may be written as

$$F(P, v - v_B) = P_0 + \sum_{n=1}^{n=\infty} k_n B_n \frac{\sin\left(\dfrac{2n\pi t}{T} + e_n + e_n'\right)}{\sin \dfrac{n\pi x_0}{l}},$$

*"On the wolf-note of the violin and 'cello," *Nature* (*London*) 29th June 1916, page 362–363. A fuller discussion of the phenomena observed at the wolf-note pitch will shortly appear in the *Philos. Mag.*

where P is the pressure of the bow, v_B its velocity, v the velocity of the bowed point, and $F(P, v - v_B)$ is the frictional force which is a function of the pressure and relative velocity. On the right-hand side, P_0 is a constant, $B_n \sin(2n\pi t/T + e_n)$ represents the nth harmonic component in the vibration of the string, x_0 is the distance of the bowed point from the fixed end, and k_n, e_n' are also constants. One side of the equation thus represents the force exerted by the bow, and the other side, the resultant of the forces required to maintain the vibration in the presence of dissipative forces.

From this result indicating the general nature of the motion at the bowed point, the next step is to deduce kinematical analysis, the characteristics of the various possible modes of vibration, and then, by summation of the series for the maintaining force, to find which of the types indicated by the kinematical analysis is dynamically possible for any given pressure or velocity of bowing. In two notes already published in this Bulletin,* I have already indicated the most convenient method of kinematical analysis. For our present purpose, it will be sufficient to remark that when the bow is applied at any point considerably removed from an end of the string, a considerable variety of possible modes of vibration is indicated. When however, the bow is applied near an end of the string, the range of possibilities becomes greatly restricted. In this case, if the motion at the bowed point be approximately representable by a two-step zig-zag vibration curve, the motion of every other point on the string is also approximately of a similar kind, as found by Helmholtz,† and the motion is uniquely determined, the only alternatives being types in which the vibration-curve at the bowed point is a four-step or a six-step zig-zag etc.

Conditions under which the motion is cyclical

The mechanical theory of the periodic types of vibration thus indicates that when the bow is applied close to an end of the string, and its pressure is insufficient to maintain the well known simple type of vibration described by Helmholtz, the motion at the bowed point must alter to a type in which it is a four-step zig-zag or a six-step zig-zag, etc. the fundamental becoming feeble and even falling out altogether. The case is quite different if the bow be applied at a point sufficiently removed from the end; the motion at the bowed point may then alter to a very considerable extent so as to adjust itself to a change in the pressure of the bow, and yet remain approximately a two-step zig-zag. In view of this, it is a significant experimental fact that cyclical changes of vibration-form occur only when the

*IV pages 1–4 and V pages 5–8.

†See also the paper by myself and another on 'Discontinuous wave motion' *Philos. Mag.* January 1916.

bow is applied with suitable pressure and velocity near an end of the string, and not when it is applied at a point sufficiently removed therefrom. We may reasonably infer that cyclical changes occur only when the pressure of bowing and other conditions are such that no one steady, stable form of vibration can be maintained.

In order to fix our ideas, we may consider a specific case in which the quantity k_1 is very large, in comparison with k_2, k_3, etc. Since k_1 relates to the fundamental component, it is evident from the expression given above, that the pressure of the bow must be very considerable in order that the ordinary type of vibration in which the fundamental is dominant might be maintained. The critical value of the pressure necessary is evidently $k_1 B_1(1-\sin e'_1)/\sin(\pi x_0/l)(\mu-\mu_A)$, where μ and μ_A are the coefficients of statical and dynamical friction, respectively. When the pressure is smaller than this critical value, the mode of vibration in which the fundamental is dominant becomes impossible. The various alternatives which then arise are, (*a*) the fundamental may fall out altogether, in which case the string would vibrate in two segments in the usual Helmholtzian mode: or (*b*), intermediate forms may arise. The contingency in (*a*) may arise, and generally does arise, if the pressure of the bow be reduced to a value not much in excess of the critical pressure for the type of vibration in two segments. The string would then settle down to a steady state of vibration. The intermediate forms of vibration referred to in (*b*), would only be possible when the pressure of the bow is less than the minimum necessary to ensure a steady vibration with dominant fundamental, but still much in excess of that required for the contingency in (*a*). It now remains to be seen why, in the conditions which exist when cyclical forms of vibration are set up, no steady state of vibration is possible.

The kinematical theory of the intermediate forms referred to in the preceding para is readily worked out, at any rate to a close approximation. They may be represented sufficiently closely by a mode of vibration in which two discontinuous changes of velocity travel along a string whose velocity-diagram consists of parallel straight lines. The discontinuous changes are, in general, unequal. The vibration-curve at the bowed point would, in general, be a four-step zig-zag in which the two ascending lines are straight and parallel, and the two descending lines are straight but not necessarily parallel to one another. In order that a steady vibration in such a mode might be possible, it is necessary that the force necessary to maintain it should be less than the statical friction in the two stages of ascent, and exactly equal to the values of the dynamical friction in the two stages of descent. As we assumed that k_1 is much larger than k_2, k_3, etc., the force required to maintain the fundamental is far in excess of those required to maintain the second, third harmonics, etc. Since a large value of k_1 is generally the result of the fundamental being strongly reinforced by the resonance of the instrument, the phase constant e'_1 would also be considerably different from the phase constants e'_2, e'_3 etc. It is readily shown that the resultant obtained by superposing these forces would not, in general, even approximately satisfy the condition for steady maintenance stated above; even if it did approximately satisfy the condition for some particular value of the bowing pressure, it would not do so for other values of the pressure, and the motion would not therefore be stable. It is thus found that for the values of the bowing pressure contemplated, none of the intermediate forms of vibration is capable of being steadily maintained. Since a finite interval of time is required for any change in the amplitude or form of vibration, either of the string or of the rest of the instrument, we see that the conditions of the case discussed are favourable for the production of cyclical forms of vibration.

Similarly when both k_1 and k_2 are very large compared with k_3, k_4, etc. or when k_2 by itself is large compared with the constants k_1, k_2, etc. and the bow is applied near an end of the string, cyclical forms of vibration are possible.

The cases in which k_3 or k_4, etc. are large compared with k_1, k_2, etc. are not of special interest. For, the amplitudes of the third and higher harmonics are small compared with that of the fundamental in the motion ordinarily maintained by bowing near an end of the string, and their influence is therefore too small to affect the possibility of a steady state of vibration.

Nature of the cyclical process

The main interest of the problem is in finding the character of the motion of the string at successive stages of the cycle, its relation to the corresponding changes in the vibration of the bridge and belly, the factors that determine these changes, and the number of periods of vibration comprised in each cycle. A rigorous mathematical formulation of all the conditions, though not impossible, would evidently be too complicated to be practically useful. The main results may however, be arrived at from broad theoretical considerations.

In the initial stage, when the bow acts upon the string *with a moderate pressure*, it would evidently be capable of setting up a form of vibration which, later on, would fail to be maintained on account of the insufficient pressure of the bow when the belly attains its maximum vibration. The change in the form of vibration of the string thus caused would, in its turn, result in a falling off of the vibration of the belly, but on account of the inertia of the belly, this change would naturally lag behind that of the string to a considerable extent. By the time the belly vibration reaches its minimum, the string would have already attained a form of vibration which, with the reduced amplitude of the bridge-vibrations, takes up considerably less energy than the bow is capable of yielding. As a result, the string commences to regain its original form of vibration, and this is subsequently followed by a revival of the vibrations of the belly. The cycle then repeats itself indefinitely.

While the description of the cyclical process given above holds good generally, individual cases differ very considerably in detail. If k_1 is large, the largest changes in amplitude occur in the fundamental component of the vibration of the string, and the vibration-curve of a point near an end of it passes by successive stages from the simple two-step to a four-step zig-zag (in which the fundamental is more or less feeble), and then back again to a two-step zig-zag. When k_2 is large, the principal changes are in the amplitude of the second harmonic, and the case is therefore somewhat more complicated than the first. When both k_1 and k_2 are large, we may have more than one kind of cyclical change possible.

As remarked above, the changes in the vibration of the belly lag behind those of the string by a considerable interval. Experiment shows this interval in a good many cases to be approximately quarter of a cycle. The total number of periods comprised in each cycle is evidently determined by the pressure of the bow, the frictional constants and the quantities k_1, k_2, etc. When k_1 or k_2 is large, the period of the cycle is practically a function of k_1 or k_2 as the case may be. It is then approximately twice the interval of time in which the vibrations of the belly would decrease from the largest to the smallest amplitude actually observed in the cycle, if the belly were first excited by steadily bowing the string and the bow were then suddenly removed.

Some experimental results

That the dissipation of energy is the controlling factor in determining the phenomena discussed in this paper, may be very prettily illustrated by loading the bridge of a 'cello with a brass clamp weighing about 44 grammes. Instead of getting two resonance-frequencies at 176 and 360 vibrations per sec. as is the case when there is no load on the bridge, we then get four resonance-points, whose frequencies are 100, 137, 184 and 233.

At the first and third of these frequencies, a very considerable vibration of the bridge and belly may be set up by bowing the string, but on removing the bow the vibration dies away quite slowly, showing that the rate of dissipation of energy is quite small at these frequencies. Cyclical effects are hardly noticeable at these frequencies, even with carefully-adjusted bowing pressure. At the two frequencies 136 and 237, however, the damping is much more marked, (though it is still considerably less than at the two corresponding frequencies 176 and 360 when there is no load on the bridge), and cyclical effects are obtained fairly easily.