Bull. Mater. Sci., Vol. 1, Numbers 3 and 4, December 1979, pp. 201–206. © Printed in India.

Critical properties of random magnetic systems

DEEPAK KUMAR
Department of Physics, University of Roorkee, Roorkee 247 672

MS received 24 October 1978

Abstract. A review of the recent applications of renormalisation group techniques to the calculations of the critical properties of random magnetic systems is presented.

1. Introduction

The theory of phase transitions and the critical phenomena in magnetic systems has now reached quite a mature stage. However, a large part of the work has been confined to certain idealistic models. For comparison with certain experiments, additional realistic features must be considered. For example, any real magnet invariably contains impurities and defects. How do such imperfections influence the ideal critical behaviour ? Such questions have only recently been tackled with the help of the powerful tool of renormalisation group. The purpose of this paper is to briefly describe some of the new developments in this area.

This paper is organised as follows. First, we describe the effects of a weak random perturbation on the critical properties of, say, a ferromagnet. The perturbation may be random local variation of exchange interaction. Next, we consider a case of strong randomness, namely, the spin-glass behaviour which occurs in systems having competing ferromagnetic and antiferromagnetic interactions.

2. Effects of the weak random perturbation

Much of the theoretical stimulus in this problem was provided by a heuristic argument due to Harris (1974), according to which a random perturbation like modulation of the exchange interactions (of short range) can alter the critical behaviour of the system only if the specific heat of the pure system is divergent at the critical point. For a system whose specific heat exponent α is negative, the random perturbation has no effect on the critical properties. This argument was later justified by explicit renormalisation group calculations. The argument itself is of interest and is therefore presented below.

Consider an alloy having a concentration x of defective bonds. Let us assume that this system has a sharp transition temperature $T_c(x)$. Now we try to see the consistency of this assumption. Consider a temperature T near $T_c(x)$, the correation length $\xi = [T - T_c(x)]^{-\nu}$. The fluctuations cause a local variation in $T_c(x)$.

The width of this variation can be estimated as follows. Divide the system into volumes Ω having ξ^d spins. Within this volume, the mean square fluctuation in the number of the defective bonds $\langle(\delta n)^2\rangle = x(1-x)\,\xi^d$. Thus the width $\triangle x$ of fluctuation in the concentration is

$$\triangle x = [x(1-x)\,\xi^d]^{1/2}/\xi^d. \tag{1}$$

Now $$\triangle T_c/T_c \approx \triangle x = [x(1-x)]^{1/2}\,\xi^{-d/2}. \tag{2}$$

In view of this fluctuation the correlation length cannot be larger than $(\triangle T/T_c)^{-\nu}$ or

$$\xi < [x(1-x)]^{-\nu/2}\,\xi^{\nu d/2}, \tag{3}$$

this requires that $\nu d/2 > 1$. Since $\alpha = 2 - \nu d$, it follows that when $\alpha < 0$, the above assumption of a sharp phase transition is a self-consistent one.

The renormalisation group calculations (Harris and Lubensky 1974; Lubensky 1975; Grinstein and Luther 1976; Aharony 1975; Krey 1975) have gone much farther than this intuitive argument. These calculations also obtain the new critical behaviour which results when $\alpha > 0$. One notable feature of the altered critical behaviour is that α becomes negative. So the specific heat of the impure system always has a cusped singularity.

For the renormalisation group calculations one usually starts with a Hamiltonian of the form

$$\mathcal{H} = \sum_{\vec{r}_1,\,\vec{r}_2} K(\vec{r}_1, \vec{r}_2)\, S(\vec{r}_1)\cdot S(\vec{r}_2) + \sum_{\vec{r}_1,\,\vec{r}_2,\,\vec{r}_3,\,\vec{r}_4} K_2(\vec{r}_1, \vec{r}_2, \vec{r}_3, \vec{r}_4)\, S(\vec{r}_1)\cdot S(\vec{r}_2)\, S(\vec{r}_3)\cdot S(\vec{r}_4) + \cdots \tag{4}$$

(where $\mathcal{H}$ stands for $-\beta H$ and H denotes the usual hamiltonian)

The coupling coefficients $K_1, K_2, \ldots$, etc. are random quantities with known probability distributions. The calculations have been performed by two methods both of which lead to the same results. The methods are : (i) recursions of the probability distribution, (ii) replica method.

In the method (i) for given values of $K_1, K_2, \cdots, K_n$, one derives the renormalised quantities $K_1', \cdots, K_n'$ in the standard way.

$$K_i' = f_i(K_1, K_2, \cdots, K_n). \tag{5}$$

Since K_i's are random quantities, the renormalised K_i's are also random. One therefore finds the probability distribution of the renormalised coefficients. This may be done through the equation

$$P'(\{K_i'\}) = \int \prod_{i=1}^{n} \delta(K_i' - f_i)\, P(\{K_1\})\, dK_i. \tag{6}$$

Thus for the random system one studies the transformations of the probability distribution of the coupling coefficients. The notion of the fixed point is generalised to the notion of the fixed point probability distribution. In practice, however, one parametrises the probability distribution by its first few moments and studies relations between the renormalised moments and the original moments.

Harris and Lubensky (1974) applied this method to the two-dimensional Ising model. They found that the pure system fixed point is stable to random perturbations, implying no change in the critical behaviour. The same method was also applied to the Landau–Ginzburg functional and an ϵ $(4-d)$ expansion was obtained for the Gaussian disorder (Lubensky 1975). Here one finds that besides the pure system fixed points, there is an additional fixed point characterised as the random fixed point. When $a > 0$, the pure Heisenberg fixed point becomes unstable and there is a cross-over to the random fixed point. In this case the critical behaviour is governed by the exponents associated with the random fixed point. These are listed in table 1. On the other hand, when $a < 0$ the pure Heisenberg fixed point remains stable.

The replica method was introduced by Edwards and Anderson (1975) to treat the spin-glass problem. Here one derives an effective non-random Hamiltonian which gives the correct averaged free energy for the random system (Grinstein and Luther 1976; Aharony 1975). The method consists in writing

$$\langle \ln z \rangle_{\mathrm{av}} = \lim_{n \to 0} \left[\frac{z^n - 1}{n} \right]_{\mathrm{av}} \tag{7}$$

$$\langle z \rangle_{\mathrm{av}} = \prod_{\alpha=1}^{n} T_{\gamma} \underset{\{S_\alpha(\vec{r})\}}{} [\exp \sum_{\alpha=1}^{n} \mathcal{H}(\{S_\alpha(\vec{r})\}]_{\mathrm{av}}$$

$$= T_{\gamma} \underset{\{S_\alpha(\vec{r})\}}{} \exp [\mathcal{H}_{\mathrm{eff}}(\{S_\alpha(\vec{r})\})]. \tag{8}$$

As an example consider the Hamiltonian

$$\mathcal{H} = \sum_{\vec{r}_1, \vec{r}_2} J(\vec{r}_1, \vec{r}_2)\, \vec{S}(\vec{r}_1) \cdot \vec{S}(\vec{r}_2). \tag{9}$$

Using the above procedure, one finds

$$\mathcal{H}_{\mathrm{eff}} = \sum_{p} \sum_{r_1, r_2} J_p [\sum_{\alpha=1}^{n} S_\alpha(\vec{r}_1) \cdot S_\alpha(\vec{r}_2)]^p, \tag{10}$$

where $J_1 = \langle J(r_1, r_2) \rangle_{\mathrm{av}}$, $J_2 = \langle J(r_1, r_2)^2 \rangle - \langle J \rangle^2 \cdots$ (11)

Table 1. Exponents for the Heisenberg and random fixed points.

	Heisenberg	Random
ν	$\frac{1}{2} + \frac{n+2}{4(n+8)}\epsilon + \frac{(n+2)(n^2+23n+60)}{8(n+8)^3}\epsilon^2$	$\frac{1}{2} + \frac{3n}{32(n-1)}\epsilon + \frac{n(127n^2-572n-32)}{4096(n-1)^3}$
η	$\frac{(n+2)}{2(n+8)^2}\epsilon^2$	$\frac{(5n-8)n}{256(n-1)^2}\epsilon^2$
a	$\frac{4-n}{2(n+8)}\epsilon - \frac{(n+2)(n^2+30n+56)}{4(n+8)^3}\epsilon^2$	$\frac{n-4}{8(n-1)}\epsilon - \frac{n(31n^2-380n-128)}{1024(n-1)^3}\epsilon^2$

Note that n refers to the dimensionality of the spin.

and J_p's refer to the pth cumulants of $J(\vec{r}_1, \vec{r}_2)$. One now performs the renormalisation group transformations on this effective Hamiltonian in the usual manner. To obtain the description of the random system, one takes the limit $n \to 0$ in the recursion relations. The procedure leads to results identical to those obtained by Harris and Lubensky (1974).

3. Critical behaviour of spin-glasses

Spin-glasses are systems in which competing ferromagnetic and antiferromagnetic interactions occur randomly with equal probabilities. So even though $\langle J(\vec{R}_1, \vec{R}_2)\rangle_{av} = 0$, a transition is supposed to occur at a temperature T_{sg} which in mean field theory is given by $k_B T_{sg} = zJ_2$, where z is the coordination number of the lattice and J_2 is defined in (11). The order parameter of the transition is essentially $\langle\, |\, \langle \vec{S}_i \rangle_T\, |^2 \,\rangle_{av}$ and is denoted by q. The mean field theory (Edwards and Anderson, 1975) gives the following exponents: $a = -1$, $\beta = 1$ and $\gamma = 1/2$. Thus the mean field exponents satisfy the scaling relation $dv = 2 - a$ when $d = 6$. The deviations from the mean field values occur when $d < 6$. An ϵ-expansion ($\epsilon = 6 - d$) was developed by Harris *et al* (1976) to obtain the deviations in the exponent values from their mean field values. From the Hamiltonian of (10) they derived a free energy functional for the spin-glass order parameter q. For m-component spins, the functional has the form

$$h = \tfrac{1}{4} mrq^2 - w(n-2)\, q^3 + n(n-1)\, uq^4, \tag{12}$$

where n denotes the number of the replicas and r, w, u are the usual parameters of the Landau–Ginzburg functional. Since q is a nonnegative quantity, this equation gives a second order transition. The exponents obtained by Harris *et al* (1976) are given in table 2. An interesting point to be noted is that $a < -1$, which implies that specific heat varies rather smoothly across the transition point. This indeed is observed in the experiments (Wenger and Keesom 1976) and in the Monte-Carlo studies of Binder and Schröder (1976).

Several authors have applied position space renormalisation group methods to this problem (Jayaprakash *et al* 1977; Young and Stinchcombe 1976; Plischke and Zobin 1977; Kinzel and Fischer 1978). However, the results of these methods are very much approximation-dependent and no definite conclusions have been derived. An interesting controversy exists regarding the spin-glass

Table 2. Exponents for the spin-glass transition in $d = 6 - \epsilon$.

$\nu = \frac{1}{2} + \frac{5m\epsilon}{12(m-1)}$	$a = -1 - \frac{3m+1}{2(2m-1)}\epsilon$
$\eta = \frac{-m\epsilon}{3(2m-1)}$	$\beta = 1 + \frac{m+1}{2(2m-1)}\epsilon$

transition in two dimensions. The calculations of Jayaprakash *et al* (1977), Plischke and Zobin (1977) and the computer simulations of Binder and Schröder (1976) show that a two-dimensional spin-glass undergoes a transition. On the other hand the calculations of Young and Stinchcombe (1976), Kinzel and Fischer (1977) and the Monte-Carlo simulations of Bray and Moore (1978) lead to the conclusion that there is no transition in two dimension.

Recently Anderson and Pound (1978) have put forward an argument to suggest that the lower critical dimensionality for the spin-glass transition is $d = 3$. This argument is based on the application of the Migdal–Kadanoff approximation to the classical spins. One can easily perform decimations on a chain of classical spins. At low temperatures, small oscillation approximation can be made to yield a particularly simple result.

$$\int \exp\left[\beta J_{12}\vec{S}_1 \cdot \vec{S}_2 + \beta J_{23}\vec{S}_2, \vec{S}_3\right] d\vec{S}_2 = \exp\left[\beta J'_{13}\, \vec{S}_1 \cdot \vec{S}_3\right] \tag{14}$$

where
$$\frac{1}{J'_{13}} = \text{sign}\,(J_{12}J_{23})\left(\frac{1}{|J_{12}|} + \frac{1}{|J_{23}|}\right). \tag{15}$$

The Migdal–Kadanoff approximation involves decimation along the chains and bond-shifting from other $d-1$ directions. Let us first apply this approximation to the pure ferromagnet, taking the scale factor to be L. This gives :

$$1/J' = L/L^{d-1}J,$$

or
$$J' = L^{d-2}J. \tag{16}$$

Thus if $d < 2$, $J' \to 0$ as $L \to \infty$. This implies that no ferromagnetic transition can occur for $d < 2$. A similar argument may now be applied to the spin-glass problem. In the bond shifting procedure, we are adding L^{d-1} bonds of random signs. The mean root square value of the sum is $L^{(d-1)/2}J$. Using this value in (16) leads to the equation

$$J' = L^{(d-3)/2}J, \tag{17}$$

which implies a lower critical dimensionality of 3 for the spin-glass transition.

Another very interesting piece of information regarding the spin-glass transition has been obtained by Fisch and Harris (1977). These authors have studied the high temperature expansion for the spin-glass order parameter susceptibility for d-dimensional hypercubic lattices. They have obtained the values of the susceptibility exponent γ_q in continuous dimensionality starting from $d = 6$. Since there is no simple field coupling to spin-glass parameter, the susceptibility is defined in the following way.

$$q = \left[\lim_{N\to\infty} \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} q_{ij}^2\right]_{\text{av}}, \tag{18}$$

$$q_{ij} = \sum_{\alpha=1}^{n} \langle S_i^\alpha S_j^\beta \rangle^2, \tag{19}$$

and
$$\chi_q = \left[\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} (q_{ij}^2 - q^2)\right]_{\text{av}}. \tag{20}$$

Table 3. Order parameter susceptibility exponent for *SG* transition.

d	γ_q	d	γ_q
6·0	1·00	4·75	2·70
5·75	1·25	4·50	3·70
5·50	1·53	4·25	7·10
5·25	1·84	4·00	∞
5·00	2·23	..	..

The results of the calculation are exhibited in table 3. At $d = 6$, $\gamma_q = 1$ as expected from the mean field calculation. But as d decreases, γ_q starts increasing, diverging at $d = 4$. The implications of this result are not quite clear. Either the order parameter q becomes irrelevant for the phase transition at $d = 4$ or perhaps there is no transition at all below $d = 4$. From experience of high temperature expansions for random ferromagnets (Rushbrook *et al* 1972), it may also be argued that the conclusions of this method regarding critical exponents of random systems are not very reliable.

We close the review at this point. The reader can now judge for himself, in what an uncertain and exciting state this field is at the moment.

References

Aharony A 1975 *Phys. Rev.* **B12** 3573
Anderson P W and Pound G M 1978 *Phys. Rev. Lett.* **40** 903
Binder K and Schröder K 1976 *Phys. Rev.* **B14** 2142
Bray A J and Moore M A 1977 *J. Phys.* **F7** L333
Edwards S F and Anderson P W 1975 *J. Phys.* **F5** 965
Fisch R and Harris A B 1977 *Phys. Rev. Lett.* **38** 785
Grinstein G and Luther A 1976 *Phys. Rev.* **B13** 1329
Harris A B 1974 *J. Phys.* **C7** 1671
Harris A B and Lubensky T C 1974 *Phys. Rev. Lett.* **33** 1540
Jayaprakash C, Chalupa C and Wortis M 1977 *Phys. Rev.* **B15** 1495
Kadanoff L P 1976 *Ann. Phys. (N.Y.)* **100** 359
Kinzel W and Fischer K H 1978 *J. Phys.* **C11** 2115
Krey U 1975 *Phys. Lett.* **A51** 189
Lubensky T C 1975 *Phys. Rev.* **B11** 3573
Plischke H and Zobin D 1977 *J. Phys.* **C10** 4571
Rusubrook *et al* 1972 *J. Phys.* **C5** 3371
Wenger and Keesom 1976 *Phys. Rev.* **B13** 4053
Young A P and Stinchcombe R 1976 *J. Phys.* **C9** 4419