Pramāṇa – J. Phys., Vol. 27, Nos 1 & 2, July & August 1986, pp. 307–320.

Interaction of second sound with acoustic waves in solids

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Abstract. An expression has been derived for the collision operator for phonons in a solid, which is valid at very low temperatures. The set of coupled equations for the elastic deformation and the phonon density or second sound has been reduced to a simple tractable form and the dispersion equation for the coupled waves consisting of the acoustic modes and second sound has been derived. It is shown that only the longitudinal mode interacts with the second sound. It is also shown that as a result of the interaction with the second sound, the longitudinal velocity along the principal axis acquires a correction term that is proportional to both γ^2 and T^4.

Keywords. Second sound; acoustic waves; phonons; two-fluid equations; collision operator.

PACS No. 63·20; 67·40

1. Introduction

Second sound is a wave-like propagation of heat in solids or liquids. It can alternatively be described as the oscillation in the fluctuation of density of the thermal phonons. The first speculation of the existence of second sound was in fact made by Nerst as early as in 1917 when he wrote: "Since in all probability heat has inertia, it is possible that at very low temperatures, an oscillatory discharge of thermal differences of potential might occur under certain circumstances". On the basis of the two-fluid theory, Tisza (1938) and Landau (1941) predicted the existence of temperature waves that propagate in superfluid helium. Peshkov (1944) predicted the existence of second sound in solids also. A detailed study of the drift velocity in a phonon gas and of second sound were made by Krumhausl and others. The first experimental observation of second sound in solids was made by Ackerman *et al* (1966) in solid ^{4}He. A detailed review of second sound in solids has been given by Ackerman and Guyer (1968). The theory of second sound in solids and related thermal conduction phenomena have been reviewed by Beck (1975).

An important progress in the development of a theory of second sound in solids was made by Goetze and Michael (1967a, b). Using Green's function formalism and phenomenological arguments, these authors derived a system of two fluid equations to describe a phonon gas and the elastic deformation in a solid. These equations at first sight appear to be mathematically complicated and probably for this reason or otherwise, no attempts have been made to solve them either analytically or numerically,

The author felicitates Prof. D S Kothari on his eightieth birthday and dedicates this paper to him on this occasion.

and apply them to solids in which second sound has been observed. It is the object of this paper to evaluate the physically important terms of these equations and to reduce the coupled set of equations in the phonon density and elastic deformation to a form that is easily amenable to either analytic or numerical solution. In §2, we have evaluated the collision operator for the phonon gas involving the anharmonicity co-efficients and given an analytic expression for this operator, that is valid at very low temperatures. In §4, we present the set of coupled equations in the elastic deformation and phonon density in a form as simple as the dispersion equation for acoustic waves in solids. It is shown that only the longitudinal mode interacts with second sound. Further, it is shown that the longitudinal velocity has a correction term proportional to T^4—a result that was experimentally observed by Franck and Hewko (1974). Our expression further shows that the correction term, apart from its proportionality to T^4, is also proportional to the square of the Gruneisen constant. Further second sound in solids should be highly anisotropic and should be dependent on the direction of propagation of the temperature wave.

2. The two-fluid equations

The two-fluid equations describing the coupled system of elastic vibrations and the phonon gas have been derived by Goetze and Michael (1967a, b) and also reviewed by Beck (1975). They are

$$\frac{\partial \phi}{\partial t}+\mathbf{v}\cdot\frac{\partial \phi}{\partial \mathbf{r}}+h_{ij}\frac{\partial^2 s_i}{\partial r_j \partial t} = L[\phi], \tag{1a}$$

$$\rho\frac{\partial^2 s_i}{\partial t^2} = F_i+[S_{ijmn}-\sum_k m(\mathbf{k})h_{ij}h_{mn}]\frac{\partial^2 s_m}{\partial r_j \partial r_n}-\sum_k h_{ij}m(\mathbf{k})\frac{\partial \phi}{\partial r_j}. \tag{1b}$$

If one substitutes $\phi, s \sim \exp[-i(\Omega t - q\cdot r)]$ these equations become

$$(-i\Omega+i\mathbf{q}\cdot\mathbf{v}_k-L)\phi = -\Omega q_r h_{rj} s_j, \tag{2a}$$

$$\rho\Omega^2 s_i = -F_i+q_j q_n \bar{S}_{ijmn} s_m + iq_r \langle h_{ir}|\phi\rangle. \tag{2b}$$

Here ϕ denotes the deviation of the density $N(\mathbf{k}, \mathbf{r}, t)$ of the phonon gas from the equilibrium phonon distribution $N_0(E)$ and is given by

$$N(\mathbf{k}, \mathbf{r}, t) = N_0(E)+m(\mathbf{k})\phi(\mathbf{k}, \mathbf{r}, t), \tag{3}$$

where

$$m(\mathbf{k}) = -\frac{\mathrm{d}N_0}{\mathrm{d}E_k} = -\beta N_0(N_0+1), \tag{4}$$

$$\beta = (1/kT) \quad \text{and} \quad E_k = \hbar\omega_k, \tag{5}$$

and the equilibrium distribution number N_0 for the phonons is given by

$$N_0(\mathbf{k}) = 1/[\exp(\beta\hbar\omega_k)-1]. \tag{6}$$

We use generally the notation of Beck (1975), but denote the energy of the phonon with momentum $\mathbf{k}$ and frequency ω_k by $\hbar\omega_k$ rather than ω_k as in Beck's article. Further, s is the elastic displacement vector and F is a possible mechanical force. Again $L[\phi]$ is the collision operator defined by

$$L[\phi] = \frac{-\pi}{8m(k)} \sum_{\mathbf{k}'} \sum_{\mathbf{k}''} \sum_{\mathbf{Q}} \left(\frac{mm'm''}{\beta} \right)^{1/2} \frac{1}{EE'E''}$$
$$\times \{2 \,|\, V_3(\mathbf{k}, \mathbf{k}', -\mathbf{k}'')|^2 \,\delta(E+E'-E'')\, \delta_{\mathbf{k}+\mathbf{k}'-\mathbf{k}'',\mathbf{Q}} [\phi + \phi' - \phi'']$$
$$+ \,|\, V_3(\mathbf{k}, -\mathbf{k}', -\mathbf{k}'')|^2 \,\delta(E-E'-E'')\, \delta_{\mathbf{k}-\mathbf{k}'-\mathbf{k}'',\mathbf{Q}} [\phi - \phi' - \phi'']\}, \quad (7)$$

where $\mathbf{Q}$ is a vector of the reciprocal lattice. The quantities h_{ij} are components of the Gruneisen parameter and are defined by

$$h_{ij}(\mathbf{k}) = -\frac{\delta E_k}{\delta u_{ij}} = -\hbar \frac{\delta \omega_k}{\delta u_{ij}} = -\hbar \omega_k \gamma_{ij}, \quad (8)$$

where u_{ij} are the components of the strain tensor and γ_{ij} are the generalized Gruneisen constants. To make the problem tractable, we assume that the Gruneisen parameter is isotropic (i.e)

$$\gamma_{ij} = \gamma \delta_{ij} \quad (9)$$

Thus $$h_{ij}(\mathbf{k}) = -\hbar \omega_k \gamma \delta_{ij}, \quad (10)$$

where γ is the Gruneisen constant.

If we replace for a moment the operator L by τ^{-1}, (2a) suggests that $\phi(\mathbf{k})$ is proportional to (a) ω_k in the first instance; (b) a function of $\mathbf{r}$ and t represented by the right hand side of (2a); and (c) by another function of the direction cosines of the vector $\mathbf{k}$, since the group velocity vector v_k on the left hand side depends on the orientation θ_k, ϕ_k of the propagation vector. So we write

$$\phi_k = \omega_k f(\theta_k, \phi_k) g(\mathbf{r}t). \quad (11)$$

This assumption is more general than the form of the distribution function assumed in the local equilibrium approximation (Beck 1975—equation (2.12)).

$$N_{\mathrm{LE}}(\mathbf{k}\mathbf{r}t) = N_0(\mathbf{k}) - m(\mathbf{k})\omega_k \frac{\delta\beta(\mathbf{r}t)}{\beta} \quad (12)$$

since it involves a factor of the angles θ_k, ϕ_k defining the direction of the vector $\mathbf{k}$.

Further the angular bracket $\langle \phi_1 | \phi_2 \rangle$ between two quantities is defined by

$$\langle \phi_1 | \phi_2 \rangle = \frac{1}{V} \sum_{\mathbf{k}} m(\mathbf{k}) \phi_1^*(\mathbf{k}) \phi_2(\mathbf{k}). \quad (13)$$

Hence

$$\langle h_{ir} | \phi \rangle = \frac{1}{V} \sum_{k} m(\mathbf{k})\, h_{ir}\, \phi_k \quad (14)$$

or

$$\langle h_{rr}|\phi\rangle = -\frac{\hbar\gamma}{V}\sum_{k} m(\mathbf{k})\omega_k\phi_k$$
$$= -\frac{\hbar\gamma}{V} g(\mathbf{r}t)\sum_{k} f(\theta_k\phi_k)m(\mathbf{k})\omega_k^2$$
$$= -\hbar\gamma g(\mathbf{r}t)\xi, \tag{15}$$

where

$$\xi = \frac{1}{V}\sum_{\mathbf{k}} m(\mathbf{k}) f(\theta_k\phi_k)\omega_k^2. \tag{16}$$

Again the modified elastic constants $\bar{S}_{ijrs}$ are defined by the equation

$$\bar{S}_{ij,rs} = S_{ij,rs} - \langle h_{ij}|h_{rs}\rangle, \tag{17}$$

where $S_{ij,rs}$ are the usual elastic constants. Substituting the expressions for h_{ij} and ξ, we find that

$$\bar{S}_{ij,rs} = S_{ij,rs} - \frac{\hbar^2\gamma^2}{V}\delta_{ij}\delta_{rs}\sum_{k} m(\mathbf{k})\omega_k^2$$
$$= S_{ij,rs} - \hbar^2\gamma^2\delta_{ij}\delta_{rs}\xi_0, \tag{18}$$

where

$$\xi_0 = \frac{1}{V}\sum_{k} m(\mathbf{k})\omega_k^2. \tag{19}$$

The quantities ξ_0 and ξ can be easily evaluated. First

$$m(\mathbf{k}) = -\beta N_0(N_0+1) = \frac{-\beta\exp(\beta\hbar\omega_k)}{[\exp(\beta\hbar\omega_k)-1]^2}. \tag{20}$$

As we are dealing with phenomena happening at very low temperatures, the Debye distribution can appropriately be used to describe the spectrum of the phonons. The sum over **k** can be transformed into an integral using the rule

$$V^{-1}\sum_{\mathbf{k}} \to \frac{1}{8\pi^3}\int \mathrm{d}\mathbf{k}$$
$$= \frac{1}{8\pi^3}\int k^2\mathrm{d}k\mathrm{d}\Omega. \tag{21}$$

Thus

$$\xi_0 = -\frac{\beta}{8\pi^3}\int \frac{\exp(\beta\hbar\omega_k)}{[\exp(\beta\hbar\omega_k-1)]^2}\,\omega_k^2 k^2\mathrm{d}k\mathrm{d}\Omega$$
$$= -\frac{\beta}{8\pi^3}\int\frac{C^2\mathrm{d}\Omega}{(\beta\hbar C)^5}\int_0^{x_{\max}}\frac{e^x}{(e^x-1)^2}\,x^4\,\mathrm{d}x. \tag{22}$$

At very low temperatures, the upper limit $x_{\max}$ can be replaced by infinity as in Debye's theory. The integral in x is then well known and has a value $4\pi^4/15$. Thus

$$\xi_0 = \frac{-\pi A_0}{30\beta^4\hbar^5}, \tag{23}$$

where

$$A_0 = \int \frac{\mathrm{d}\Omega}{C^3(\theta_k,\phi_k)}. \tag{24}$$

Again

$$\begin{aligned}\xi &= \frac{1}{V}\sum_{\mathbf{k}} m(\mathbf{k}) f(\theta_k, \phi_k)\omega_k^2 \\ &= \frac{-\pi A_1}{30\beta^4\hbar^5},\end{aligned} \tag{25}$$

where

$$A_1 = \int \frac{f(\theta_k, \phi_k)}{C^3(\theta_k, \phi_k)} \mathrm{d}\Omega. \tag{26}$$

3. The collision operator

The major difficulty in solving the two-fluid equations is in obtaining an estimate for the collision operator, which appears as a complicated expression with a double summation, double delta function and anharmonicity coefficients. However with an approximation for $V^3(\mathbf{k}_1 j_1, \mathbf{k}_2 j_2, \mathbf{k}_3 j_3)$ due to Klemens (see Maradudin and Fein 1962), the various terms in this expression can be evaluated—some exactly, some approximately—as we shall show below. Using a computer to evaluate integrals over the angular variables θ_k, ϕ_k numerically, it is possible to calculate the collision operator fully at low temperatures. Now Klemen's approximation may be written as (see Maradudin and Fein 1962)

$$\begin{aligned}&V^3(\mathbf{k}_1 j_1, \mathbf{k}_2 j_2, \mathbf{k}_3 j_3) \\ &= \frac{\hbar^{3/2}}{2^{3/2}6\sqrt{N}}\Delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\frac{\phi(\mathbf{k}_1 j_1\, \mathbf{k}_2 j_2 \mathbf{k}_3 j_3)}{[\omega(\mathbf{k}_1 j_1)\omega(\mathbf{k}_2 j_2)\omega(\mathbf{k}_3 j_3)]^{1/2}},\end{aligned} \tag{27}$$

where N denotes the number of unit cells in the crystal and further

$$|\phi(\mathbf{k}_1 j_1\, \mathbf{k}_2 j_2\, \mathbf{k}_3 j_3)|^2 = \frac{48\gamma^2}{MC^2}\omega_{j_1}^2(\mathbf{k}_1)\omega_{j_2}^2(\mathbf{k}_2)\omega_{j_3}^2(\mathbf{k}_3). \tag{28}$$

In the above, C is a typical sound wave velocity and γ is the Gruneisen constant. Further, if v_a is the volume of the unit cell, one may write $(M|v_a) = \rho$ where ρ is the density of the crystal.

The expression for $L(\phi)$ in the curly bracket contains two terms, each of which is a product of two delta factors. Consider for example the first term in the curly bracket. This term has two delta factors and will be non-vanishing only if

$$\mathbf{k}+\mathbf{k}'-\mathbf{k}''=0 \text{ or } \mathbf{Q} \tag{29a}$$

and
$$E+E'-E''=\hbar\{\omega(\mathbf{k})+\omega(\mathbf{k}')-\omega(\mathbf{k}'')\}=0. \tag{29b}$$

At very low temperatures, only the acoustic modes are excited and we shall use the dispersion relation of the form

$$\omega(\mathbf{k})=Ck, \tag{30}$$

where the sound wave velocity C is a function θ_k, ϕ_k of the propagation vector $\mathbf{k}$. Substituting the expression for $V^{(3)}$ from (27) and (28) taking into account the two delta factors and summing over k'', we find that

$$\begin{aligned}L[\phi]=\frac{-\pi\gamma^2}{48\,NMC^2\beta^{1/2}}\sum_{\mathbf{k}'}\Bigg\{&2\left[\frac{m(\mathbf{k}')m(\mathbf{k}+\mathbf{k}')}{m(\mathbf{k})}\right]^{1/2}\\&\times[\phi(\mathbf{k})+\phi(\mathbf{k}')-\phi(\mathbf{k}+\mathbf{k}')]\,\delta(E+E'-E'')\\&+\left[\frac{m(\mathbf{k}')m(\mathbf{k}-\mathbf{k}')}{m(\mathbf{k})}\right]^{1/2}[\phi(\mathbf{k})-\phi(\mathbf{k}')\\&-\phi(\mathbf{k}-\mathbf{k}')]\,\delta(E-E'-E'')\Bigg\}.\end{aligned} \tag{31}$$

We shall first evaluate the second summation, as this can be calculated exactly without approximation. We shall consider the case $\mathbf{Q}=0$, but our argument is not restricted to normal processes only and can be extended to Umklapp processes too. The delta factors lead to the conservation laws

$$\mathbf{k}-\mathbf{k}'-\mathbf{k}''=0 \tag{32a}$$

and
$$\omega_k-\omega_{k'}-\omega_{k''}=0. \tag{32b}$$

Now from (20) we have

$$\left[\frac{m(\mathbf{k}')m(\mathbf{k}-\mathbf{k}')}{m(\mathbf{k})}\right]^{1/2}=\frac{i\beta^{1/2}[1-\exp(-\beta\hbar\omega_k)]}{[1-\exp(-\beta\hbar\omega_{k'})][1-\exp(-\beta\hbar\omega_{k''})]} \tag{33}$$

Now

$$\begin{aligned}&\sum_{k'}\left[\frac{m(\mathbf{k}')m(\mathbf{k}-\mathbf{k}')}{m(\mathbf{k})}\right]^{1/2}\phi(\mathbf{k}'')\,\delta(\omega_k-\omega_{k'}-\omega_{k''})\\&=\frac{V}{8\pi^3}\,i\beta^{1/2}[1-\exp(-\beta\hbar\omega_k)]\\&\times\int\frac{\phi(\mathbf{k}'')\,\delta(\omega_k-\omega_{k'}-\omega_{k''})}{(1-\exp(-\beta\hbar\omega_{k'}))\,[1-\exp(-\beta\hbar\omega_{k''})]}\,\mathrm{d}\mathbf{k}'.\end{aligned} \tag{34}$$

In this integral, if we make the change of variables $\mathbf{k}'' = \mathbf{k} - \mathbf{k}'$, the variables $\mathbf{k}'$ and $\mathbf{k}''$ get interchanged in the integrand, which is symmetric in these variables. Hence this transformation does not alter the integrand including the delta factor, while on the other hand $\mathrm{d}\mathbf{k}'' = -\mathrm{d}\mathbf{k}'$. So

$$I = -\frac{V}{8\pi^3} i\beta^{1/2}[1-\exp(-\beta\hbar\omega_k)] \times \int \frac{\phi(\mathbf{k}'')\delta(\omega_k - \omega_{k'} - \omega_{k''})}{[1-\exp(-\beta\hbar\omega_{k'})][1-\exp(-\beta\hbar\omega_{k''})]} \mathrm{d}\mathbf{k}''. \tag{35}$$

Hence

$$\sum_{\mathbf{k}'} \left[\frac{m(\mathbf{k}')m(\mathbf{k}-\mathbf{k}')}{m(\mathbf{k})}\right]^{1/2} [\phi(\mathbf{k}') + \phi(\mathbf{k}'')]\,\delta\hbar(\omega_k - \omega_{k'} - \omega_{k''}) = 0, \tag{36}$$

as the two integrals are of opposite sign and cancel each other.

Finally consider

$$\begin{aligned}\sum_{\mathbf{k}'} &\left[\frac{m(\mathbf{k}')m(\mathbf{k}-\mathbf{k}')}{m(\mathbf{k})}\right]^{1/2} \phi(\mathbf{k})\,\delta(\omega_k - \omega_{k'} - \omega_{k''}) \\ &= i\beta^{1/2}\phi(\mathbf{k})[1-\exp(-\beta\hbar\omega_k)] \\ &\quad\times \sum_{\mathbf{k}'} \frac{\delta(\omega_k - \omega_{k'} - \omega_{k''})}{[1-\exp(-\beta\hbar\omega_{k'})][1-\exp(-\beta\hbar\omega_{k''})]}.\end{aligned} \tag{37}$$

Now

$$\begin{aligned}S &= \frac{1}{[1-\exp(-\beta\hbar\omega_{k'})][1-\exp(-\beta\hbar\omega_{k''})]} \\ &= \sum_n \sum_m \exp(-\beta\hbar)(n\omega_{k'} + m\omega_{k''}).\end{aligned}$$

In view of the delta factor, $\omega_{k'} + \omega_{k''} = \omega_k$: one can separate out terms $n = m$ in the double summation and this term gives a contribution $\exp(-n\beta\hbar\omega_k)$. Thus if $\omega_{k'} + \omega_{k''} = \omega_k$

$$\begin{aligned}S &= \sum_n \exp(-n\beta\hbar\omega_k) + \sum_{n>m} \exp[-\beta\hbar(n\omega_{k'} + m\omega_{k''})] \\ &\quad + \sum_{n<m} \exp[-\beta\hbar(n\omega_{k'} + m\omega_{k''})] \\ &= [1-\exp(-\beta\hbar\omega_k)]^{-1} + \sum_{n>m} \exp(-m\beta\hbar\omega_k)\exp[-(n-m)\omega_{k'}] \\ &\quad + \sum_{n<m} \exp(-n\beta\hbar\omega_k)\exp[-(m-n)\omega_{k''}].\end{aligned} \tag{38}$$

The difference between the two terms lies in the exponential function, which involves

$\omega_{k'}$ in one term and $\omega_{k''}$ in the other. Hence

$$\sum_{k'}\left[\frac{m(\mathbf{k}')m(\mathbf{k}-\mathbf{k}')}{m(\mathbf{k})}\right]^{1/2}\phi(\mathbf{k})\,\delta(\omega_k-\omega_{k'}-\omega_{k-k'})$$

$$=i\beta^{1/2}\phi(\mathbf{k})\sum_{k'}\delta(\omega_k-\omega_{k'}-\omega_{k''})$$

$$+\frac{i\beta^{1/2}}{8\pi^3}V\phi(\mathbf{k})[1-\exp(-\beta\hbar\omega_k)]$$

$$\times\left\{\sum_{n>m}\exp(-m\beta\hbar\omega_k)\int\exp[-(n-m)\,\omega_{k'}]\,\delta\,(\omega_k-\omega_{k'}-\omega_{k''})\,\mathrm{d}\mathbf{k}'\right.$$

$$\left.+\sum_{m>n}\exp(-n\beta\hbar\omega_k)\int\exp[-(n-m)\omega_{k''}]\,\delta\,(\omega_k-\omega_{k'}-\omega_{k''})\,\mathrm{d}\mathbf{k}'\right\}.$$

By changing the integration variable from $\mathbf{k}'$ to $\mathbf{k}''$ by means of the substitution $\mathbf{k}''=\mathbf{k}-\mathbf{k}'$, the delta factor remains unaltered while $\mathrm{d}\mathbf{k}''=-\mathrm{d}\mathbf{k}'$. By renaming the summation indices n and m into m and n in one of the two terms above, we see that the two integrals cancel each other. Thus

$$I=\sum_{k'}\left\{\frac{m(\mathbf{k}')m(\mathbf{k}-\mathbf{k}')}{m(\mathbf{k})}\right\}^{1/2}\phi(\mathbf{k})\delta\hbar\,(\omega_k-\omega_{k'}-\omega_{k''})$$

$$=\hbar^{-1}\,i\beta^{1/2}\,\phi(\mathbf{k})\sum_{k'}\delta\,(\omega_k-\omega_{k'}-\omega_{k''}).\tag{39}$$

Let us suppose that out of the $3N$ wave vectors $\mathbf{k}$, the relation $\omega_k-\omega_{k'}-\omega_{k''}=0$ is satisfied for $3M$ values, and we shall write $Z=(M|N)$. Then

$$I=\hbar^{-1}i\beta^{1/2}\,\phi(\mathbf{k})Z.\tag{40}$$

From the values of the elastic constants and the structure of the crystal, Z can be evaluated with a computer. Though the above result has been derived for $\mathbf{Q}=0$, it is formally true even if $\mathbf{k}''=\mathbf{k}-\mathbf{k}'+\mathbf{Q}$ where $\mathbf{Q}$ is a reciprocal lattice vector as the argument will not be altered.

We shall next consider the three sums contained in the first term of (31). Here the conservation laws are

$$\omega_k+\omega_{k'}-\omega_{k''}=0,\tag{41a}$$

$$\mathbf{k}+\mathbf{k}'-\mathbf{k}''=0.\tag{41b}$$

Again

$$\left[\frac{m(\mathbf{k}')m(\mathbf{k}+\mathbf{k}')}{m(\mathbf{k})}\right]^{1/2}=\frac{i\beta^{1/2}[1-\exp(-\beta\hbar\omega_k)]\exp(-\beta\hbar\omega_{k'})}{[1-\exp(-\beta\hbar\omega_{k'})][1-\exp(-\beta\hbar\omega_{k+k'})]}.\tag{42}$$

Unlike the previous case, the terms containing $\phi(\mathbf{k}')$ and $\phi(\mathbf{k}'')$ do not cancel here, and it is not feasible to evaluate the integral exactly because of the delta factor. We shall therefore use the approximation,

$$k''=|\mathbf{k}+\mathbf{k}'|\cong k+k'\cos\theta,\tag{43}$$

where θ is the angle between the vectors **k** and **k**$'$. Such approximations are used in other contexts too, for example in evaluating the attenuation of acoustic waves as a result of three phonon processes (Truell *et al* 1969). By expanding $|\mathbf{k}+\mathbf{k}'|$ as a series involving Legendre functions, one can of course improve the accuracy.

Now

$$\sum_{k'}\left[\frac{m(\mathbf{k}')m(\mathbf{k}+\mathbf{k}')}{m(\mathbf{k})}\right]^{1/2}\phi(\mathbf{k})\,\delta\hbar\,(\omega_k+\omega_{k'}-\omega_{k+k'})$$

$$=\frac{V}{8\pi^3\hbar}\,i\beta^{1/2}[1-\exp(-\beta\hbar\omega_k)]\,\phi(\mathbf{k})\times I_1, \tag{44}$$

where

$$I_1=\frac{\displaystyle\int\exp(-\beta\hbar\omega_{k'})\,\delta(\omega_k+\omega_{k'}-\omega_{k+k'})\,\mathrm{d}\mathbf{k}'}{[1-\exp(-\beta\hbar\omega_{k'})]\,[1-\exp(-\beta\hbar\omega_{k+k'})]}. \tag{45}$$

Now

$$\begin{aligned}\omega_k+\omega_{k'}-\omega_{k+k'}&=C_{\mathbf{k}}k+C_{\mathbf{k}'}k'-C_{\mathbf{k}+\mathbf{k}'}|\mathbf{k}+\mathbf{k}'|\\&=C_{\mathbf{k}}k+C_{\mathbf{k}'}k'-C_{\mathbf{k}+\mathbf{k}'}(k+k'\cos\theta)\\&=(C_{\mathbf{k}}-C_{\mathbf{k}+\mathbf{k}'})k+k'(C_{\mathbf{k}'}-C_{\mathbf{k}+\mathbf{k}'}\cos\theta)\\&=(C_{\mathbf{k}+\mathbf{k}'}\cos\theta-C_{\mathbf{k}'})\left\{\frac{(C_{\mathbf{k}}-C_{\mathbf{k}+\mathbf{k}'})}{(C_{\mathbf{k}+\mathbf{k}'}\cos\theta-C_{\mathbf{k}'})}k-k'\right\}.\end{aligned} \tag{46}$$

Hence

$$\delta(\omega_k+\omega_{k'}-\omega_{k+k'})=(C_{\mathbf{k}+\mathbf{k}'}\cos\theta-C_{\mathbf{k}'})^{-1}\,\delta(\psi k-k'), \tag{47}$$

where

$$\psi=(C_{\mathbf{k}}-C_{\mathbf{k}+\mathbf{k}'})/(C_{\mathbf{k}+\mathbf{k}'}\cos\theta-C_{\mathbf{k}'}). \tag{47a}$$

The delta factor in the integral (45) will be non-vanishing if

(1) $(C_{\mathbf{k}}-C_{\mathbf{k}+\mathbf{k}'})$ and $(C_{\mathbf{k}+\mathbf{k}'}\cos\theta-C_{\mathbf{k}'})$

are of the same sign and (2) if $k'=\psi k$. Hence only those elements of the solid angle $\mathrm{d}\Omega_k$ in the integral contribute, for which

$(C_{\mathbf{k}}-C_{\mathbf{k}+\mathbf{k}'})$ and $(C_{\mathbf{k}+\mathbf{k}'}\cos\theta-C_{\mathbf{k}'})$

are of the same sign. The integration over **k**$'$ can now be performed very easily in (45) because of the delta factor and we find that

$$I_1=\frac{\displaystyle\int(C_{\mathbf{k}+\mathbf{k}'}\cos\theta-C_{\mathbf{k}'})^{-1}\exp(-\beta\hbar C_{\mathbf{k}'}k\psi)\,k^2\psi^2\,\mathrm{d}\Omega_{k'}}{[1-\exp(-\beta\hbar C_{\mathbf{k}'}k\psi)]\,[1-\exp(-\beta\hbar C_{\mathbf{k}+\mathbf{k}'}(C_{\mathbf{k}}k+C_{\mathbf{k}'}k\psi)]}, \tag{48}$$

where the integration is to be performed only over those regions of the unit sphere for which $(C_{\mathbf{k}}-C_{\mathbf{k}+\mathbf{k}'})$ and $(C_{\mathbf{k}+\mathbf{k}'}\cos\theta-C_{\mathbf{k}'})$ are of the same sign. Consider next

$$I_2=\frac{\int \exp(-\beta\hbar\omega_{k'})\,\phi(\mathbf{k}')\,\delta(\omega_k+\omega_{k'}-\omega_{k+k'})\,\mathrm{d}\mathbf{k}'}{[1-\exp(-\beta\hbar\omega_{k'})][1-\exp(-\beta\hbar\omega_{k+k'})]}. \tag{49}$$

If we substitute as in (11) viz,

$$\phi(\mathbf{k}')=C_{\mathbf{k}'}k'f(\theta_{k'},\phi_{k'})g, \tag{50}$$

We find that

$$I_2=\frac{g\int \exp(-\beta\hbar C_{\mathbf{k}'}k')(C_{\mathbf{k}'}f)\,\delta(\omega_k+\omega_{k'}-\omega_{k+k'})k'^3\,\mathrm{d}k'\,\mathrm{d}\Omega_{k'}}{[1-\exp(-\beta\hbar\omega_{k'})][1-\exp(-\beta\hbar\omega_{k+k'})]}. \tag{51}$$

Evaluating the integral over $\mathbf{k}'$ by substituting (47) for the delta factor, we get

$$I_2=\frac{g\int (C_{\mathbf{k}+\mathbf{k}'}\cos\theta-C_{k'})^{-1}\exp(-\beta\hbar C_{\mathbf{k}'}k\psi)\,(C_{\mathbf{k}'}f_{k'})k^3\psi^3\,\mathrm{d}\Omega_{k'}}{[1-\exp(-\beta\hbar C_{\mathbf{k}'}k\psi)]\{1-\exp[-\beta\hbar k(C_{\mathbf{k}}+C_{\mathbf{k}'}\psi)]\}}. \tag{52}$$

As before the above integral should be evaluated only over those regions for which

$$(C_{\mathbf{k}}-C_{\mathbf{k}+\mathbf{k}'})\quad\text{and}\quad(C_{\mathbf{k}+\mathbf{k}'}\cos\theta-C_{\mathbf{k}'})$$

are of the same sign. Consider finally the integral

$$I_3=\frac{\int \exp(-\beta\hbar\omega_{k'})\,\phi(\mathbf{k}'')\,\delta(\omega_k+\omega_{k'}-\omega_{k+k'})\,\mathrm{d}\mathbf{k}'}{[1-\exp(-\beta\hbar\omega_{k'})][1-\exp(-\beta\hbar\omega_{k''})]} \tag{53}$$

Here
$$\begin{aligned}\phi(\mathbf{k}'')&=f(\theta_{k''},\phi_{k''})\omega_{k''}\,g(\mathbf{r},t)\\&=f(\theta_{k''},\phi_{k''})(\omega_k+\omega_{k'})\,g(\mathbf{r},t).\end{aligned} \tag{54}$$

We might mention that $f(\theta_{k''},\phi_{k''})$ is a function of the angular variables (θ_k,ϕ_k) and $(\theta_{k'},\phi_{k'})$. Hence

$$\begin{aligned}I_3&=\frac{\int \exp(-\beta\hbar\omega_{k'})\,(f_{k''}g)\,(C_{\mathbf{k}}k+C_{\mathbf{k}'}k')\,\delta(\omega_k+\omega_{k'}-\omega_{k''})k'^2\,\mathrm{d}\Omega_{k'}\,\mathrm{d}k'}{[1-\exp(-\beta\hbar\omega_{k'})][1-\exp(-\beta\hbar\omega_{k''})]}\\&=\frac{g\int (C_{\mathbf{k}+\mathbf{k}'}\cos\theta-C_{k'})^{-1}\exp(-\beta\hbar C_{k'}k\psi)\,f_{k''}(C_{\mathbf{k}}k+C_{\mathbf{k}'}k\psi)k^2\psi^2\,\mathrm{d}\Omega_{k'}}{[1-\exp(-\beta\hbar C_{\mathbf{k}'}k\psi)]\{1-\exp[-\beta\hbar k(C_{\mathbf{k}}+C_{\mathbf{k}'}\psi)]\}}.\end{aligned} \tag{55}$$

Thus

$$\sum_{k'}\left[\frac{m(\mathbf{k}')m(\mathbf{k}+\mathbf{k}')}{m(\mathbf{k})}\right]^{1/2}[\phi(\mathbf{k})+\phi(\mathbf{k}')-\phi(\mathbf{k}+\mathbf{k}')]\,\delta(\omega_k+\omega_{k'}-\omega_{k+k''})$$

$$=\frac{V}{8\pi^3}i\beta^{1/2}\,[1-\exp(-\beta\hbar\omega_k)]\,[I_1\phi(\mathbf{k})+I_2-I_3]. \tag{56}$$

This sum is proportional to k^3. We had considered only normal processes for which $\mathbf{Q}=0$, but the argument can be extended to Umklapp processes too. For this, one should replace $\mathbf{k}$ by $(\mathbf{k}+\mathbf{Q})$ and sum over $\mathbf{Q}$. Normally it is sufficient to sum over values of the $\mathbf{Q}$ vectors in the first supercell centred around the origin in the reciprocal space. The integral over θ' and ϕ' can be performed numerically with a computer. Combining (31), (39), (40) and (56), we find that the collision operator is given by

$$L[\phi]=\frac{-\pi\gamma^2 i}{48\,NM\,C^2\hbar}\left\{\phi(\mathbf{k})Z+\frac{V}{4\pi^3}[1-\exp(-\beta\hbar\omega_k)]\,[I_1\phi(\mathbf{k})+I_2-I_3]\right\}. \tag{57}$$

4. Coupled wave equations for second sound and acoustic waves

The modified elastic constants have been defined in (18). For cubic crystals, we have

$$\bar{S}_{11,11}=\bar{C}_{11}=C_{11}-\hbar^2\gamma^2\xi_0 = C_{11}+\frac{\pi\hbar^2\gamma^2A_0}{30\,\hbar^5}(kT)^4, \tag{58}$$

$$\bar{S}_{11,22}=\bar{S}_{11,33}=\bar{C}_{12} = C_{12}+\frac{\pi\hbar^2\gamma^2A_0}{30\,\hbar^5}(kT)^4, \tag{59}$$

$$\bar{S}_{11,23}=\bar{S}_{11,31}=\bar{S}_{11,12}=0, \tag{60}$$

$$\bar{S}_{12,12}=\bar{S}_{23,23}=\bar{S}_{31,31}=\bar{C}_{44}=C_{44}. \tag{61}$$

We denote the modified elastic constants by a bar. We may note that $\bar{C}_{11}$ and $\bar{C}_{12}$ have a correction term that is proportional to T^4. Most of the experiments on the second sound at low temperatures have been carried out for solid He^4, which belongs to the hexagonal class. For crystals belonging to the hexagonal system, the elastic constants are given by

$$\bar{C}_{11}=C_{11}-\hbar^2\gamma^2\xi_0, \tag{62a}$$

$$\bar{C}_{33}=C_{33}-\hbar^2\gamma^2\xi_0;\quad \bar{C}_{13}=C_{13}-\hbar^2\gamma^2\xi_0, \tag{62b}$$

$$\bar{C}_{12}=C_{12}-\hbar^2\gamma^2\xi_0;\quad \bar{C}_{44}=C_{44}, \tag{62c}$$

$$\bar{C}_{66}=C_{66}=\tfrac{1}{2}(\bar{C}_{11}-\bar{C}_{12})=\tfrac{1}{2}(C_{11}-C_{12}). \tag{62d}$$

Thus as a result of the interaction of the phonon gas with the elastic deformation, the constants $C_{11}, C_{12}, C_{13}, C_{33}$ get modified by the presence of an additional term that is proportional to T^4.

Let us take the angular average of (2a) and cancel the factor ω_k throughout. We shall further write

$$\langle \mathbf{q}\cdot\mathbf{v}_k \rangle = \frac{1}{4\pi}\int (\mathbf{q}\cdot\mathbf{v}_k) f(\theta_k, \phi_k)\,\mathrm{d}\Omega_k; \tag{63a}$$

$$p = \frac{1}{4\pi}\int f(\theta_k, \phi_k)\,\mathrm{d}\,\Omega_k. \tag{63b}$$

Also

$$\langle L(\phi) \rangle = i\gamma^2 \omega_k g S. \tag{63c}$$

Then (2a) becomes

$$\Omega\hbar\gamma q_r s_r + i(p\Omega - \langle \mathbf{q}\cdot\mathbf{v}_k \rangle + S\gamma^2) g = 0. \tag{64}$$

In the absence of any external forces, the coupled set of equations for a wave propagating along a direction (lmn) or $(q_1/q, q_2/q, q_3/q)$ in a cubic crystal are given by

$$\rho V^2 s_1 = \{\bar{C}_{11} l^2 + \bar{C}_{44}(m^2+n^2)\} s_1 + (\bar{C}_{12}+\bar{C}_{44}) lm\, s_2 + (\bar{C}_{12}+\bar{C}_{44})\, ln\, s_3 - \frac{il\gamma\hbar}{q}\xi g \tag{65}$$

where

$$\xi = \frac{-\pi(kT)^4}{30\hbar^5} A_1, \tag{66}$$

and two similar equations.

We shall write $(\Omega/q) = V$ and $\mathbf{n} = (\mathbf{q}/q)$. Eliminating s_1, s_2, s_3 and g from (65) and (64), we get

$$\begin{vmatrix} \bar{C}_{11} l^2 + \bar{C}_{44}(m^2+n^2) - \rho V^2 & (\bar{C}_{12}+\bar{C}_{44}) lm & (\bar{C}_{12}+\bar{C}_{44}) ln & -\frac{i\gamma\hbar}{q}\xi l \\ (\bar{C}_{12}+\bar{C}_{44}) ln & \bar{C}_{11} m^2 + \bar{C}_{44}(n^2+l^2) - \rho V^2 & (\bar{C}_{12}+\bar{C}_{44}) ln & -\frac{i\gamma\hbar}{q}\xi m \\ (\bar{C}_{12}+\bar{C}_{44}) ln & (\bar{C}_{12}+\bar{C}_{44}) mn & \bar{C}_{11} n^2 + \bar{C}_{44}(l^2+m^2) - \rho V^2 & -\frac{i\gamma\hbar}{q}\xi n \\ \hbar\gamma l V & \hbar\gamma m V & \hbar\gamma n V & i\left(pV - \langle \mathbf{n}\cdot\mathbf{V}_k \rangle + \frac{S}{q}\gamma^2\right) \end{vmatrix} = 0. \tag{67}$$

For propagation along the principal axis or the direction (100), this equation simplifies much and is given by

$$(\bar{C}_{44} - \rho V^2)^2 \left\{ (\bar{C}_{11} - \rho V^2)\left(pV - \langle \mathbf{n}\cdot\mathbf{v}_k \rangle + \frac{S\gamma^2}{q} \right) + \frac{\hbar^2\gamma^2\xi}{q} V \right\} = 0. \tag{68}$$

Two roots of this equation are given by $(\bar{C}_{44} - \rho V^2) = 0$. These are the two transverse modes. Since $\bar{C}_{44} = C_{44}$, the coupling between the acoustic waves and the phonon gas does not affect the velocities of the two transverse modes and these are still given by the classical value $(C_{44}/\rho)^{1/2}$. The term in the curly bracket shows that the longitudinal mode interacts with the second sound. This is a cubic equation in V (not ρV^2) and its roots will give the velocity of the longitudinal mode as well as the velocity of the second sound, along with the *damping*. We do not propose to solve this equation numerically here but without solving this equation directly, one can approximately obtain an expression for the longitudinal velocity. From (68) we have

$$\rho V^2 = \bar{C}_{11} + \frac{\hbar^2 \gamma^2 \xi V}{q\{pV - \langle \mathbf{n} \cdot \mathbf{v}_k \rangle + S\gamma^2/q\}}. \tag{69}$$

To find out the velocity of the longitudinal mode, we solve the above equation by iteration and substitute a trial solution $V_0 = (C_{11}/\rho)^{1/2}$ on the right-hand side. Then writing $\xi = -BT^4$, we get

$$\rho V^2 = \bar{C}_{11} - \frac{BT^4 \hbar^2 \gamma^2 (C_{11}/\rho)^{1/2}}{q[p(C_{11}/\rho)^{1/2} - \langle \mathbf{n} \cdot \mathbf{v}_k \rangle + S\gamma^2/q]} \tag{70}$$

The above equation shows that the longitudinal velocity has a correction term which is proportional to both T^4 and γ^2. Franck and Hewko (1974) measured the longitudinal sound wave velocity as a function of temperature for ^{4}He from 0·75 K to nearly 4 K. They found that the velocity always increases with falling temperature in accordance with an equation of the form $C = (C_0 - AT^4)$. Our result gives an explanation for this experimental observation regarding the dependence of the correction term on the fourth power of T and also shows that the correction term should be proportional to γ^2 besides.

Franck and Hewko carried out the measurements in fact for HCP ^{4}He, though (70) has been derived for cubic crystals. But the structure of the equation continues to be the same for hexagonal crystals too, as can be seen from the values of the modified elastic constants and the dispersion equation along (0 0 1) direction for hexagonal crystals. Thus as a result of the coupling between the phonon gas and the elastic deformation, the longitudinal wave velocity gets modified by the presence of an additional term that is proportional to both T^4 as well as γ^2. Secondly the second sound interacts with the longitudinal mode only for propagation along the principal axes.

A casual look at the expression for the collision operator suggests that $L(\phi)$ is strongly dependent on the direction of propagation of phonons. It is well known now that at very low temperatures, phonons propagate ballistically, and are focussed in certain directions. (Maris 1971; Jacob Philip and Viswanathan 1978; Northrop and Wolfe 1980). Obviously, the second sound which depends on the fluctuations in the number density of the phonons, should also be correlated with phonon focussing and should be prominently observable in directions where there is an abundance of phonons. A thorough numerical investigation of the collision operator and the coupled dispersion equation would throw more light on the directional dependence of second sound in solids but this will be dealt with elsewhere.

References

Ackerman C C, Bertman B, Fairbank H A and Guyer R A 1966 *Phys. Rev. Lett.* **16** 789
Ackerman C C and Guyer R A 1968 *Ann. Phys.* **50** 128

Beck H 1975 *Dynamical properties of solids* (eds) G H Horton and A A Maradudin (Amsterdam: North Holland) Vol. 2, Ch. 4.
Franck J P and Hewko R A D 1974 *Low temperature physics: Quantum crystals and magnetism* (eds) K D Timmerhaus, W J O'Sullivan and E F Hammel (New York and London: Plenum)
Goetze W and Michel K H 1967a *Phys. Rev.* **156** 963
Goetze W and Michel K H 1967b *Phys. Rev.* **157** 738
Jacob Philip and Viswanathan K S 1978 *Phys. Rev.* **B17** 4769
Landau L D 1941 Zh E.T.F. **11** 592
Maradudin A A and Fien A E 1962 *Phys. Rev.* **128** 2589
Maris H J 1971 *J. Acoust. Soc. Am.* **50** 812
Nerst W 1917 *Gie Theoretics Chen. Groundlagen des neuen Warmes atzes* (Halle: Knapp)
Northrop G A and Wolfe J P 1980 *Phys. Rev.* **B22**
Peshkov V 1944 *J. Phys.* (*Moscow*) **8** 131
Tisza L 1938 *C R Acad Sci.* (*Paris*) **207** 1035, 1186
Truell R, Elbaum C and Chich B B 1969 *Ultrasonic methods in Solid State Physics* (New York: Academic Press) p. 407