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Ordinary and extraordinary cyclotron waves in metals

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Abstract. Dispersion equations for the ordinary and extraordinary cyclotron waves propagating perpendicular to the magnetic field in metals in the critical region where the wavelength is comparable to the electron Larmor radius are derived as an infinite but rapidly converging power series expansion in $\bar{\delta}(=\omega/\Omega-M)$. Numerical studies for the cyclotron wave propagation near the first seven resonances are carried out. The non-local behaviour of those waves in the critical region $0{\cdot}1 \leqslant kR \leqslant 3{\cdot}0$ is studied. For the ordinary waves the first few resonances show significant dispersion than those near higher resonances which are dispersion-free. Only one extraordinary wave propagates near the fundamental cyclotron frequency. For the higher resonances, two modes propagate near each of the resonant frequencies, of which one mode remains constant for all values of kR whereas the second mode shows significant dispersion. But beyond the fifth resonance both the modes are dispersion-free.

Keywords. Cyclotron resonance; cyclotron waves in metals; perpendicular propagation; ordinary wave; extraordinary wave.

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1. Introduction

In the presence of a strong magnetic field, a metal or a doped semiconductor can support a variety of electromagnetic or plasma oscillations. The best known electromagnetic excitation in a solid state plasma is the helicon, which is a circularly-polarized wave propagating parallel to the magnetic field. Helicons can also propagate at small angles to the magnetic field and are characterized by the fact that their velocities are very small, often of the order of the sound wave velocity in solids. This feature bestows on them an ability to interact strongly with the phonons. Vast literature is at present available both on the helicons as well as on their interaction with the phonons (Viswanathan 1975, 1979; Sekhar and Viswanathan 1976; Idiculla and Viswanathan 1980, 1981). The subject has also been reviewed by Kaner and Skobov (1971) and Platzman and Wolff (1973).

While the helicons have very low frequencies compared to the electron cyclotron frequency, the solid state plasma can also support a class of high-frequency waves known as the cyclotron waves. These waves propagate very easily at right angles to the magnetic field in the neighbourhood of the electron cyclotron frequency or its harmonics and have velocities which are comparable to the particle velocities in the metal. Further, the propagation characteristics of the cyclotron waves depend strongly on the Coulomb interactions among the electrons in the metals and on the non-local properties of the conductivity tensor. For these reasons they can be used as probes of the many-body effects in metals.

Kaner and Skobov (1964) first pointed out the possibility of propagating electromagnetic waves at right angles to the magnetic field in the vicinity of cyclotron resonances. The cyclotron waves were first experimentally observed in alkali metals by Walsh and Platzman (1965). As stated earlier, these waves depend sensitively on the finite-**k** or non-local properties of the conductivity tensor. The dispersion characteristics of the cyclotron waves were studied by Kaner and Skobov (1971) both in the long-wavelength ($kR \ll 1$) and in the short-wavelength ($kR \gg 1$) limits. A search of the literature on cyclotron waves suggests that much work remains to be done on the nature of the waves in the critical region where the wavelength is of the order of the electron Larmor radius. The solution of the dispersion equations for the ordinary as well as the extraordinary waves near cyclotron resonance requires much analytical as well as computational work in this domain. Besides, when the cyclotron waves propagate obliquely to the magnetic field, though very nearly perpendicular to it, the Doppler-like term in the denominator (of the conductivity tensor σ_{ij}) introduces finite regions of Landau damping, magnetic Landau damping and Doppler-shifted cyclotron damping. Very little is at present known as to how such collisionless damping acts to destroy the propagation of the cyclotron waves and it is our aim to address ourselves to some of these problems. In this paper, we solve the dispersion equations for the ordinary as well as the extraordinary waves in metals propagating near cyclotron resonances and study the non-local behaviour of those waves for wavelengths which are comparable to the electron Larmor radius. Those results are presented in the next two sections.

2. Ordinary cyclotron wave propagation

We shall choose the z-axis to coincide with the static magnetic field. Then the well-known dispersion equation for the ordinary wave (for derivation see Appendix A) is

$$\sigma_{zz} = 0. \tag{1}$$

The components of the conductivity tensor are well-known and are reproduced in textbooks or review articles by various authors. In this paper we use the expressions for those as given by Platzman and Wolff (1973).

For propagation perpendicular to the magnetic field, we have

$$\sigma_{zz} = iN \sum_{n=0}^{\infty} \left[\frac{\tilde{\omega}}{(\tilde{\omega}^2 - n^2)(1+\delta_{n0})} \int_0^{\pi} J_n^2(b) \cos^2\theta \sin\theta \, d\theta \right]. \tag{2}$$

Here

$$\tilde{\omega} = \frac{\omega + i(\tau)^{-1}}{\Omega}, \tag{3}$$

$$b = k(v_F/\Omega) \sin\theta, \tag{4}$$

$$N = (3n_0 e^2/m^*\Omega) \tag{5}$$

and

$$\delta_{n0} = \begin{cases} 1; & n = 0 \\ 0; & n \neq 0. \end{cases} \tag{6}$$

In the above equations ω, τ^{-1}, Ω, k, v_F, n_0 and m^* stand for the frequency of the electromagnetic wave, the collision frequency of electrons with defects or impurities, the cyclotron frequency, the magnitudes of the wave vector and the Fermi velocity, the particle density and the effective mass of the particle respectively. Replacing (v_F/Ω) by R (the Larmor radius) we get

$$b = kR \sin\theta. \tag{7}$$

The expression for the square of the Bessel function is (Watson 1958)

$$J_n^2(b) = \sum_{s=0}^{\infty} \left[\frac{(-1)^s (2n+2s)!}{s!(2n+s)! [(n+s)!]^2} \cdot \frac{b^{2n+2s}}{2^{2n+2s}} \right]. \tag{8}$$

Substituting the above expression in (2) and using the formula

$$\int_0^{\pi} \sin^{2n+2s+1}\theta \cos^2\theta \, d\theta = \frac{[(n+s)!]^2 \, 2^{2n+2s+1}}{(2n+2s+1)(2n+2s+3)(2n+2s)!}, \tag{9}$$

we find that the dispersion equation for ordinary cyclotron wave becomes

$$\sum_{n=0}^{\infty} \sum_{s=0}^{\infty} P_{ns} \left(\frac{\tilde{\omega}}{\tilde{\omega}^2 - n^2} \right) = 0, \tag{10}$$

where

$$P_{ns} = \frac{2(-1)^s (kR)^{2n+2s}}{(1+\delta_{n0})\, s!\,(2n+s)!\,(2n+2s+1)(2n+2s+3)}. \tag{11}$$

The dispersion equation is in the form of an infinite power series in kR and holds good for any value of k.

As stated earlier, the cyclotron waves propagate very freely near the cyclotron resonances. For electrons in a metal at very low temperatures, τ is of the order of 10^{-9} to 10^{-8} sec and Ω is of the order of 10^{11} rad/sec. Thus $\Omega\tau \gg 1$ and for this reason we can safely ignore collisions among the particles (electrons). We shall now investigate how the non-local effects modify the dispersion of the waves.

Near the Mth resonance, we shall assume

$$\omega = M\Omega + \delta, \tag{12}$$

or

$$\tilde{\omega} = M + \tilde{\delta}. \tag{13}$$

Here

$$\tilde{\omega} = \omega/\Omega \tag{14}$$

and

$$\tilde{\delta} = \delta/\Omega. \tag{15}$$

It is presumed that $\tilde{\delta}$ is small compared to unity, or $|\tilde{\delta}| < 1$.

Expanding (10) as a power series in $\tilde{\delta}$, the dispersion equation for the ordinary wave takes the form

$$\sum_{j=0}^{\infty} C_{j+1} \tilde{\delta}^j = 0, \tag{16}$$

where the coefficients C_{j+1} are given in Appendix B.

Normally $\tilde{\delta}$ is small so that the infinite series expression in (16) can be truncated with the first two terms. This gives a linear equation of the form

$$C_1 + C_2\tilde{\delta} = 0 \tag{17}$$

leading to

$$\tilde{\delta} = -(C_1/C_2). \tag{18}$$

Occasionally, when $\tilde{\delta}$ becomes comparable to unity though less than one, it is advisable, in the interest of accuracy to take the third term of (16) also and solve instead the quadratic equation

$$C_1 + C_2\tilde{\delta} + C_3\tilde{\delta}^2 = 0. \tag{19}$$

Equations (18) and (19) were solved for different values of kR in steps of 0·1 in the interval $kR = 0{\cdot}1$ to $kR = 3{\cdot}0$ for the first seven resonances using a digital computer. Due to the rapid convergence of the three infinite series in the expression for C_{j+1}, we considered only terms upto the order of 12th power in kR for all the three series. As a further check we extended our calculations to terms of the order of 26th power in kR and solved (18) and (19). The result was the reassurance of the correctness of our first approximation. The computer was instructed to print the values of kR and $\tilde{\omega}$ for each value of kR in the interval.

In our calculations one of the two real roots of the quadratic equation always happened to be greater than unity and was therefore discarded.

The calculations using the linear and the quadratic equations gave identical results for resonances higher than or equal to 2. For the first resonance the quadratic approximation gave results that agreed better with a direct solution (as elaborated below) of the dispersion equation (10).

To check the accuracy of the numerical results, we solved the dispersion equation directly by another independent method. The dispersion equation for the ordinary wave may also be written in the form

$$\sum_{n=0}^{\infty} U_n \left(\frac{\tilde{\omega}}{\tilde{\omega}^2 - n^2} \right) = 0 \tag{20}$$

where

$$U_n = \sum_{s=0}^{\infty} P_{ns}. \tag{21}$$

To study the dispersion equation for the nth resonance, we retained the first $(n+m)$ terms in (20) where m is small. The resulting algebraic equation can be solved directly with the aid of a computer.

For studying the first five resonances we retained the first six terms of the dispersion equation (20). Then (20) reduces to the simplified form

$$V_0\tilde{\omega}^{10} + V_1\tilde{\omega}^8 + V_2\tilde{\omega}^6 + V_3\tilde{\omega}^4 + V_4\tilde{\omega}^2 + V_5 = 0, \tag{22}$$

where the coefficients V_0 to V_5 are given in Appendix B. Here due to the rapid convergence of the terms of the infinite power series in kR for U_n, we retained only terms upto $(kR)^{10}$ for evaluating U_0 to U_5 and V_0 to V_5.

A computer program was written to solve the algebraic equation (22) for different values of kR in steps of 0·1 in the interval $kR = 0{\cdot}1$ to $kR = 3{\cdot}0$.

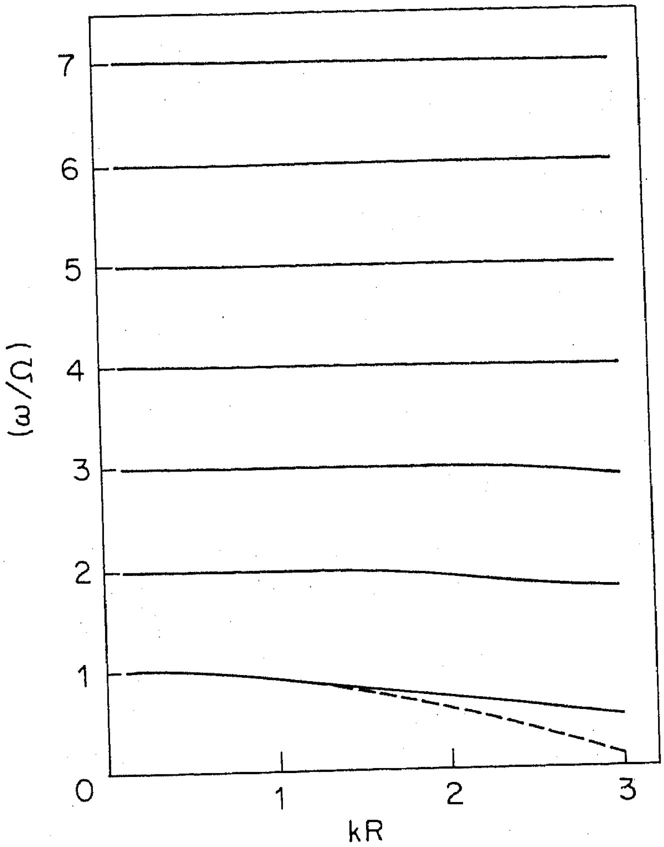

Figure 1. Dispersion curves for the ordinary cyclotron waves in metals propagating near the first seven cyclotron resonances.

The dispersion curves for the propagation of ordinary cyclotron waves near the first seven resonances are shown in figure 1.

It is found that the linear approximation (dashed line) is inadequate for the first resonance. Hence for the first resonance we used the other computational methods described above. For the next six resonances, even the linear approximation agrees with the solution of the quadratic equation or with the direct numerical method of solution of the dispersion equation.

The curves in figure 1 show that the frequency generally decreases as kR increases. This decrease is more pronounced for the first resonance but not so for the higher resonances.

3. Extraordinary cyclotron wave propagation

We shall next consider propagation of the extraordinary electromagnetic waves with frequencies close to the cyclotron frequency or its harmonics.

The dispersion equation for the extraordinary wave (for derivation see Appendix A) is

$$\sigma_{xx}\sigma_{yy}+\sigma_{xy}^2=0. \tag{23}$$

Assuming as before that the wave is propagating perpendicular to the impressed static magnetic field, the three relevant components of the conductivity tensor, namely,

σ_{xx}, σ_{yy} and σ_{xy} are given by (Platzman and Wolff 1973)

$$\sigma_{xx} = iN \sum_{n=1}^{\infty} \left[n^2 \left(\frac{\tilde{\omega}}{\tilde{\omega}^2 - n^2} \right) \cdot \int_0^{\pi} \frac{J_n^2(b)}{b^2} \sin^3\theta \, d\theta \right], \tag{24}$$

$$\sigma_{yy} = iN \left(\frac{1}{2\tilde{\omega}} \right) \cdot \int_0^{\pi} J_0'^2(b) \sin^3\theta \, d\theta$$
$$+ iN \sum_{n=1}^{\infty} \left[\left(\frac{\tilde{\omega}}{\tilde{\omega}^2 - n^2} \right) \cdot \int_0^{\pi} J_n'^2(b) \sin^3\theta \, d\theta \right] \tag{25}$$

and

$$\sigma_{xy} = N \sum_{n=0}^{\infty} \left[\frac{\tilde{\omega}^2}{(\tilde{\omega}^2 - n^2)(1 + \delta_{n0})} \cdot \int_0^{\pi} \frac{J_n(b) J_n'(b) \sin^3\theta \, d\theta}{b} \right]. \tag{26}$$

Using the series expansions for the squares and products of the Bessel functions or their derivatives in the integrals occurring in (24) to (26) and integrating term by term we can express the integrals as a power series in the parameter kR. Then the expressions for σ_{xx}, σ_{yy} and σ_{xy} expressed as power series in kR are

$$\sigma_{xx} = 2iN \sum_{n=1}^{\infty} \sum_{s=0}^{\infty} \left[\frac{n^2 \tilde{\omega} (-1)^s (kR)^{2n+2s-2}}{(\tilde{\omega}^2 - n^2)\, s!\, (2n+s)!\, (2n+2s+1)} \right], \tag{27}$$

$$\sigma_{yy} = \left(\frac{2iN}{\tilde{\omega}} \right) \sum_{s=0}^{\infty} \left[\frac{(-1)^s (kR)^{2s+2}}{s!\, (s+1)!\, (2s+3)(2s+5)} \right]$$
$$+ iN \sum_{n=1}^{\infty} \sum_{s=0}^{\infty} \left[\left(\frac{\tilde{\omega}}{\tilde{\omega}^2 - n^2} \right) \cdot \frac{(-1)^s (n+s)(kR)^{2n+2s-2}}{s!(2n+s-2)!\,(2n+2s-1)(2n+2s+1)} \right]$$
$$- 2iN \sum_{n=1}^{\infty} \sum_{s=0}^{\infty} \left[\left(\frac{\tilde{\omega}}{\tilde{\omega}^2 - n^2} \right) \cdot \frac{(-1)^s (n+s)(kR)^{2n+2s}}{s!\,(2n+s)!\,(2n+2s+1)(2n+2s+3)} \right]$$
$$+ iN \sum_{n=1}^{\infty} \sum_{s=0}^{\infty} \left[\left(\frac{\tilde{\omega}}{\tilde{\omega}^2 - n^2} \right) \cdot \frac{(-1)^s (n+s+2)(kR)^{2n+2s+2}}{s!\,(2n+s+2)!\,(2n+2s+3)(2n+2s+5)} \right] \tag{28}$$

and

$$\sigma_{xy} = N \sum_{s=0}^{\infty} \left[\frac{(-1)^s s (kR)^{2s-2}}{s!\, s!\, (2s+1)} \right]$$
$$+ 2N \sum_{n=1}^{\infty} \sum_{s=0}^{\infty} \left[\left(\frac{\tilde{\omega}^2}{\tilde{\omega}^2 - n^2} \right) \cdot \frac{(-1)^s (n+s)(kR)^{2n+2s-2}}{s!\,(2n+s)!\,(2n+2s+1)} \right]. \tag{29}$$

To solve the dispersion equation, we look for solutions for the frequency of the extraordinary wave lying close to the cyclotron frequency or its harmonics. Writing as in (12) to (15) and substituting the equation for $\tilde{\omega}$ in (27) to (29), we find that in the neighbourhood of cyclotron resonances the three relevant components of the conductivity tensor can be expressed as a power series in kR. Thus we have,

$$\sigma_{xx} = iN (\tilde{\delta})^{-1} \sum_{j=0}^{\infty} A_{j+1} \tilde{\delta}^j, \tag{30}$$

$$\sigma_{yy} = iN(\tilde{\delta})^{-1} \sum_{j=0}^{\infty} B_{j+1}\,\tilde{\delta}^{j}, \tag{31}$$

and

$$\sigma_{xy} = N(\tilde{\delta})^{-1} \sum_{j=0}^{\infty} D_{j+1}\,\tilde{\delta}^{j}, \tag{32}$$

where the expressions for A_{j+1}, B_{j+1} and D_{j+1} are given in Appendix B.

Substituting (30) to (32) in the dispersion equation (23) and simplifying, the latter assumes the form

$$\sum_{j=0}^{\infty} W_j\tilde{\delta}^{j} = 0 \tag{33}$$

where

$$W_j = \sum_{s=1}^{j+1} (D_s D_{j+2-s} - A_s B_{j+2-s}). \tag{34}$$

For numerical study of the extraordinary wave propagation near cyclotron resonances we proceeded as follows.

As a first approximation we truncated the dispersion equation (33) with the quadratic term and solved the equation

$$W_0 + W_1\tilde{\delta} + W_2\tilde{\delta}^2 = 0. \tag{35}$$

A computer program was written for calculating the terms A_1, A_2, A_3, B_1, B_2, B_3, D_1, D_2 and D_3 and hence for W_0, W_1 and W_2 of the quadratic equation (35). Due to the rapid convergence of the series expansions for A_{j+1}, B_{j+1} and D_{j+1} in the expressions for W_j we considered only terms upto the order of 12th power in kR.

Equation (35) was solved through the computer for various values of kR in steps of 0·1 in the range $kR = 0{\cdot}1$ to $kR = 3{\cdot}0$ for the first seven resonances. This yielded two real roots for all resonances. However, one of the two real roots for propagation near the first resonance was discarded as it violated the required condition $|\tilde{\delta}| < 1$ even from $kR = 0{\cdot}1$. It follows that there are two distinct modes propagating near any subharmonic except near the fundamental resonance frequency where there exists only one wave.

For the second and third resonances, one of the two roots violated the condition $|\tilde{\delta}| < 1$ for $kR > 1{\cdot}5$ for the second resonance and $kR > 2{\cdot}9$ for the third resonance. So in order to obtain correct numerical results, we had to proceed to equations of higher order.

It was seen that for propagation near the second resonance the value of $\tilde{\delta}$ tends to unity in the range $1{\cdot}5 < kR \leq 3{\cdot}0$. As a result, the dispersion equation has a slower convergence in this region. Therefore an accurate solution of (33) in this region necessitated the consideration of a fairly large number of terms. In fact we had to solve an equation of order fourteen in $\tilde{\delta}$ (in which the various terms contained powers of kR of order 12) for waves propagating near the second resonance to obtain real roots satisfying the condition $|\tilde{\delta}| < 1$. This was achieved by suitably modifying our computer program.

Also for propagation near the third resonance we have to solve a dispersion equation of the third or higher order in $\tilde{\delta}$. So we solved a fourth order equation.

The fact that the solution of the cubic and the fifth order approximations of the

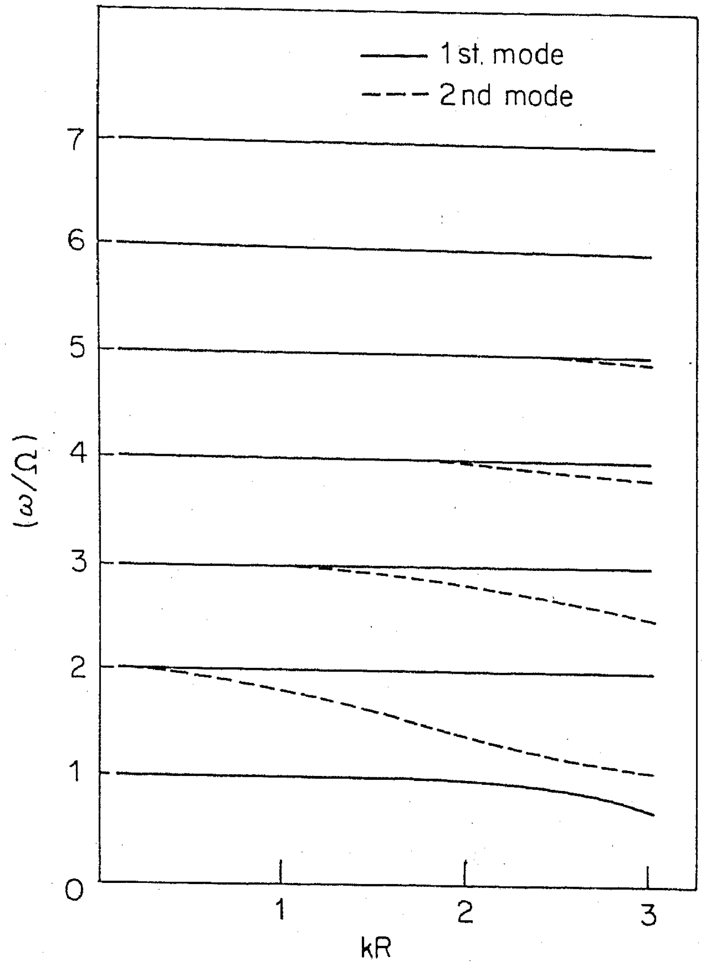

Figure 2. Dispersion curves for the extraordinary cyclotron waves in metals propagating near the first seven cyclotron resonances. For the sixth and seventh resonances the frequencies of both the modes almost coincide. Hence for the two resonances both the modes are shown together as one continuous line.

dispersion equation (33) for propagation near the fundamental cyclotron frequency yielded only one real root confirms our earlier finding that only one wave propagates in the vicinity of the fundamental cyclotron frequency.

In figure 2 we give the dispersion curves for the propagation of extraordinary cyclotron waves near the first seven harmonics.

Near the first harmonic there exists only a single wave represented by a continuous line. The frequency of the wave remains almost constant at Ω upto $kR = 1{\cdot}5$. Thereafter the frequency of the wave decreases as kR increases.

For the second and higher harmonics, two modes generally propagate near each of the resonant frequencies. Of these one mode, denoted by a continuous line has a constant frequency in the range for kR whereas the other mode denoted by a dashed line shows dispersion. As can be seen from the figure, the dispersion is strongest for the second mode near the second harmonic. For the third, fourth and fifth harmonics, the frequency of the second mode almost coincides with that of the first mode upto a certain value of kR and starts branching off afterwards. For the sixth and seventh harmonics, the frequency of the two modes almost coincides throughout the entire range of values of kR that we calculated.

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Appendix A

For a discussion of the dispersion behaviour of the high frequency waves (HFW) we concentrate on the bulk dielectric properties of the metal (i.e. solid state plasma).

It is well known that for propagation accurately perpendicular to the static magnetic field, the HFW are undamped if the collisions are neglected. (At low temperatures in pure samples $\tau \sim 10^{-9}$–10^{-8} sec and $\Omega \sim 10^{11}$ rad/sec. This is the range of parameters where HFW are observed. Thus, in this regime $\Omega\tau \gg 1$ and collisions are unimportant).

Assuming a perturbing field of the form $\mathbf{E} = \mathbf{E}_0 \cdot \exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)]$, the Maxwell's equations yield

$$\mathbf{k} \times (\mathbf{k} \times \mathbf{E}_0) + (\omega/c)^2 \varepsilon \cdot \mathbf{E}_0 = 0, \tag{A.1}$$

where we have defined the magnetic field $\mathbf{B}$ to be parallel to the z-axis and the wave vector to be parallel to the x-axis

$$(\mathbf{z} \parallel \mathbf{B} \text{ and } \mathbf{x} \parallel \mathbf{k}).$$

In (A.1), $\mathbf{k}$ is the wave vector, ω the frequency of the HFW and ε is the dielectric tensor defined as

$$\varepsilon_{\alpha\beta}(\mathbf{k}, \omega, \mathbf{B}) = \delta_{\alpha\beta} + (4\pi i/\omega)\,\sigma_{\alpha\beta}(\mathbf{k}, \omega, \mathbf{B}). \tag{A.2}$$

In (A.2), the first term $\delta_{\alpha\beta}$ is the displacement current and the second term is a complex tensor contribution due to the magnetized conduction electrons where $\sigma_{\alpha\beta}(\mathbf{k}, \omega, \mathbf{B})$ is the wave-vector-, frequency-, and field-dependent conductivity tensor.

Equation (A.1) will have a non-trivial solution when the determinant of the coefficients vanishes, i.e.,

$$\begin{vmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} - (kc/\omega)^2 & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} - (kc/\omega)^2 \end{vmatrix} = 0. \tag{A.3}$$

But in our coordinate system $\mathbf{z} \| \mathbf{B}$ and $\mathbf{x} \| \mathbf{k}$. Then using the symmetry of the magnetoconductivity tensor, we have $\varepsilon_{xz} = \varepsilon_{yz} = \varepsilon_{zx} = \varepsilon_{zy} = 0$. Then (A.3) reduces to

$$\begin{vmatrix} \varepsilon_{xx} & \varepsilon_{xy} & 0 \\ \varepsilon_{yx} & \varepsilon_{yy} - (kc/\omega)^2 & 0 \\ 0 & 0 & \varepsilon_{zz} - (kc/\omega)^2 \end{vmatrix} = 0. \tag{A.4}$$

This yields us the familiar dispersion relation

$$[\varepsilon_{zz} - (kc/\omega)^2][\varepsilon_{xx}\,\varepsilon_{yy} + \varepsilon_{xy}^2 - \varepsilon_{xx}\cdot(kc/\omega)^2] = 0. \tag{A.5}$$

The first root,

$$\varepsilon_{zz} = (kc/\omega)^2 \tag{A.6}$$

is the ordinary wave which is purely transverse in character ($\mathbf{E} \| \mathbf{z}$). The other root,

$$(\varepsilon_{xx}\,\varepsilon_{yy} + \varepsilon_{xy}^2)/\varepsilon_{xx} = (kc/\omega)^2 \tag{A.7}$$

is the extraordinary wave which is not purely transverse in character but has a weak longitudinal component as well.

Using (A.2) in (A.6) we get the dispersion relation for the ordinary wave as

$$(kc/\omega)^2 = 1 + (4\pi i/\omega)\,\sigma_{zz}(\mathbf{k}, \omega, \mathbf{B}). \tag{A.8}$$

The experimental conditions under which cyclotron waves are observed lead to a simplification of (A.8). In the range of parameters where the HFW are observed one has

$$\omega_p^2/\omega^2 \sim 10^{10} \gg (kc/\omega)^2 \sim 10^5 \gg 1$$

(where $\omega \sim 10^{11}$ rad/sec and $\omega_p = (4\pi ne^2/m^*)^{1/2} \sim 10^{16}$ rad/sec in the plasma frequency) and the term $(4\pi i/\omega)\cdot\sigma_{zz} \sim O(\omega_p/\omega)^2 \sim 10^{10}$. Therefore one can neglect the first term on the right hand side of (A.8) coming from the displacement current in Maxwell's equations. In other words, because of the extremely high plasma frequency in metals, the response due to the conduction electrons is dominant, and so one can neglect the contribution $\delta_{\alpha\beta}$ due to the displacement current in the expression for $\varepsilon_{\alpha\beta}$ [equation (A.2)]. Thus, for metals, one can define $\varepsilon_{\alpha\beta}$ as

$$\varepsilon_{\alpha\beta}(\mathbf{k}, \omega, \mathbf{B}) = (4\pi i/\omega)\cdot\sigma_{\alpha\beta}(\mathbf{k}, \omega, \mathbf{B}). \tag{A.9}$$

Again, in the regime where cyclotron waves are observed, $(kc/\omega)^2 \sim 10^5$ so that solutions of (A.8) are given to $O(kc/\omega_p)^2 \sim 10^{-5}$ by the zeros of the conductivity,

$$\sigma_{zz} = 0 \tag{A.10}$$

which is thus the dispersion relation for the ordinary wave.

Similarly using (A.9) in (A.7) we have for the extraordinary wave,

$$(kc/\omega)^2 = [(\sigma_{xx}\sigma_{yy} + \sigma_{xy}^2)/\sigma_{xx}]\cdot(4\pi i/\omega). \tag{A.11}$$

Hence solutions of (A.11) are given to $O(kc/\omega_p)^2 \sim 10^{-5}$ by

$$(\sigma_{xx}\sigma_{yy} + \sigma_{xy}^2)/\sigma_{xx} = 0. \tag{A.12}$$

Hence

$$\sigma_{xx}\sigma_{yy}+\sigma_{xy}^2=0 \tag{A.13}$$

is the dispersion relation for the extraordinary wave.

Appendix B

$$V_0 = 1/3. \tag{B.1}$$

$$V_1 = (kR)^2/15-(55/3). \tag{B.2}$$

$$V_2 = (1023/3)-(18/5)\,(kR)^2+(kR)^4/35. \tag{B.3}$$

$$V_3 = (kR)^6/63-(10/7)\,(kR)^4+(323/5)\,(kR)^2-(7645/3). \tag{B.4}$$

$$V_4 = (21076/3)-(6676/15)\,(kR)^2+(769/35)\,(kR)^4 -(41/63)\,(kR)^6+(1/99)\,(kR)^8. \tag{B.5}$$

$$V_5 = (1/143)\,(kR)^{10}-(25/99)\,(kR)^8+(400/63)\,(kR)^6 -(720/7)(kR)^4+(960)(kR)^2-4800. \tag{B.6}$$

$$A_1 = \sum_{s=0}^{\infty}\frac{M^2(-1)^s(kR)^{2M+2s-2}}{s!\,(2M+s)!\,(2M+2s+1)}. \tag{B.7}$$

$$B_1 = \sum_{s=0}^{\infty}\frac{(-1)^s(M+s)(kR)^{2M+2s-2}}{2s!\,(2M+s-2)!\,(2M+2s-1)\,(2M+2s+1)} -\sum_{s=0}^{\infty}\frac{(-1)^s(M+s)(kR)^{2M+2s}}{s!\,(2M+s)!\,(2M+2s+1)\,(2M+2s+3)} +\sum_{s=0}^{\infty}\frac{(-1)^s(M+s+2)(kR)^{2M+2s+2}}{2s!\,(2M+s+2)!\,(2M+2s+3)\,(2M+2s+5)}. \tag{B.8}$$

$$C_1 = \sum_{s=0}^{\infty}\frac{(-1)^s(kR)^{2M+2s}}{s!\,(2M+s)!\,(2M+2s+1)\,(2M+2s+3)}. \tag{B.9}$$

$$D_1 = \sum_{s=0}^{\infty}\frac{(-1)^sM(M+s)(kR)^{2M+2s-2}}{s!\,(2M+s)!\,(2M+2s+1)}. \tag{B.10}$$

$$D_2 = \sum_{n=1}^{\infty}{}'\sum_{s=0}^{\infty}\frac{2M^2(-1)^s(n+s)(kR)^{2n+2s-2}}{(M^2-n^2)s!\,(2n+s)!\,(2n+2s+1)} +\sum_{s=0}^{\infty}\frac{(-1)^s s(kR)^{2s-2}}{s!\,s!\,(2s+1)}+1{\cdot}5\sum_{s=0}^{\infty}\frac{(-1)^s(M+s)(kR)^{2M+2s-2}}{s!\,(2M+s)!\,(2M+2s+1)} \tag{B.11}$$

where Σ', stands for the summation over all values of $n\neq M$. For $j\geqslant 0$,

$$Z_j = (-1)^j/M^{j+1}, \tag{B.12}$$

$$Q_j^1 = (-1)^j/(2M)^{j+1}, \tag{B.13}$$

$$Q_j^2 = (-1)^{j-1}/(2^{j+1} M^j) \tag{B.14}$$

and

$$G_j^1 = 2 \cdot \sum_{r=0}^{L} \frac{(-1)^j M^{j+1-2r} n^{2r} (j+1)!}{(M^2-n^2)^{j+1} (j+1-2r)! (2r)!}, \tag{B.15}$$

where L is the integer part of the positive real number $(j+1)/2$. For $j \geqslant 1$,

$$A_{j+1} = \left[\sum_{n=1}^{\infty}{}' \sum_{s=0}^{\infty} \frac{(-1)^s n^2 (kR)^{2n+2s-2}}{s!(2n+s)!(2n+2s+1)} \right] G_{j-1}^1 + \left[\sum_{s=0}^{\infty} \frac{(-1)^s M^2 (kR)^{2M+2s-2}}{s!(2M+s)!(2M+2s+1)} \right] Q_{j-1}^1, \tag{B.16}$$

$$\begin{aligned} B_{j+1} = {} & \left[\sum_{s=0}^{\infty} \frac{2(-1)^s (kR)^{2s+2}}{s!(s+1)!(2s+3)(2s+5)} \right] Z_{j-1} \\ & + \left\{ \sum_{n=1}^{\infty}{}' \left[\sum_{s=0}^{\infty} \frac{(-1)^s (n+s)(kR)^{2n+2s-2}}{2s!(2n+s-2)!(2n+2s-1)(2n+2s+1)} \right.\right. \\ & - \sum_{s=0}^{\infty} \frac{(-1)^s (n+s)(kR)^{2n+2s}}{s!(2n+s)!(2n+2s+1)(2n+2s+3)} \\ & \left.\left. + \sum_{s=0}^{\infty} \frac{(-1)^s (n+s+2)(kR)^{2n+2s+2}}{2s!(2n+s+2)!(2n+2s+3)(2n+2s+5)} \right] \right\} G_{j-1}^1 \\ & + \left\{ \sum_{s=0}^{\infty} \left[\frac{(-1)^s (M+s)(kR)^{2M+2s-2}}{2s!(2M+s-2)!(2M+2s-1)(2M+2s+1)} \right.\right. \\ & - \frac{(-1)^s (M+s)(kR)^{2M+2s}}{s!(2M+s)!(2M+2s+1)(2M+2s+3)} \\ & \left.\left. + \frac{(-1)^s (M+s+2)(kR)^{2M+2s+2}}{2s!(2M+s+2)!(2M+2s+3)(2M+2s+5)} \right] \right\} Q_{j-1}^1, \end{aligned} \tag{B.17}$$

$$\begin{aligned} C_{j+1} = {} & \sum_{s=0}^{\infty} \left[\frac{(-1)^s (kR)^{2s}}{(s!)^2 (2s+1)(2s+3)} \right] Z_{j-1} \\ & + \sum_{s=0}^{\infty} \left[\frac{(-1)^s (kR)^{2M+2s}}{s!(2M+s)!(2M+2s+1)(2M+2s+3)} \right] Q_{j-1}^1 \\ & + \left[\sum_{n=1}^{\infty}{}' \sum_{s=0}^{\infty} \frac{(-1)^s (kR)^{2n+2s}}{s!(2n+s)!(2n+2s+1)(2n+2s+3)} \right] G_{j-1}^1, \end{aligned} \tag{B.18}$$

$$D_{j+2} = \left[\sum_{n=1}^{\infty}{}' \sum_{s=0}^{\infty} \frac{(-1)^s (n+s)(kR)^{2n+2s-2}}{s!(2n+s)!(2n+2s+1)} \right] G_j^2 + \left[\sum_{s=0}^{\infty} \frac{(-1)^s (M+s)(kR)^{2M+2s-2}}{s!(2M+s)!(2M+2s+1)} \right] Q_j^2 \tag{B.19}$$

and

$$G_j^2 = (G_j^1) M + G_{j-1}^1. \tag{B.20}$$