PITCH ANGLE SCATTERING INTO LOSS CONE BY CYCLOTRON RESONANCE

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Received May 22, 1972

1. Introduction

Several physical processes involving wave-particle interactions are responsible for the breakdown of the adiabatic invariants and the consequent scattering of the charged particles into the loss cone[1-3] (Brown, 1966, Mc Ilwain, 1966, Roberts, 1966). As is well known, charged particles in the radiation zones have three kinds of periodic motion, and any resonant wave-particle interaction associated with these periodicities could cause the breakdown of the adiabatic invariants. Of the different types of waves observed in the magnetosphere, the whistler mode has the right frequency to cause cyclotron resonance with electrons gyrating about a line of force. The longitudinal adiabatic invariant breaks down when hydromagnetic disturbances having power at the bounce frequency of a particle interact with it resonantly. The bounce-resonance interaction could change the particle's pitch angle and lead to the loss of the particles in the trapped zone by pitch angle diffusion.

The resonant interaction of the whistler mode with the cyclotron motion of the electrons has earlier been discussed by Cornwall[4,5] (1964, 1965). Kennel and Petschek[7] (1966). Roberts[3] (1968) has reviewed in detail both the bounce-resonance scattering by hydro-magnetic waves as well as the cyclotron resonance scattering by whistler mode disturbances. The method developed by him is essentially a single particle picture dealing with the interaction of an electron with a wave under conditions of resonance. Alternatively, the problem could also be treated statistically, starting from the Vlasov equation. As a result of the cyclotron resonance by the whistler mode, waves could exchange energy with particles and scatter them into the loss cone. One could obtain the proportion of particles scattered into the loss cone by integrating over the volume of the loss cone in the velocity space, the first order correction to the distribution function. Generally, loss by pitch angle scattering is a diffusion process. The theory of radial diffusion of charged particles in the magnetosphere has earlier been treated by Falthammar[6] (1963) and reviewed by Walt[8] (1971). When the spectrum of the disturbance consists of a large number of waves, the particles could suffer collision with the waves, and the random displacements thus suffered by the particles could cause a diffusion into the loss cone. In Section 4, we have derived the expressions for the diffusion coefficients for interaction of circularly polarised whistler modes with electrons. It is shown that the expression for the parallel component of the diffusion coefficient by this method agrees with the coefficient derived by Roberts earlier.

2. CYCLOTRON RESONANCE BY WHISTLER MODE

The collisionless Boltzmann transport equation for the electrons is given by

$$\frac{\partial f}{\partial t} + \nu \cdot \nabla f - \frac{e}{m}\left(\mathbf{E} + \frac{\nu \times \mathbf{B}}{c}\right) \cdot \frac{\partial f}{\partial \nu} = 0. \tag{1}$$

In this section we shall study the interaction between a single wave propagating along a line of force and the particles gyrating about the line. Later in section 4 we study the wave-particle interaction between the particles and a disturbance, that would be analysed into a spectrum of waves and the consequent diffusion of the particles in the velocity space.

We shall assume that the magnetic field acting on a charged particle is the dipole field and the perturbation field arising from the passage of the whistler mode. Then

$$\mathbf{B} = b + \mathbf{B}_d. \tag{2}$$

where B_d is the dipole field and b is the perturbation field produced by the whistler. The electric field $\mathbf{E}$ as well as the magnetic perturbation b arising from the passage of the whistler are supposed to be small. We shall choose the z direction as coinciding with the direction of the dipole field at any point so that:

$$\mathbf{B}_d = B_d e_z. \tag{3}$$

Let us now linearise the equation (1) and write

$$f = f_0 + f_1 \tag{4}$$

where f_0 is the solution of the equation

$$\frac{\partial f_0}{\partial t} + \boldsymbol{v} \cdot \nabla f_0 - \frac{e}{m} \frac{\boldsymbol{v} \times \mathbf{B}_d}{c} \cdot \frac{\partial f_0}{\partial \boldsymbol{v}} = 0 \tag{5}$$

and f_1 is the first order perturbation in f; it is of the same order as that of b and $\mathbf{B}$.

Since

$$\frac{e}{m} \frac{\boldsymbol{v} \times \mathbf{B}_d}{c} = \Omega_c \boldsymbol{v} \times e_z \tag{6}$$

where Ω_c is the cyclotron frequency of the electron, the equation satisfied by f_1 is given by

$$\frac{\partial f_1}{\partial t} + \boldsymbol{v} \cdot \nabla f_1 - \frac{e}{m}\left(\mathbf{E} + \frac{\boldsymbol{v} \times b}{c}\right) \cdot \frac{\partial f_0}{\partial \boldsymbol{v}} - \Omega_c\, \boldsymbol{v} \times e_z \cdot \frac{\partial f_1}{\partial \boldsymbol{v}} = 0. \tag{7}$$

Now, if θ is the polar angle

$$\boldsymbol{v} \times e_z \cdot \frac{\partial f_1}{\partial \boldsymbol{v}} = -\frac{\partial f_1}{\partial \theta}. \tag{8}$$

We shall assume that f_1, E and b

behave as

$$e^{\,i(\omega t - k.r)}$$

and set

$$f_1 = g e^{\,i(\omega t - k.r)}. \tag{9}$$

The whistler mode is a circularly polarised electromagnetic wave and for definiteness, we shall first consider a wave with right hand polarisation with respect to the magnetic field, so that we can write

$$\mathrm{E}_y = i\mathrm{E}_x. \tag{10}$$

We can further write

$$g = g e^{i\theta}. \tag{11}$$

The equation satisfied by f_1 is then given by

$$-\,i(\omega - k \cdot \boldsymbol{v} - \Omega_c) f_1 = \frac{e}{m}\left(\mathbf{E} + \frac{\boldsymbol{v} \times b}{c}\right) \cdot \frac{\partial f_0}{\partial \boldsymbol{v}}$$

or,

$$f_1 = \frac{\frac{ie}{m}\left(\mathbf{E} + \frac{\boldsymbol{v} \times \boldsymbol{b}}{c}\right) \cdot \frac{\partial f_0}{\partial \boldsymbol{v}}}{\omega - \boldsymbol{k} \cdot \boldsymbol{r} - \Omega_c} . \tag{12}$$

By using Maxwell's equation

$$\text{Curl } \mathbf{E} = -\frac{1}{c}\frac{\partial \boldsymbol{b}}{\partial t}, \tag{13}$$

connecting the electric and magnetic disturbances in (12), we find that the first order correction to the distribution function may be reduced to the form

$$f_1 = \frac{ie\mathrm{E}_\perp}{m} \frac{\left[\left(1 - \frac{kv_\parallel}{\omega}\right)\frac{\partial f_0}{\partial v_\perp} + \frac{kv_\perp}{\omega}\frac{\partial f_0}{\partial v_\parallel}\right]}{(\omega - \boldsymbol{k} \cdot \boldsymbol{v} - \Omega_c)} \tag{14}$$

where $\mathrm{E}_\perp$ is the component of E along $v_\perp$. One can see that the vanishing of the denominater in (14) leads to cyclotron resonance and the condition for this is:

$$\omega - \boldsymbol{k} \cdot \boldsymbol{v} = \Omega_c. \tag{15}$$

The scalar product $\boldsymbol{k} \cdot \boldsymbol{r}$ could be equal to $\pm kv_\parallel$ depending on whether the particle moves along or opposite to the direction of propagation of the wave. Generally, the frequency of the whistler mode is lower than the cyclotron frequency so that we can choose the negative sign. If ω_c is the non-relativistic electron cyclotron frequency, then

$$\Omega_c = \omega_c (1 - \beta^2)^{\frac{1}{2}} = \omega_c/\gamma \tag{16}$$

where

$$\beta = \frac{v}{c} \quad \text{and} \quad \gamma = (1 - \beta^2)^{-\frac{1}{2}} = \frac{m}{m_0}.$$

Further, let α be the pitch angle so that

$$v_\parallel = v \cos \alpha. \tag{17}$$

Then (15) becomes

$$\omega (1 + n\beta \cos \alpha) = \frac{\omega_c}{\gamma} \tag{18}$$

where $n = ck/\omega$ and is the refractive index for whistler mode propagation.

Hence

$$f_1 = \frac{ie\,E_\perp}{m_0\,\omega_c} \frac{\left[\left(1 - \frac{kv_\parallel}{\omega}\right)\frac{\partial f_0}{\partial v_\perp} + \frac{v_\perp k}{\omega}\frac{\partial f_0}{\partial v_\parallel}\right]}{\frac{\gamma\omega}{\omega_c}(1 + n\beta\cos\alpha) - 1}. \tag{19}$$

Using the well-known relation

$$\frac{1}{x} = P\left[\frac{1}{x}\right] - i\pi\delta(x) \tag{20}$$

where P [1/x] denotes the principal part of 1/x, we find that

$$f_1 = \frac{ieE_\perp}{m_0\omega_c}\left[\left(1 - \frac{kv_\parallel}{\omega}\right)\frac{\partial f_0}{\partial v_\perp} + \frac{v_\perp k}{\omega}\frac{\partial f_0}{\partial v_\parallel}\right]$$

$$\times \left\{P\left[\frac{1}{\frac{\gamma\omega}{\omega_c}(1 + n\beta\cos\alpha) - 1}\right]\right.$$

$$\left. - i\pi\delta\left[\frac{\gamma\omega}{\omega_c}(1 + n\beta\cos\alpha) - 1\right]\right\}. \tag{21}$$

The two terms in (21) correspond to the cases when there is no resonance and at resonance respectively. By definition of the principal part, the first term excludes a narrow region containing resonance. The second term gives the contribution arising from resonance alone and we shall consider this case only. If we denote by f_{1R} the change in the distribution function arising from resonant wave-particle interactions, we have

$$f_{1R} = \frac{e\pi E_\perp}{m_0\omega_c}\left[\left(1 - \frac{kv_\parallel}{\omega}\right)\frac{\partial f_0}{\partial v_\perp} + \frac{kv_\perp}{\omega}\frac{\partial f_0}{\partial v_\parallel}\right]$$

$$\times \delta\left[\frac{\gamma\omega}{\omega_c}(1 + n\beta\cos\alpha) - 1\right]. \tag{22}$$

3. Scattering into Loss Cone

Certain processes even in gaseous plasmas, can be profitably expressed in a quantum mechanical languages. The theory of waves in plasmas from a quantum mechanical viewpoint has been reviewed by Harris[9] (1969). Under certain conditions exchange of energy can take place between waves and particles, and by receiving energy from the wave, a particle can directly get scattered into the loss cone. While diffusion arises from multiple collisions resulting into a random walk of the particle, processes involving exchange of energy or momentum can directly scatter the particle into the loss cone and can be expected to take place in a characteristic time τ such that $\omega\tau \ll 1$. In this case we can get the proportion of the particles scattered into the loss cone by integrating f_{1R} throughout the volume of the loss cone, and one can write

$$\delta n = \int_{\alpha<\alpha_c} f_{1R}\, dv \tag{23}$$

where α_c is the critical pitch angle for a particle along a line of force. This critical pitch angle is a characteristic property of the line of force about which the particle gyrates. When $\alpha < \alpha_c$, the particle will be scattered into the loss cone and will be lost. For a given n and β let α_0 be the value of the pitch angle such that

$$\frac{\gamma\omega}{\omega_c}(1 + n\beta \cos \alpha_0) = 1. \tag{24}$$

The delta term in (23) will make a contribution to the integral only if α_0 lies within the range of integration. For a particle to be in the loss cone, it is necessary that

$$\alpha_0 < \alpha_c \quad \text{or} \quad \cos \alpha_0 > \cos \alpha_c.$$

Hence

$$n\beta \cos \alpha_0 = [y(1 - \beta^2)^{\frac{1}{2}} - 1] > n\beta \cos \alpha_c \tag{25}$$

where

$$y = \omega_c/\omega.$$

On eradicating the radicals and simplifying, we get

$$f(\beta) = \beta^2 (n^2 \cos^2 \alpha_c + y^2) + 2np \cos \alpha_c + (1 - y^2) < 0. \tag{26}$$

Since resonance takes place for frequencies less than the electron cyclotron frequency, $y > 1$. The two roots β_1 and β_2 of the equation $f(\beta) = 0$ are given by

$$\left.\begin{matrix}\beta_1\\ \beta_2\end{matrix}\right\} = \frac{-n \cos \alpha_c \pm y(y^2 + n^2 \cos^2 \alpha_c - 1)^{\frac{1}{2}}}{y^2 + n^2 \cos^2 \alpha_c}. \tag{27}$$

Of these β_1 is negative and β_2 is positive. Since β is positive, the expression $f(\beta)$ will be negative only if $0 < \beta < \beta_2$. Thus the upper limit for the velocity of any particle in the loss cone is $c\beta_2$. To get an idea of the value of the integral (23), we shall choose a simple Maxwellian distribution for f_0 so that

$$f_0 = Ce^{-\tilde{\beta}v^2/2}$$

where

$$\tilde{\beta} = \frac{m_0}{kT} \tag{28}$$

and we shall further neglect relativistic corrections so that $\gamma = 1$. Then we find that

$$\delta n = -\frac{2\pi^2 eE_\perp \tilde{b}}{m_0\omega_c} \int_0^{c\beta_2} v^3 e^{-\tilde{\beta}v^2/2}\left[1 - \frac{(y-1)^2}{n^2\beta^2}\right] dv$$

$$= \frac{2\pi^2 eE_\perp^{\,b}}{m_0\omega_c n^2}\left[\frac{2n^2}{\tilde{\beta}}\{1 - e^{-\tilde{\beta}v^2/2}(1 + \tfrac{1}{2}\tilde{\beta}\, v_2^2\}\right.$$

$$\left. + c^2(y-1)^2\{e^{-\tilde{\beta}v_2^2/2} - 1\}\right]. \tag{29}$$

The number of particles lost is a function of several parameters such as the temperature, the frequency of the wave and the cyclotron frequency. It can be seen from (33) that the maximum of δn treated as a function of ω occurs when $\omega = \omega_c$. This means $v_\parallel = 0$ or the particle is located at the mirror point. The most favourable position for scattering into the loss cone as a result of resonant exchange of energy between the whistler and the particle is the mirror point along a line of force.

4. Diffusion into the Loss Cone

In this section we shall consider the diffusion of particles into the loss cone or more generally in the velocity space as a result of interaction with waves having the characteristics of whistler modes. We shall assume that the particles are subjected to an electromagnetic disturbance consisting of a large number of waves. We shall now write

$$f = F_0 + \delta f \tag{30}$$

where δf is the fluctuation in the distribution function arising from the perturbation created by the passage of the waves. It is assumed that the average value $\langle \delta f \rangle$ of δf is equal to zero so that

$$\langle f \rangle = F_0. \tag{31}$$

As the electric field is created by the waves, we have

$$\langle \mathbf{E} \rangle = 0 \quad \text{so that} \quad \mathbf{E} = \delta \mathbf{E}. \tag{32}$$

Making these substitutions in (1), we get

$$\frac{\partial F_0}{\partial t} + \boldsymbol{v} \cdot \nabla F_0 - \frac{e}{m} \frac{\boldsymbol{v} \times \mathbf{B}_0}{c} \cdot \frac{\partial F_0}{\partial \boldsymbol{v}}$$

$$= \frac{e}{m} \frac{\partial}{\partial \boldsymbol{v}} \cdot \langle \delta \mathbf{E} \delta f \rangle + \frac{1}{c} \left\langle \boldsymbol{v} \times \boldsymbol{b} \cdot \frac{\partial}{\partial \boldsymbol{v}} \delta f \right\rangle. \tag{33}$$

We shall now analyse δf into a series of circularly polarised waves, and we shall assume that F_0 is the constant term in the fourier series for f. Then

$$\delta f = \sum_k{}' f_{\mathbf{k}p}\, e^{-i(\omega_k t - \mathbf{k}.\boldsymbol{r})} \tag{34}$$

where the prime indicates that the term $k = 0$ is left out of the summation. The suffix p is used to indicate that we are dealing with circularly polarised waves. We could thus set:

$$f_{\mathbf{k}p} = f_{\mathbf{k}} e^{i\theta}. \tag{35}$$

The reality condition for δf then implies that

$$f_{-\mathbf{k}p} = f_{\mathbf{k}} e^{-i\theta} \;;\; f_{-\mathbf{k}} = f_{\mathbf{k}} \quad \text{and} \quad \omega_{-\mathbf{k}} = -\,\omega_{\mathbf{k}}. \tag{36}$$

As the electric field $\delta \mathbf{E}$ arises from the passage of transverse waves, we likewise write

$$\delta \mathbf{E} = \sum_k{}' \mathbf{E}_{\mathbf{k}p} e^{-(i\omega_k t - \mathbf{k}.\boldsymbol{r})} \tag{37}$$

where

$$\mathbf{E}_{\mathbf{k}p} = \mathbf{E}_{\mathbf{k}} e^{i\theta} \quad \text{and} \quad \mathbf{E}_{-\mathbf{k}p} = \mathbf{E}_{\mathbf{k}} e^{-i\theta}.$$

Similarly the magnetic disturbance $\boldsymbol{b}$ can be expressed as

$$\boldsymbol{b} = \Sigma\, \boldsymbol{b}_{\mathbf{k}p} e^{-i(\omega_k t - \mathbf{k}.\boldsymbol{r})} \tag{38}$$

where as before

$$b_{kp} = b_k e^{i\theta} \quad \text{and} \quad b_{-k} = b_k e^{-i\theta}.$$

From Maxwell's equation:

$$\text{Curl } \mathbf{E} = -\frac{1}{c}\frac{\partial \boldsymbol{b}}{\partial t}$$

we get

$$\boldsymbol{b}_{kp} = \frac{c}{\omega_k}(\boldsymbol{k} \times \mathbf{E}_{kp}). \tag{39}$$

Now

$$\langle \delta\mathbf{E}\delta f\rangle = \sum_{kk'} \langle \mathbf{E}_{k'p'} e^{-i(\omega_{k'}t - \mathbf{k}'.\boldsymbol{r})} f_{kp} e^{-i(\omega_k t - \boldsymbol{k}.\boldsymbol{r})}\rangle$$

$$= \sum_k f_k \mathbf{E}_{-k}. \tag{40}$$

Similarly one can show that

$$\left\langle \boldsymbol{v} \times \boldsymbol{b} \cdot \frac{\partial}{\partial \boldsymbol{v}} \delta f\right\rangle = \sum_k \left\langle \boldsymbol{v} \times \frac{c}{\omega_k}(\boldsymbol{k} \times \mathbf{E}_{-k}) \cdot \frac{\partial f_k}{\partial \boldsymbol{v}}\right\rangle. \tag{41}$$

The Fourier component f_k can now be evaluated by the same method by which f_1, as given by (12), was evaluated from the linearised first order equation (7). We get

$$f_k = \frac{\dfrac{ie}{m}\left(\mathbf{E}_k + \dfrac{\boldsymbol{v} \times \boldsymbol{b}_k}{c}\right) \cdot \dfrac{\partial \mathrm{F}_0}{\partial \boldsymbol{v}}}{\omega - \boldsymbol{k} \cdot \boldsymbol{v} - \Omega_c}$$

$$= \frac{ie}{m\mathrm{D}_k}\left[\left(\mathrm{E}_k \cdot \frac{\partial f_0}{\partial \boldsymbol{v}_\perp}\right)\left(1 - \frac{v_\parallel k}{\omega_k}\right) + \left(\frac{\boldsymbol{v}_\perp \cdot \mathbf{E}_k}{\omega_k}\right) k \frac{\partial f_0}{\partial v_\parallel}\right] \tag{42}$$

where

$$\mathrm{D}_k = \omega_k - \boldsymbol{k} \cdot \boldsymbol{v} - \Omega_c. \tag{43}$$

If we denote by J the expression on the right-hand side of (33) so that

$$\mathrm{J} = \frac{e}{m}\left[\frac{\partial}{\partial \boldsymbol{v}} \cdot \langle \delta\mathbf{E}\delta f\rangle + \frac{1}{c}\left\langle \boldsymbol{v} \times \boldsymbol{b} \cdot \frac{\partial}{\partial \boldsymbol{v}} \delta f\right\rangle\right] \tag{44}$$

then on substituting from (40) and (41) for the averages

$\langle \delta\mathbf{E}\delta f\rangle$

and

$$\left\langle \boldsymbol{v} \times \boldsymbol{b} \cdot \frac{\partial}{\partial \boldsymbol{v}} \delta f \right\rangle,$$

we get

$$= \frac{ie^2}{m^2} \sum_k \frac{\partial}{\partial \boldsymbol{v}_\perp} \cdot \left\{ \frac{\mathbf{E}_{-k}}{\mathrm{D}_k} \left[\left(\mathbf{E}_k \cdot \frac{\partial \mathrm{F}_0}{\partial \boldsymbol{v}_\perp} \right) \left(1 - \frac{k v_{\parallel}}{\omega_k} \right) + \frac{\boldsymbol{v}_\perp \cdot \mathbf{E}_k}{\omega_k} \right. \right.$$

$$\left. \left. \times k \frac{\partial \mathrm{F}_0}{\partial v_{\parallel}} \right] \right\} + \frac{ie^2}{m^2} \sum_k \frac{1}{\omega_k \mathrm{D}_k} \left[(\mathbf{E}_{-k} \cdot \boldsymbol{v}_\perp) k \frac{\partial}{\partial v_{\parallel}} \right.$$

$$\left. \times \left\{ \mathbf{E}_k \cdot \frac{\partial \mathrm{F}_0}{\partial \boldsymbol{v}_\perp} \left(1 - \frac{k v_{\parallel}}{\omega_k} \right) + \frac{\boldsymbol{v}_\perp \cdot \mathbf{E}_k}{\omega_k} k \frac{\partial \mathrm{F}_0}{\partial v_{\parallel}} \right\} \right]$$

$$- \frac{ie^2}{m^2} \sum_k \frac{1}{\omega_k \mathrm{D}_k} (k v_{\parallel}) \mathbf{E}_{-k} \cdot \frac{\partial}{\partial \boldsymbol{v}_\perp} \left\{ \left(\mathbf{E}_k \cdot \frac{\partial \mathrm{F}_0}{\partial \boldsymbol{v}_\perp} \right) \right.$$

$$\left. \times \left(1 - \frac{k v_{\parallel}}{\omega_k} \right) + \frac{(\boldsymbol{v}_\perp \cdot \mathbf{E}_k)}{\omega_k} k \frac{\partial \mathrm{F}_0}{\partial v_{\parallel}} \right\}. \tag{45}$$

If $\boldsymbol{e}_r$ and e_θ are the unit vectors in polar coordinates, we have

$$\frac{\partial \mathrm{F}_0}{\partial \boldsymbol{v}_\perp} = \frac{\partial \mathrm{F}_0}{\partial \boldsymbol{v}_\perp} \boldsymbol{e}_r + \frac{1}{v_\perp} \frac{\partial \mathrm{F}_0}{\partial \theta} \boldsymbol{e}_\theta. \tag{46}$$

It is reasonable to assume that F_0 is independent of the polar angle θ, then

$$\frac{\partial \mathrm{F}_0}{\partial \boldsymbol{v}_\perp} = \frac{\partial \mathrm{F}_0}{\partial \boldsymbol{v}_\perp} e r. \tag{47}$$

We can now write the right-hand side of (45) as

$$\mathrm{J} = \frac{\partial}{\partial \boldsymbol{v}_\perp} \cdot \left(\mathbf{D}_{\perp\perp} \cdot \frac{\partial \mathrm{F}_0}{\partial \boldsymbol{v}_\perp} \right) + \frac{\partial}{\partial v_{\parallel}} \left(\mathbf{D}_{\perp\parallel} \cdot \frac{\partial \mathrm{F}_0}{\partial \boldsymbol{v}_\perp} \right)$$

$$+ \frac{\partial}{\partial \boldsymbol{v}_\perp} \cdot \left(\mathbf{D}_{\parallel\perp} \cdot \frac{\partial \mathrm{F}_0}{\partial v_{\parallel}} \right) + \frac{\partial}{\partial v_{\parallel}} \left(\mathrm{D}_{\parallel,\parallel} \frac{\partial \mathrm{F}_0}{\partial v_{\parallel}} \right) \tag{48}$$

where the diffusion coefficients can be seen to be

$$\mathbf{D}_{\perp\perp} = \frac{ie^2}{m^2} \sum_k \frac{\mathbf{E}_{-k} \mathbf{E}_k}{\mathrm{D}_k} \left(1 - \frac{v_{\parallel} k}{\omega_k} \right)^2. \tag{49}$$

$$\mathbf{D}_{\perp\parallel} = \mathrm{D}_{\parallel\perp} = \frac{ie^2}{m^2} \sum_k \frac{(\mathbf{E}_{-k} \cdot \boldsymbol{v}_\perp)}{\omega_k \mathrm{D}_k} k \left(1 - \frac{k v_{\parallel}}{\omega_k} \right) \mathbf{E}_k \tag{50}$$

and

$$D_{\parallel,\parallel} = \frac{ie^2}{m^2} \sum_k \frac{1}{\omega_k D_k} (\mathbf{E}_{-k} \cdot \boldsymbol{v}_\perp)\, k^2 \frac{(\boldsymbol{v}_\perp \cdot \mathbf{E}_k)}{\omega_k}. \tag{51}$$

The contribution to the diffusion coefficients arising from the resonant wave-particle interaction can be obtained by making use of the prescription (20) for the expression $1/D_k$ in the denominator. We find that the resonant contribution to the parallel component $D_{\parallel,\parallel}$, is given by:

$$D^R_{\parallel,\parallel} = \pi \left(\frac{e}{m}\right)^2 \sum_k \frac{1}{\omega_k^2} (\mathbf{E}_k \cdot \boldsymbol{v}_\perp)^2 k^2$$

$$= \pi \left(\frac{e}{m}\right)^2 \sum_k \frac{k^2}{\omega_k^2} \cdot E_k^2\, v^2_\perp \cos^2 \phi_k \tag{52}$$

where ϕ_k is the angle between E_k and $v_\perp$ and ω_k satisfied the resonance condition (18). Since this phase angle is a random variable, we shall replace it by its average value. We get

$$D^R_{\parallel,\parallel} = \frac{\pi}{2} \left(\frac{e}{m}\right)^2 \sum_k \frac{k^2 E_k^2}{\omega_k^2} v_\perp^2$$

$$= \frac{\pi}{2} \left(\frac{e}{mc}\right)^2 \sum_k b_k^2\, v_\perp^2 \tag{53}$$

from the equation (39) connecting the amplitudes of the electric and magnetic vectors. Generally the summation over k can be replaced by an integration using a suitable frequency distribution function. The pitch angle diffusion coefficient has earlier been derived by Roberts (1968). His method consists in calculating the change $\Delta\mu$ in the cosine of the electron's pitch angle and then evaluating the Fokker-Planck diffusion Coefficient, which is equal to $\langle(\Delta\mu)^2\rangle/t$. The expression given by him is

$$\langle(\Delta\mu)^2\rangle/t = \frac{1}{2}\left(\frac{e}{mc}\right)^2 (1 - \mu^2)\, b^2. \tag{54}$$

By replacing $v_\perp^2$ by $v^2(1-\mu^2)$ we find that both the formulae agree except for the factor v^2, which is due to the fact that we have calculated the diffusion in velocity space whereas the coefficient given by Roberts is the diffusion in pitch angle.

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