THE PROPAGATION OF ULTRASONIC WAVES IN PIEZOELECTRIC SEMICONDUCTORS

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Abstract

The equations of wave propagation in piezoelectric semiconductors have been derived for a frame of reference in which the principal axes concide with the crystallographic axes. It is shown that generally the dispersion relation is given by a determinant of order six but under condition wherein the plasma modes are not excited, it could be reduced to a determinant of order five, which is equivalent to the one given by Hutson and White. The dispersion relation for hybrid waves which couple acoustic phonons with plasmons has beer derived and this is shown to be given by a determinental equation of order four.

1. Introduction

The propagation of ultrasonic waves in piezoelectric semiconductors has been the subject of many theoretical and experimental investigations during the last decade. The amplification of ultrasonic waves in photoconductive, Cadmium Sulphide has been studied experimentally by Hutson, Mc Fee and White. They have demonstrated that the application of a dc electric field pulse could cause the interacting charge carriers to drift in the direction of wave propagation faster than the stress wave velocity, and the electric field can feed energy to the acoustic field, thereby causing the waves to grow.

The dispersion relation for long acoustic waves travelling in a piezo electric crystal has earlier been derived by Hutson and White. They have found that the dispersion equation is given by a determinant of order five. Solutions of this equation generally give hybrid waves, which couple the elastic and electromagnetic wave fields. Of these, three solutions correspond mainly to elastic waves, with some energy being carried by the accompanying electric field, and the remaining two solutions transport energy as electromagnetic waves while a fraction of the energy is being carried out by the accompanying elastic wave field. Generally the elastic and the electromagnetic waves are coupled together and the division of energy between

the elastic and electrical components as well as the velocity of propagation will be determined by the piezoelectric coefficients. Hutson and White have derived the wave equation for propagation of ultrasonic waves along an arbitrary direction. They have derived this for a frame of reference in which the x_1 axis coincides with the direction of propagation, but generally the elastic as well as the piezoelectric constants are referred with reference to the crystallographic axes. Further the equation of elastic wave propagation in crystals are commonly studied with reference to the crystallographic axes, with the direction cosines of the propagation vector occuring as parameters. We have therefore derived the equations of wave propagation in piezoelectric crystals for a frame of reference in which the principal axes coincide with the crystallographic axes. It is shown that generally the dispersion equation is determined by a determinant of order six and under certain conditions in which the plasma modes are not excited, the dispersion equation can be reduced to a determinant of order five, which is equivalent to the one given by Hutson and White.

We have also studied the interaction between longitudinal electrostatic oscillations or plasma modes with elastic waves. The dispersion relation for hybrid waves which couple acoustic phonons with plasmons has been derived in section 4 and this is shown to be given by a determinental equation of order four.

2. WAVE PROPAGATION IN PIEZOELECTRIC CRYSTALS

Let us denote by T_{ij} and S_{ij} the stress and strain tensors respectively. If u_1, u_2 and u_3 denote the components of the displacements at a point (x_1, x_2, x_3), the components of the strain tensor are defined by

$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{1}$$

We assume the displacements to be small so that stress and strain are related to each other by Hooke's law:

$$T_{kl} = C_{ijkl}S_{ij} \tag{2}$$

where C_{ijkl} are the elastic constants and they satisfy the usual relations for permutation of the indices.

If however the substance is piezoelectric, stress can produce electricity and conversely the application of a voltage will produce stress. The relation between the stress tensor and the components of the electric vector

introduces the third order piezoelectric tensor, which we shall denote by $\tilde{e}_{ijk}$. For a piezoelectric crystal, the stress-strain relations assume the form

$$T_{kl} = C_{ijkl}S_{ij} - \tilde{e}_{mkl}E_m. \tag{3}$$

Further piezoelectricity modifies the relation between the electric displacement and the electric vector to the form

$$D_n = \epsilon_{mn}E_m + \tilde{e}_{nij}S_{ij} \tag{4}$$

where ϵ_{mn} is the dielectric tensor.

The electromagnetic variables are related to each other by means of Maxwell's equations which, in MKS units, we write as

$$\text{Curl } E = -\frac{\partial B}{\partial t} \tag{5 a}$$

$$\text{Curl } H = \frac{\partial D}{\partial t} + j \tag{5 b}$$

$$\text{Div } B = 0. \tag{5 c}$$

Besides, the electric vector E and the current density j are related by Ohm's law, which in its generalised form may be written as

$$j_i = \sigma_{ij}E_j \tag{6}$$

where σ_{ij} is the conductivity tensor. It is generally a function of several parameters like the plasma frequency ω_p, the wave vector $\vec{k}$, the ratio of the electron drift velocity to the sound wave velocity, etc.

Again

$$B = \mu_0 H \tag{7}$$

where μ_0 is the magnetic permeability. The equations of motion for elastic vibrations are given by

$$\frac{\rho\partial^2 u_k}{\partial t^2} = \frac{\partial T_{kl}}{\partial x_l}. \tag{8}$$

We shall assume that the elastic displacement u_k as well as the components of the electric vector have space-time variations of the following form:

$$u_i = \bar{U}_i e^{-i(\omega t - k.r)}$$

$$E_i = \bar{E}_i e^{-i(\omega t - k.r)}. \tag{9}$$

Substituting (9) in (8), one gets

$$(C_{ijkl}k_jk_l - \rho\omega^2\delta_{ki})\, u_i + i\tilde{e}_{ikl}k_l E_i = 0. \tag{10}$$

Next, taking the curl of Maxwell's equation (5 *a*) and substituting (5 *b*) in it, we get.

$$\text{Curl Curl } E = -\mu_0\left(\frac{\partial^2 D}{\partial t^2} + \frac{\partial}{\partial t}\sigma E\right). \tag{11}$$

For wave propagation, this equation transforms into

$$\vec{k} \times (\vec{k} \times \vec{E}) = -\mu_0\omega^2 (\alpha E + \bar{e}S) \tag{12}$$

where the dyadic α is defined by

$$\alpha = \epsilon + \frac{i\sigma}{\omega}$$

and

$$(\bar{e}S)_i = \tfrac{1}{2}\, i\tilde{e}_{imn}\,(u_m k_n + u_n k_m). \tag{13}$$

The components of the vector equation (12) can be written as

$$\frac{i\mu_0\omega^2}{2}(\bar{e}_{lij} + \bar{e}_{lji})\, k_j u_i + (k_i k_l - k^2\delta_{il} + \mu_0\omega^2\alpha_{li})\, E_i = 0. \tag{14}$$

Since $\vec{k} \times \vec{E} = k \times E_\perp$, we may also write (12) as

$$k^3 E_\perp = \mu_0\omega^2 (\alpha E + \bar{e}S). \tag{15}$$

By taking the scalar product of this equation with $\vec{k}$, we find that

$$(\alpha E + \bar{e}S)\cdot\vec{k} = 0. \tag{16}$$

The equation (10) and (14) constitute a set of six equations in the six variables u_1, u_2, u_3, E_2, E_3, E_1. By eliminating these variables from the equations, we get the determinental equation of order six given in Table 1. This equation gives the dispersion equation for waves that are a mixture of elastic, electromagnetic and electrostatic oscillations.

At this stage, we may point out that the equations (10) and (12) can alternatively be derived from the following Lagrangian density:

$$L = \tfrac{1}{2}\rho\dot{U}^2 - \tfrac{1}{2}\, C_{ijkl}S_{kl}S_{ij} + E^\dagger\,\bar{e}S + \tfrac{1}{2}\,E^\dagger\,\epsilon E - \tfrac{1}{2}\,H\cdot B + j\cdot A.$$

The various terms in this expression can be interpreted physically. The first two terms give the Langragian for the elastic vibrations. The term

TABLE I

$C_{1j1l}k_jk_l - \rho\omega^2$	$C_{3j1l}k_jk_l$	$i\tilde{e}_{11l}k_l$	$i\tilde{e}_{21l}k_l$	$i\tilde{e}_{31l}k_l$
$C_{1j2l}k_jk_l$	$C_{3j2l}k_jk_l$	$i\tilde{e}_{12l}k_l$	$i\tilde{e}_{22l}k_l$	$i\tilde{e}_{32l}k_l$
$C_{1j3l}k_jk_l$	$C_{3j3l}k_jk_l - \rho\omega^2$	$i\tilde{e}_{13l}k_l$	$i\tilde{e}_{23l}k_l$	$i\tilde{e}_{33l}k_l$
$i\mu_0\omega^2\tilde{e}_{11j}k_j$	$i\mu_0\omega^2\tilde{e}_{13j}k_j$	$k_1^2 - k^2 + \mu_0 \times \omega^2\alpha_{11}$	$k_1k_2 + \mu_0\omega^2\alpha_{12}$	$k_1k_3 + \mu_0\omega^2\alpha_{13}$
$i\mu_0\omega^2\tilde{e}_{21j}k_j$	$i\mu_0\omega^2\tilde{e}_{23j}k_j$	$k_3k_1 + \mu_0\omega^2\alpha_{21}$	$k_2^2 - k^2 + \mu_0 \times \omega^2\alpha_{22}$	$k_2k_3 + \mu_0\omega^2\alpha_{23}$
$i\mu_0\omega^2\tilde{e}_{31j}k_j$	$i\mu_0\omega^2\tilde{e}_{33j}k_j$	$k_3k_1 + \mu_0\omega^2\alpha_{31}$	$k_3k_2 + \mu_0\omega^2\alpha_{32}$	$k_3^2 - k^2 + \mu_0 \times \omega^2\alpha_{33}$

$= 0$

$E^{+}\epsilon S$ gives the interaction energy between the elatic modes and the electromagnetic field, which are coupled through the third order piezoelectric tensor. The last three terms give the Lagrangian density for the electromagnetic field, when currents are present in the system.

One can verify that the equations (10) and (12) follow directly from the Euler equations given by

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial(\partial U_i/\partial t)} + \sum_j \frac{\partial}{\partial x_j}\frac{\partial L}{\partial(\partial U_i/\partial x_j)} = \frac{\partial L}{\partial U_i}$$

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial(\partial A_i/\partial t)} + \sum_\mu \frac{\partial}{\partial x_\mu}\frac{\partial L}{\partial(\partial A_i/\partial x_\mu)} = \frac{\partial L}{\partial A_i}. \qquad (18)$$

COUPLING BETWEEN ACOUSTIC AND ELECTROMAGNETIC WAVES

We shall consider in this section the interaction between the acoustic wave and electromagnetic waves brought through the piezoelectric coupling tensor. We shall assume that wave propagation takes place in the direction of the wave vector k. The waves observed in the photoconductive Cadmium Sulphide by Hutson *et al.* are of this kind and the dispersion equation for such hybrid waves can be obtained by eliminating the longitudinal component of the electrical vector from the equations of motion. We shall find it convenient to express the components of the electrical vector in a co-ordinate frame S′ having axes along e_1', e_2', e_3' such that e_3' coincides with the direction of the wave vector k. The old frame of reference S, which is generally related to the crystallographic axes, will be retained for expressing the components of the elastic displacement u. Let E_1', E_2', E_3', denote the components of $\vec{E}$ in the frame S′; it is clear than that E_1' and E_2' are the transverse components of $\vec{E}$. Let the transformation law between (E_1', E_2', E_3') and ($E_1E_2E_3$) be given by

$$\begin{pmatrix} E_1' \\ E_2' \\ E_3' \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}. \qquad (19)$$

We shall denote the components of the matrix $\alpha = (\epsilon + i\sigma/\omega)$ in the new frame S′ by α'_{ij}.

TABLE II

$$
\begin{array}{|ccccc|}
k_j k_l \left(C_{1jil} + \dfrac{\tilde{e}'_{31l}\tilde{e}'_{3j1}}{\alpha'_{33}} \right) & \left(C_{2jil} + \dfrac{\tilde{e}'_{31l}\tilde{e}'_{3j2}}{\alpha'_{33}} \right) \times k_j k_l & \left(C_{3jil} + \dfrac{\tilde{e}'_{31l}\tilde{e}'_{3j3}}{\alpha'_{33}} \right) \times k_j k_l & ik_l \left(\tilde{e}'_{11l} - \dfrac{\alpha'_{31}}{\alpha'_{33}} \tilde{e}'_{31l} \right) & ik_l \left(\tilde{e}'_{21l} - \dfrac{\alpha'_{32}}{\alpha'_{33}} \tilde{e}'_{31l} \right) \\
\left(C_{1j2l} + \dfrac{\tilde{e}'_{32l}\tilde{e}'_{3j1}}{\alpha'_{33}} \right) \times k_j k_l & \left(C_{2j2l} + \dfrac{\tilde{e}'_{32l}\tilde{e}'_{3j2}}{\alpha'_{33}} \right) \times (k_j k_l) - \rho\omega^2 & \left(C_{3j2l} + \dfrac{\tilde{e}'_{32l}\tilde{e}'_{3j3}}{\alpha'_{33}} \right) \times k_j k_l & ik_l \left(\tilde{e}'_{12l} - \dfrac{\alpha'_{31}}{\alpha'_{33}} \tilde{e}'_{32l} \right) & ik_l \left(\tilde{e}'_{22l} - \dfrac{\alpha'_{32}}{\alpha'_{33}} \tilde{e}'_{32l} \right) \\
\left(C_{1j3l} + \dfrac{\tilde{e}'_{33l}\tilde{e}'_{3j1}}{\alpha'_{33}} \right) \times k_j k_l & \left(C_{2j3l} + \dfrac{\tilde{e}'_{33l}\tilde{e}'_{3j2}}{\alpha'_{33}} \right) \times k_j k_l & \left(C_{3j3l} + \dfrac{\tilde{e}'_{33l}\tilde{e}'_{3j3}}{\alpha'_{33}} \right) \times (k_j k_l) - \rho\omega^2 & ik_l \left(\tilde{e}'_{13l} - \dfrac{\alpha'_{31}}{\alpha'_{33}} \tilde{e}'_{33l} \right) & ik_l \left(\tilde{e}'_{23l} - \dfrac{\alpha'_{32}}{\alpha'_{33}} \tilde{e}'_{33l} \right) \\
i \left(\tilde{e}'_{11n} - \dfrac{\alpha'_{13}}{\alpha'_{33}} \tilde{e}'_{31n} \right) k_n & ik_n \left(\tilde{e}'_{12n} - \dfrac{\alpha'_{13}}{\alpha'_{33}} \tilde{e}'_{32n} \right) & ik_n \left(\tilde{e}'_{13n} - \dfrac{\alpha'_{13}}{\alpha'_{33}} \tilde{e}'_{33n} \right) & \left(\alpha'_{11} - \dfrac{\alpha'_{13}\alpha'_{31}}{\alpha'_{33}} - \dfrac{k^2}{\mu_0\omega^2} \right) & \left(\alpha_{12}' - \dfrac{\alpha'_{13}\alpha'_{32}}{\alpha_{33}'} \right) \\
ik_n \left(\tilde{e}'_{21n} - \dfrac{\alpha'_{23}}{\alpha'_{33}} \tilde{e}'_{31n} \right) & ik_n \left(\tilde{e}'_{22n} - \dfrac{\alpha'_{23}}{\alpha'_{33}} \tilde{e}'_{32n} \right) & ik_n \left(\tilde{e}'_{23n} - \dfrac{\alpha'_{23}}{\alpha'_{33}} \tilde{e}'_{33n} \right) & \left(\alpha'_{21} - \dfrac{\alpha'_{23}\alpha'_{31}}{\alpha'_{33}} \right) & \left(\alpha'_{22} - \dfrac{\alpha'_{23}\alpha'_{32}}{\alpha'_{33}} - \dfrac{k^2}{\mu_0\omega^2} \right)
\end{array}
= 0
$$

Obviously

$$\alpha'_{ij} = \tilde{A}_{ik}\alpha_{kl}A_{lj} \tag{20}$$

where $\tilde{A}$ is the transpose of A.

Since the piezoelectric constants are dependent linearly on the components of the electric displacement, they transform linearly with respect to the above transformation. Hence

$$\tilde{e}'_{ikl} = A_{ij}\tilde{e}_{jkl}. \tag{21}$$

In this the component i corresponding to the electric vector alone has been transformed.

From the equation (16), it now follows:

$$E_3' = \frac{1}{\alpha'_{33}}\{(\alpha'_{31}E_1' + \alpha'_{32}E_2') + \tilde{e}'_{3mn}S_{mn}\}. \tag{22}$$

We shall assume that $\alpha'_{33} \neq 0$.

Since $\alpha_{33}' = 0$ gives the dispersion equation for the propagation of plasma waves in the dielectric, this condition implies that plasma oscillations are not excited or are filtered out.

Now

$$\begin{aligned}\tilde{e}_{ikl}k_lE_i &= \tilde{e}_{ikl}k_l(A^{-1})_{ij}E_j' \\ &= (A_{ji}\tilde{e}_{ikl})\,k_lE_j' \\ &= \tilde{e}_{jkl}k_lE_j' \qquad (23)\\ &= k_lE_1'\left(\tilde{e}'_{1kl} - \frac{\alpha'_{31}}{\alpha'_{33}}\tilde{e}'_{3kl}\right) \\ &\quad + k_lE_2'\left(\tilde{e}'_{2kl} - \frac{\alpha'_{32}}{\alpha'_{33}}\tilde{e}'_{3kl}\right) \\ &\quad - \frac{i\tilde{e}'_{3kr}k_r\tilde{e}'_{3mn}(k_mu_n + k_nu_m)}{2\alpha'_{33}}. \qquad (24)\end{aligned}$$

Thus substituting (24) in (10) and rearranging the terms we get:

$$\begin{aligned}&u_i\left[C_{ijkl}k_jk_l - \rho\omega^2\delta_{ki} + \frac{\tilde{e}'_{3kr}k_rk_m\tilde{e}'_{mi}}{\alpha'_{33}}\right] \\ &\quad + ik_lE_1'\left(\tilde{e}'_{1kl} - \frac{\alpha'_{31}}{\alpha'_{33}}\tilde{e}'_{3kl}\right) + ik_lE_2'\left(\tilde{e}'_{2kl} - \frac{\alpha'_{32}}{\alpha'_{33}}\tilde{e}'_{3kl}\right) \\ &\quad = 0,\end{aligned}$$

Again on substituting (22)

$$i\left[\tilde{e}'_{imn}-\frac{\alpha'_{is'}}{\alpha'_{33'}}\tilde{e}'_{3mn}\right]k_n u_m+\left(\alpha'_{is}-\frac{\alpha'_{i3}}{\alpha'_{33}}\alpha'_{3s}-\frac{k^2}{\mu_0\omega^2}\delta_{is}\right)E_S'=0$$

$$(i, S=1, 2). \tag{26}$$

By giving i the values 1 and 2, we get two equations in u, E_1' and E_2'.

Equations (25) and (26) constitute a set of five equations in the five variables u_1, u_2, u_3, E_1' and E_2'. On eliminating them, we get the determinental equation given in Table II. This equation determines the frequency of propagation for any given wave vector k. As this is an equation of the fifth degree in ω^2, it has five roots. Of these, three are of acoustic nature and the remaining two describe electromagnetic waves, but generally these are coupled to each other. As the matrix α is complex, the elements of the equation in Table II are in general complex and besides this equation is not symmetric. Hence the roots are not generally real but are necessarily complex. The waves are hence attenuated. If however $(v_d/s)\geqslant 1$ one obtains growing waves.

We may mention that the condition $\alpha'_{33}\neq 0$ is necessary in order to obtain hybrid waves that couple elastic and electromagnetic modes only. This condition implies that longitudinal oscillations are not excited. In order to effectively filter out the plasma oscillations, it is also necessary that α'_{33} is as large as possible. The equation in Table II can alternatively be obtained by elementary determinental operations. Let O denote the operator, given by

$$O=\left[r_1-\frac{i\tilde{e}_{111}}{ka_{11}}\frac{k^2}{\mu_0\omega^2}r_6\right]+\left[r_2-\frac{k^2}{\mu_0\omega^2}\frac{i\tilde{e}_{121}}{ka_{11}}r_6\right]$$

$$+\left[r_3-\frac{i\tilde{e}_{131}}{ka_{11}}\frac{k^2}{\mu_0\omega^2}r_6\right]+\left[r_4-\frac{a_{21}}{a_{11}}r_6\right]+\left[r_5-\frac{a_{31}}{a_{11}}r_6\right] \tag{A}$$

where r_i denotes the i-th row of the determinant. The operator does not alter the value of the determinant in Table I, since it effects only transposition of the rows. The first five rows of the determinant contain zero as the last element. Expanding now the determinant along the last column, we obtain

$$a_{33}D_5=0$$

where D_5 denotes the determinant given in Table II. Since $a_{33}\neq 0$, the dispersion equation governing hybrid waves that couple elastic and electromagnetic modes is given by $D_5=0$.

4. INTERACTION BETWEEN ACOUSTIC MODES AND PLASMA OSCILLATIONS

We shall next eliminate the transverse components of the electric vector from equations (10), (15) and (16). We would then be obtaining four equations in the elastic displacement u and the longitudinal component of the electric vector. As the displacement of the electric vector and the direction of propagation of the wave become the same, these equations should describe the coupling between the plasma modes and sound waves. The components of the equation (15) along the axes e_1' and e_2', which are perpendicular to the direction of propagation, are given by

$$\left(\frac{k^2}{\mu_0\omega^2} - \alpha'_{11}\right) E_1' - \alpha'_{12}E_2' = \alpha'_{13}E_3' + \tilde{e}'_{1mn}S_{mn} \tag{27 a}$$

$$-\alpha'_{21}E_1' + \left(\frac{k^2}{\mu_0\omega^2} - \alpha'_{22}\right) E_2' = \alpha'_{23}E_3' + \tilde{e}'_{2mn}S_{mn} \tag{27 b}$$

where the components of $(\tilde{S})_i$ are given by (13) and in particular

$$S_{mn} = \tfrac{1}{2}i\,(u_m k_n + u_n k_m). \tag{28}$$

We shall now denote the matrix in the left hand side by β so that

$$\beta_{ij} = \frac{k^2}{\mu_0\omega^2}\,\delta_{ij} - \alpha_{ij}' \tag{29}$$

and we shall further set $\gamma = \beta^{-1}$. (30)

Then if determinant $|\beta| \neq 0$, we have

$$\begin{aligned} E_1' &= \gamma_{11}(\alpha'_{13}E_3' + \tilde{e}_{1mn}S_{mn}) + \gamma_{12}(\alpha'_{23}E_3' + \tilde{e}_{2mn}S_{mn}) \\ E_2' &= \gamma_{21}(\alpha'_{13}E_3' + \tilde{e}'_{1mn}S_{mn}) + \gamma_{22}(\alpha'_{23}E_3' + \tilde{e}'_{2mn}S_{mn}) \end{aligned} \tag{31}$$

i.e.,

$$E_\perp = b_{\perp 1}u_1 + b_{\perp 2}u_2 + b_{\perp 3}u_3 + b_{\perp 4}E_3.$$

If we substitute these in equation (16), we get

$$\begin{aligned} &iu_i k_j\,[\tilde{e}'_{3ij} + \tilde{e}'_{1ij}(\alpha'_{31}\gamma_{11} + \alpha'_{32}\gamma_{21}) + \tilde{e}'_{2ij}(\alpha'_{31}\gamma_{12} + \alpha'_{32}\gamma_{22})] \\ &\quad + E_3'\,[\alpha'_{33} + \alpha'_{13}(\alpha'_{31}\gamma_{11} + \alpha'_{32}\gamma_{21}) + \alpha'_{23}(\alpha'_{31}\gamma_{12} \\ &\quad + \alpha'_{32}\gamma_{22})] = 0. \end{aligned} \tag{32}$$

Again

$$\begin{aligned} &i\,\varepsilon_{ikl}k_l E_i \\ &\quad = i\tilde{e}'_{jkl}k_l E_j' \end{aligned}$$

TABLE III

$$
\begin{vmatrix}
C_{1j1l}k_jk_l - \rho\omega^2 + i\tilde{e}'_{11l}k_lb_{11} + i\tilde{e}'_{21l}k_lb_{21} & C_{2j1l}k_jk_l + i\tilde{e}'_{11l}k_lb_{12} + i\tilde{e}'_{21l}k_lb_{22} & C_{3j1l}k_jk_l + i\tilde{e}'_{11l}k_lb_{13} + i\tilde{e}'_{21l}k_lb_{23} & i(\tilde{e}'_{31l} + \tilde{e}'_{11l}b_{14} + \tilde{e}'_{21l}b_{24})k_l \\
C_{2j1l}k_jk_l + i\tilde{e}'_{12l}k_lb_{11} + i\tilde{e}'_{22l}k_lb_{21} & C_{2j2l}k_jk_l - \rho\omega^2 + i\tilde{e}'_{12l}k_lb_{12} + i\tilde{e}'_{22l}k_lb_{22} & C_{3j2l}k_jk_l + i\tilde{e}'_{12l}k_lb_{13} + i\tilde{e}'_{22l}k_lb_{23} & i(\tilde{e}'_{32l} + \tilde{e}'_{12l}b_{14} + \tilde{e}'_{22l}b_{24})k_l \\
C_{3j1l}k_jk_l + i\tilde{e}'_{13l}k_lb_{11} + i\tilde{e}'_{23l}k_lb_{21} & C_{3j2l}k_jk_l + i\tilde{e}'_{13l}k_lb_{12} + i\tilde{e}'_{23l}k_lb_{22} & C_{3j3l}k_jk_l - \rho\omega^2 + i\tilde{e}'_{13l}k_lb_{13} + i\tilde{e}'_{23l}k_lb_{23} & i(\tilde{e}'_{33l}k_l + \tilde{e}'_{13l}k_lb_{14} + i\tilde{e}'_{23l}k_lb_{24}) \\
i\mu_0\omega^2k_j\tilde{e}'_{31j} + (k_1k_3 - \mu_0\omega^2\alpha'_{31})b_{11} + (k_2k_3 - \mu_0\omega^2\alpha'_{32}) \times b_{21} & i\mu_0\omega^2k_j\tilde{e}'_{32j} + (k_1k_3 + \mu_0\omega^2\alpha'_{31})b_{12} + (k_2k_3 + \mu_0\omega^2\alpha'_{32}) \times b_{22} & i\mu_0\omega^2k_j\tilde{e}'_{33j} + (k_1k_3 + \mu_0\omega^2\alpha'_{31})b_{13} + (k_2k_3 + \mu_0\omega^2\alpha'_{32}) \times b_{23} & (k_3^2 - k^2 + \mu_0\omega^2\alpha'_{33}) + (k_1k_3 + \mu_0\omega^2\alpha'_{31}) \times b_{14} + (k_2k_3 + \mu_0\omega^2\alpha'_{32})b_{24}
\end{vmatrix} = 0
$$

where $E_\perp = b_{\perp i}u_i = b_{\perp 1}u_1 + b_{\perp 2}u_2 + b_{\perp 3}u_3 + b_{\perp 4}E_4$

$\perp = 1, 2.$

$$= i\tilde{e}_{3kl}k_l E_3' + i\tilde{e}'_{1kl}k_l [E_3' (\gamma_{11}\alpha'_{13} + \gamma_{12}\alpha'_{23}) + S_{mn} (\gamma_{11}\tilde{e}'_{1mn} + \gamma_{12}\tilde{e}'_{2mn})] + i\tilde{e}'_{2kl}k_l) [E_3' (\gamma_{21}\alpha'_{13} + \gamma_{22}\alpha'_{23}) + S_{mn} (\gamma_{21}\tilde{e}'_{1mn} + \gamma_{22}\tilde{e}'_{2mn})]$$

$$= iE_3'k_l [(\tilde{e}'_{3kl} + \tilde{e}'_{1kl} (\gamma_{11}\alpha'_{13} + \gamma_{12}\alpha'_{23}) + \tilde{e}'_{2kl} (\gamma_{21}\alpha'_{13} + \gamma_{22}\alpha'_{23})] - u_i k_l k_j [\tilde{e}'_{1kl} (\gamma_{11}\tilde{e}_{1ij} + \gamma_{12}\tilde{e}'_{2ij}) + \tilde{e}'_{2kl} (\gamma_{21}\tilde{e}'_{1ij} + \gamma_{22}\tilde{e}'_{2ij})]. \quad (33)$$

On substituting this in equation (10), we get

$$u_i [C_{iikl}k_j k_l - \rho\omega^2\delta_{ki} - k_j k_l \{\tilde{e}'_{1kl} (\gamma_{11}\tilde{e}'_{1ij} + \gamma_{12}\tilde{e}'_{2ij}) + \tilde{e}'_{2kl} (\gamma_{21}e'_{1ij} + \gamma_{22}\tilde{e}'_{2ij})\}] + iE_3'k_l [\tilde{e}'_{3kl} + \tilde{e}'_{1kl} (\gamma_{11}\alpha'_{13} + \gamma_{12}\alpha'_{23}) + \tilde{e}'_{2kl} (\gamma_{21}\alpha'_{13} + \gamma_{22}\alpha_{23}')] = 0. \quad (34)$$

The condition for the validity of the dispersion equation is that $|\beta| \neq 0$, which implies that electromagnetic waves are not propagated in the semiconductor, under these conditions.

Equations (34) and (32) constitute a set of four equations in the four variables u_1, u_2, u_3 and E_3, and eliminating these from them, we obtain the determinental equation given in Table III. These equations involve only the component of the electric vector along the wave vector and for this reason, the waves should represent plasma oscillations. The equation (35) gives the dispersion relation for hybrid waves, which are a mixture of elastic vibrations and plasma oscillations. As the elements of the matrix in Table III are complex and the matrix is not symmetric, the roots of this equation are generally complex and the waves should either be growing or decaying.

REFERENCES

Hutson, A. R., Mc Fee, J. H. and White, D. L. — *Phys. Rev. Letters*, 1961, **7**, 237.

——— and White, D. L., — *J. Applied Physics*, 1962, **33**, 40–47.

Steele and Vural .. *Wave Interactions in Solid State Plasmas*, McGraw-Hill, New York, 1969, pp. 134–44.

White, D. L. .. *J. Appl. Phys.*, 1962, **33**, 2547–54,