TURBULENT SHOCK WAVES IN A COLLISION-FREE PLASMA

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ABSTRACT

The paper deals with the structure of collisionless shocks arising from turbulent wave-particle interactions. The conditions under which wave-particle interaction effects could become significant leading to growing waves and a shock are discussed. Using the Mott-Smith expression for the zero-order distribution functions for the ions within the shock, the dielectric constant as well as the integral representing the wave-particle interaction term in the Lenard-Balescu equation are evaluated for a collisionless plasma. An expression is given for the ion distribution function within the shock.

It is shown that the component of the pressure tensor perpendicular to the direction of flow of the plasma leads to a new kind of viscosity term arising from the interaction of the particles with the growing waves and this provides a dissipative mechanism to account for the conversion of the kinetic energy of the incoming plasma into the thermal energy of the hot ionised gas behind the shock.

1. INTRODUCTION

COLLISIONLESS shocks have attracted the attention of a number of workers and though the subject is of recent origin, the literature on it has already become unwieldy.[1-6] A well-known example of a collision-free shock is the bow shock produced in front of the magnetosphere by the solar wind blowing at hypersonic speeds. Satellite observations have confirmed that the thickness of the shock front is of the order of a few hundreds of kilometres, which is much smaller than the mean free path ($\sim 10^{13}$ cm.) in the solar plasma. Collisionless shocks can be produced in the laboratory also in shock tubes. The ratio of the electron to the ion temperature plays a significant role in the formation of such shocks. Anderson *et al.*[7] have shown that shocks are produced when the ratio T_e/T_i is increased to about 8 or 10 by cooling the ions. Besides, these shocks could occur in a Stellerator in controlled fusion experiments and a knowledge of the properties of these shocks could provide information about the instabilities that plague such experiments and ways to suppress them.

It is now generally agreed that plasma instabilities play a vital role in determining the structure of collisionless shocks and the shock itself is a consequence of the wave-particle interactions. When plasma waves, which arise as a result of the thermal fluctuations, become unstable, they grow in amplitude generating a turbulent region behind the shock layer. These waves scatter and trap the particles, and provide a mechanism for converting the kinetic energy of the supersonic cool plasma into the thermal energy of the hot downstream plasma. A theory of the turbulent shock waves generated by wave-particle interaction has been given by Tidman.[8] The present paper is a development of this work and we provide an expression for the distribution function for the plasma in the thin shock layer by evaluating the integral representing the wave-particle interactions in the transport equation. When once the distribution function is known, several microscopic properties of the plasma such as the growth rate of non-linear ion waves, the thickness of the shock-front, the pressure tensor, etc., can be evaluated. In Section 4, we evaluate the dispersion equation by using the Mott-Smith function for the zero-order distribution function. Since the wave-particle interaction converts the kinetic energy of the incoming plasma into the thermal energy of the gas behind the shock, it is a dissipative mechanism and one should be able to associate a coefficient of viscosity with it. Section 6 deals with the pressure tensor and the viscosity coefficient.

2. THE WAVE-PARTICLE INTERACTION

The kinetic equation for the averaged distribution function in a collisionless plasma under conditions of weak turbulence is given by

$$\frac{\partial f}{\partial t} + v \cdot \nabla f + \langle \mathbf{E} \rangle \cdot \frac{\partial f}{\partial v} = S_{ef} \tag{1}$$

where S_{ef} represents the collisions between the particles and the waves, and is given by*

$$S_{ef} = \frac{-2e^4}{m^2} \frac{\partial}{\partial v} \cdot \int \frac{\delta\,[k \cdot v - v']}{k^4 \,|\, \epsilon(k, k \cdot v)\,|^2} \left\{ f(v)\, k \cdot \frac{\partial f(v')}{\partial v'} - f(v')\, k \cdot \frac{\partial f(v)}{\partial v} \right\} k\,dk\,dv'. \tag{2}$$

* The symbols v and v' appearing in these equations are vectors.

In the above, $\epsilon(\boldsymbol{k}, \omega)$ represents the dielectric constant of the plasma.

We shall consider in this paper shocks corresponding to a steady state, and we shall further assume that there is no average electric field $\langle E \rangle$ and further that the flow is one-dimensional. In this case, equation (1) reduces to

$$v_x \frac{df}{dx} = S_{ef}. \tag{3}$$

we shall rewrite the above equation through a set of non-dimensional variables $\tilde{x}$, $\tilde{v}$ by introducing two scale lengths r_0 and v_0 for the linear dimensions of the system and thermal velocities. By means of the transformation

$$x = r_0 \tilde{x}$$

$$v = v_0 \tilde{v}$$

and

$$f(v) = \frac{F(\tilde{v})}{v_0^3}. \tag{4}$$

we find that the equation (3) can be reduced to the following form after some simplifications:

$$\tilde{v}_x \frac{\partial F}{\partial \tilde{x}} = -\lambda \frac{\partial}{\partial \tilde{v}} \cdot \int \frac{k \delta [\boldsymbol{k} \cdot \tilde{v} - \tilde{v}']}{k^4 | \mathcal{E}(k, v_0 k \cdot \tilde{v})|^2} \left\{ F(\tilde{v}) \, \boldsymbol{k} \cdot \frac{\partial F(\tilde{v}')}{\partial \tilde{v}'} - F(\tilde{v}') \, \boldsymbol{k} \cdot \frac{\partial F(\tilde{v})}{\partial \tilde{v}} \right\} dk d\tilde{v}' \tag{5}$$

where

$$\lambda = \frac{2e^4 r_0}{m^2 v_0^4}. \tag{6}$$

The parameter λ is normally very small, but under conditions wherein electrostatic waves grow non-linearly and wave-particle interaction effects become significant, it can be of the order of magnitude of unity. For example, let us consider the typical case of the shock wave produced by the flow of the solar wind past the magnetosphere. In this case, observations suggest that the thickness of the shock is of the order of 1,000 km. so that we could take $r_0 \sim 1{,}000$ km. We further take $v_0 = 10^6$ cm./sec., which corresponds to the thermal velocity of the electrons. Then a simple calculation shows that for electrons

$$\lambda_e = 12 \cdot 83.$$

In view of the factor m^2 in the denominator, the value of λ for protons is much smaller, but if we take $v_0^e \approx 10\, v_0^i$ corresponding to the case in which the electron temperature is much higher than the ion temperature, λ_i also is of the order of unity. Thus it is clear that under certain set of conditions which depend on the ratio (r_0/v_0^4), the parameter λ is not negligible both for electrons as well as for ions and the wave-particle interaction can be significant under turbulent conditions.

3. THE DIELECTRIC CONSTANT

Let us denote by v_1 and v_2 the stream velocity of the ions in front of the shock and just behind the shock. The upstream velocity v_1 is assumed to be hypersonic. The shock is supposed to be a thin layer where the incoming plasma interpenetrates with the hot subsonic plasma of the downstream ragion. We shall choose the direction of flow of the plasma as the x-axis and consider the case of a normal shock. Following Tidman, we choose the distribution function for the ions inside or near the shock layer as a bimodal Mott-Smith function given by

$$\langle f_i \rangle = \frac{n_1(x)}{(2\pi)^{3/2} V_1^3} \exp. \left[- \frac{(v - v_1)^2}{2V_1^2} \right] + \frac{n_2(x)}{(2\pi)^{3/2} V_2^3} \exp. \left[- \frac{(v - v_2)^2}{2V_2^2} \right] \tag{7}$$

where V_1 and V_2 denote the thermal velocity of the protons in front of and behind the shock. If N_1 and N_2 denote the average density of the ions in the upstream and downstream plasma, then we have the boundary conditions

$$n_1(+\infty) = n_2(-\infty) = 0; \quad n_1(-\infty) = N_1; \quad n_2(+\infty) = N_2.$$

As stated earlier we shall suppose that the electron temperature is considerably higher than the ion temperature so that the electron gas is subsonic even in front of the shock and does not suffer appreciable velocity jump across the shock. For future calculations, the distribution function for the electron gas will be taken as

$$\langle f_e \rangle = \frac{n_e(x)}{(2\pi)^{3/2} V_e^3(x)} \exp. \left[- \frac{(v - v_e(x))}{2V_e^2(x)} \right] \tag{8}$$

where

$$n_e(-\infty) = N_1; \quad n_e(+\infty) = N_2; \quad v_e(-\infty) = v_1;$$
$$v_e(+\infty) = v_2.$$

The integral on the right-hand side of (2) involves the dielectric function $\mathcal{E}(k, \omega)$ in the denominator and before evaluating it, it is necessary to obtain an expression for $\mathcal{E}(k, \omega)$. This is given by

$$\mathcal{E}(k, \omega) = 1 + \frac{4\pi e^2}{M_i k^2} G_i + \frac{4\pi e^2}{m k^2} G_e \tag{9}$$

where

$$G_i = \int \frac{k_x\left(\frac{\partial f_i}{\partial v_x}\right)}{\omega - k_x v_x}\, dv \tag{10 a}$$

and

$$G_e = \int \frac{k_x\left(\frac{\partial f_e}{\partial v_x}\right)}{\omega - k_x v_x}\, dv. \tag{10 b}$$

We shall first evaluate G_i, the ion contribution to the dielectric constant. Substituting from (7) for f_i in (10 a), we get

$$G_i = \sum_{\alpha=1,2} \frac{n_\alpha(x)}{(2\pi)^{3/2} V_\alpha^5} \int \frac{k_x(v_x - v_{x\alpha})}{k_x v_x - \omega}$$
$$\times \exp.\left\{-\frac{1}{2V_\alpha^2}\left[(v_x - v_{x\alpha})^2 + v_y^2 + v_z^2\right]\right\} dv$$
$$= \sum_\alpha \frac{n_\alpha(x)}{(2\pi)^{\frac{1}{2}} V_\alpha^3} \int \frac{k_x(v_x - v_{x\alpha})}{k_x v_x - \omega} \exp.\left[-\frac{(v_x - v_{x\alpha})^2}{2V_\alpha^2}\right] dv_x \tag{11 a}$$

on performing the integrations with respect to v_y and v_z. Hence

$$G_i = \sum_\alpha \frac{n_\alpha(x)}{(2\pi)^{\frac{1}{2}} V_\alpha^3} \left\{\sqrt{2\pi}\, V_\alpha + (\omega - k_\alpha v_{x\alpha})\right.$$
$$\left. \times \int \frac{\exp.\left[\frac{-(v_x - v_{x\alpha})^2}{(2V_\alpha^2)}\right]}{(k_x v_x - \omega)}\, dv_x\right\}. \tag{11 b}$$

We now define the dispersion function $Z(\zeta)$ by means of the integral

$$Z(\zeta) = \pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} \frac{e^{-x^2}}{x - \zeta}\, dx. \tag{12}$$

Then writing

$$\zeta_a = \frac{\omega - k_x v_{xa}}{\sqrt{2}\, V_a k_x}$$

one finds easily

$$G_i = \sum_{a=1,2} n_a(x) \left\{ \frac{1}{V_a^2} + \frac{(\omega - k_x v_{xa})}{\sqrt{2}\, V_a^3 k_x} Z(\zeta_a) \right\}. \tag{13}$$

The integral for G_e can be evaluated similarly using the distribution function f_e for the electrons and one finds that the dielectric constant $\mathcal{E}(k, \omega)$ is given by

$$\mathcal{E}(k, \omega) = 1 + \sum_{a=1,2} \left\{ \frac{\omega^2{}_{pa}}{k^2 V_a^2} + \frac{\omega^2{}_{pa} \zeta_a Z(\zeta_a)}{k^2 V_a^2} \right\} + \frac{\omega^2{}_{pe}}{k^2 V_e^2} \zeta_e Z(\zeta_e) \tag{14}$$

where $\omega^2{}_{pa}$ and $\omega^2{}_{pe}$ are the plasma frequencies for the ions and the electrons.

4. THE DISPERSION EQUATION

The dispersion equation is now given by

$$\mathcal{E}(k, \omega) = 0. \tag{15}$$

The solution of the above equation gives the wavelengths and associated frequencies of the longitudinal electrostatic waves in the plasma. Before proceeding with the dispersion equation using the Mott–Smith function, it will be useful to recall briefly the results of the two stream instability in a plasma and the nature of growing waves. It is well known that for a distribution function having the form

$$f = \frac{n_0}{2} [\delta(v - v_1) + \delta(v - v_2)] \tag{16}$$

the dispersion equation is given by

$$\frac{1}{\omega_p^2} = \frac{y^2 + t^2}{(y^2 - t^2)^2} \tag{17}$$

where

$$y = \omega - \frac{k(v_1 + v_2)}{2} \tag{18}$$

and

$$t = \frac{k(v_1 - v_2)}{2}.$$

The roots of this equation are

$$\left.\begin{matrix} 2y_1^2 \\ 2y_2^2 \end{matrix}\right\} = \omega_p^2 \left\{ 1 + \frac{2t^2}{\omega_p^2} \pm \sqrt{1 + \frac{8t^2}{\omega_p^2}} \right\}. \tag{19}$$

Of these two roots, y_1^2 is always positive. Since we are interested in growing waves, we shall consider in detail y_2^2; this will be negative if

$$1 + \frac{8t^2}{\omega_p^2} > \left(1 + \frac{2t^2}{\omega_p^2}\right)^2$$

or if

$$k < \frac{2\omega_p}{v_1 - v_2}. \tag{20}$$

Thus waves whose wavelengths are greater than $\pi(v_1 - v_2)/\omega_p$ will be unstable and can grow in amplitude. This result will be useful to have a qualitative picture of growing waves and to understand fully the dispersion equation (15), which takes into account of the thermal agitation of the particles.

If for any real value of the wavelength of an electrostatic wave, the frequency ω turns out to be complex, the waves are either damped or growing waves. We are interested only in the latter type of waves and for these, the imaginary part of ω, which gives the growth rate of the wave, should be positive.

Let us now write

$$\omega = \omega_r + i\omega_i \tag{21 a}$$

$$\zeta_a = \zeta_{ar} + i\omega_i\zeta_{ai} \tag{21 b}$$

$$Z(\zeta_a) = Z_{ar} + iZ_{ai} \tag{21 c}$$

where

$$\zeta_{ar} = \frac{\omega_r - k_x v_{xa}}{\sqrt{2}\, k_x V_a}\ (a = 1, 2, e) \tag{22 a}$$

and

$$\zeta_{ai} = \frac{1}{\sqrt{2}\, k_x V_a}\,. \tag{22 b}$$

Substituting these in (15), and equating the real and imaginary parts of the equation to zero, one obtains

$$1 + \sum_a \frac{\omega^2{}_{pa}}{k^2 V_a{}^2} + \sum_a \frac{\omega^2{}_{pa}}{k^2 V_a{}^2} (\zeta_{ar} Z_{ar} - \omega_i \zeta_{ai} Z_{ai}) = 0 \tag{23 a}$$

$$\sum_a \frac{\omega^2{}_{pa}}{k^2 V_a{}^2} (\zeta_{ar} Z_{ai} + \omega_i \zeta_i Z_{ar}) = 0 \tag{23 b}$$

where the summation for a runs over the indices 1, 2 and the electrons.

These are a set of simultaneous equations in ζ_r (or ω_r) and ω_i, and a solution of these two equations gives the growth rate of the wave as well as the frequency ω_r corresponding to any wave having the wavelength k_x.

To solve the equations (23), it is necessary to obtain analytical expressions for Z_r and Z_i.

Now[10]

$$Z(\zeta) = \pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} \frac{e^{-x^2}}{x - \zeta}\, dx.$$

$$= \pi^{\frac{1}{2}}\, i e^{-\zeta^2} [1 - erf(-i\zeta)] \tag{24}$$

and

$$erf(-i\zeta) = erf(\omega_i \zeta_i - i\zeta_r)$$

$$= erf(\omega_i \zeta_i) + \frac{e^{-\omega_i^2 \zeta_i^2}}{2\pi \omega_i \zeta_i} [(1 - \cos 2\omega_i \zeta_i \zeta_r) - i \sin 2\omega_i \zeta_i \zeta_r]$$

$$+ \frac{2}{\pi} e^{-\omega_i^2 \zeta_i^2} \sum_{n=1}^{\infty} \frac{e^{-n^2/4}}{n^2 + 4\omega_i^2 \zeta_i^2}$$

$$\times [f_n(\zeta_i, \zeta_r) + i g_n(\zeta_i, \zeta_r)] + \epsilon \tag{25}$$

where

$$f_n = 2\omega_i\zeta_i - 2\omega_i\zeta_i \cosh n\zeta_r \cos 2\omega_i\zeta_i\zeta_r + n \sinh n\zeta_r \sin 2\omega_i\zeta_i\zeta_r$$

$$g_n = -\left[2\omega_i\zeta_i \cosh n\zeta_r \sin 2\omega_i\zeta_i\zeta_r + n \sinh n\zeta_r \cosh 2\omega_i\zeta_i\zeta_r\right] \quad (26)$$

and

$$|\epsilon| \approx 10^{-16} |erf(-i\zeta)|.$$

It follows after some simplification that the real and imaginary parts of $Z(\zeta)$ are given by the expressions

$$Z_r = \pi^{\frac{1}{2}} e^{-\zeta_r^2+\zeta_i^2\omega_i^2} \Bigg[\sin 2\omega_i\zeta_i\zeta_r \Bigg\{ 1 - erf(\omega_i\zeta_i)$$

$$- \frac{e^{-\omega_i^2\zeta_i^2}}{2\pi\omega_i\zeta_i}(1 - \cos 2\omega_i\zeta_i\zeta_r)$$

$$- \frac{2}{\pi} e^{-\omega_i^2\zeta_i^2} \sum_{n=1}^{\infty} \frac{e^{-n^2/4}}{n^2 + 4\omega_i^2\zeta_i^2} f_n \Bigg\}$$

$$- \cos 2\omega_i\zeta_i\zeta_r \Bigg\{ \frac{e^{-\omega_i^2\zeta_i^2}}{2\pi\omega_i\zeta_i} \sin 2\omega_i\zeta_i\zeta_r - \frac{2}{\pi} e^{-\omega_i^2\zeta_i^2}$$

$$\times \sum_{n=1}^{\infty} \frac{e^{-n^2/4}}{n^2 + 4\omega_i^2\zeta_i^2} g_n \Bigg\} \Bigg] \quad (27)$$

$$Z_i = \pi^{\frac{1}{2}} e^{-\zeta_r^2+\zeta_i^2\omega_i^2} \Bigg[\cos 2\omega_i\zeta_i\zeta_r \Bigg\{ 1 - erf(\omega_i\zeta_i)$$

$$- \frac{e^{-\omega_i^2\zeta_i^2}}{2\pi\omega_i\zeta_i}(1 - \cos 2\omega_i\zeta_i\zeta_r)$$

$$- \frac{2}{\pi} e^{-\omega_i^2\zeta_i^2} \sum_{n=1}^{\infty} \frac{e^{-n^2/4}}{n^2 + 4\omega_i^2\zeta_i^2} f_n \Bigg\}$$

$$+\sin 2\omega_i\zeta_i\zeta_r \left\{ \frac{e^{-\omega_i^2\zeta_i^2}}{2\pi\omega_i\zeta_i} \sin 2\omega_i\zeta_i\zeta_r \right.$$

$$\left. -\frac{2}{\pi} e^{-\omega_i^2\zeta_i^2} \sum_{n=1}^{\infty} \frac{e^{-n^2/4}}{n^2+4\omega_i^2\zeta_i^2} g_n \right\}. \tag{28}$$

The equations (23) together with the expressions (26), (27) and (28) for Z_r and Z_i provide a basis for the numerical evaluation of ω_r and ω_i for any given wavelength. They could throw light on the damping of the waves as well as their growth rate near a shock.

Another method of solving the equations (18) is to assume that ω_i is much smaller than ω_r, expand Z (ζ) as a Taylor series ζ_r and $\omega_i\zeta_i$ truncate the series with the first two or three terms. In this way one could obtain analytic expressions for ω_i. The equations determining ω_i and ζ_{ar} can be found to be given by

$$\omega_i^2 = \frac{\sum \frac{\omega^2 p_a}{V_a^2} \zeta_{ai} [Z(\zeta_{ar}) + \zeta_{ar} Z'(\zeta_{ar})]}{\sum \frac{\omega^2 p_a}{V_a^2} \zeta^3{}_{ai} \left[\frac{1}{2} Z''(\zeta_{ar}) + \frac{\zeta_{ar}}{6} Z'''(\zeta_{ar})\right]}$$

$$= \frac{4\left[1 - \sum_a \frac{\omega^2 p_a}{k^2 V_a^2} + \sum_a \frac{\omega^2 p_a}{k^2 V_a^2} \zeta_{ar} Z(\zeta_{ar})\right]}{\sum_a \frac{\omega^2 p_a}{k^4 V_a^4} [Z'(\zeta_{ar}) + \zeta_{ar} Z''(\zeta_{ar})]}. \tag{29}$$

5. THE ION DISTRIBUTION FUNCTION

In order to evaluate the ion distribution function within the shock-front, we write

$$F(\tilde{v}) = F_0(\tilde{v}) + F_1(\tilde{v}) \tag{30}$$

and substitute $F_0(\tilde{v})$ for the distribution function on the right-hand side of (5). $F_0(\tilde{v})$ is chosen to be a Mott–Smith distribution function given by (7). It is first necessary to express this function in non-dimensional variables. Let us now write

$$V_1 = a_1 v_0; \quad V_2 = a_2 v_0; \quad \bar{N}_1 = \left(\frac{v_1}{v_0}\right) \text{ and } \bar{N}_2 = \left(\frac{v_2}{v_0}\right). \tag{31}$$

The numbers $\bar{N}_1$ and $\bar{N}_2$ are of the order of Mach numbers in front of and behind the shock. We find that

$$F_0(\tilde{v}) = v_0^3 f$$
$$= \frac{n_1(\tilde{x})}{(2\pi)^{3/2} a_1^3} e^{-(\tilde{v}-N_1)^2/2a_1^2} + \frac{n_2(\tilde{x})}{(2\pi)^{3/2} a_2^3} e^{-(\tilde{v}-N_2)^2/2a_2^2}$$
$$= G_1(\tilde{v}) + G_2(\tilde{v}) \tag{32}$$

where G_1 and G_2 denote respectively the first and second terms of the above expression. The kinetic equation determining the ion distribution function is then given by

$$\frac{\partial F}{\partial \tilde{x}} = -\frac{\lambda}{\tilde{v}_x}\frac{\partial}{\partial \tilde{v}_x}\int \frac{1}{k^4_x}\frac{\delta(\tilde{v}_x - \tilde{v}_x')}{|\epsilon(k, v_0 k_x \tilde{v}_x)|^2} P dk_x d\tilde{v}_x' \, d\tilde{v}_y' \, d\tilde{v}_z' \tag{33}$$

where

$$P = F_0(\tilde{v})\, \boldsymbol{k} \,.\, \frac{\partial F_0(\tilde{v}')}{\partial \tilde{v}'} - F(\tilde{v}')\, \boldsymbol{k} \,.\, \frac{\partial F_0(\tilde{v})}{\partial \tilde{v}'}$$
$$= k_x \left\{ F_0(\tilde{v}) \frac{\partial F_0(\tilde{v}')}{\partial \tilde{v}_x'} - F(\tilde{v}') \frac{\partial F_0(\tilde{v})}{\partial \tilde{v}_x} \right\}. \tag{34}$$

With the expression (32) for $F_0(\tilde{v})$ it is easy to evaluate the quantity P as well as the integral $\int P d\tilde{v}_y' \, d\tilde{v}_z'$. These are given by

$$P = k_x \left\{ -\left(G_1(v) + G_2(\tilde{v})\right)\left[\sum_{\alpha=1}^{2} \frac{(\tilde{v}_x' - N_\alpha)}{a_\alpha^2} G_\alpha(\tilde{v}')\right] \right.$$
$$\left. + \left(G_1(\tilde{v}') + G_2(\tilde{v}')\right)\left[\sum_{\alpha} \frac{(\tilde{v}_x - N_\alpha)}{a_\alpha^2} G_\alpha(\tilde{v})\right]\right\} \tag{35}$$

and

$$\int P d\tilde{v}_y' \, d\tilde{v}_z'$$
$$= k_x \left\{ \frac{(\tilde{v}_x - \tilde{v}_x')}{a_1^2} G_1(\tilde{v})\, g_1(\tilde{v}_x') + \frac{(\tilde{v}_x - \tilde{v}_x')}{a_1^2} G_2(\tilde{v})\, g_2(\tilde{v}_x') \right.$$
$$+ \left[\frac{(\tilde{v}_x - N_1)}{a_1^2} - \frac{(\tilde{v}_x' - N_2)}{a_2^2}\right] G_1(\tilde{v})\, g_2(\tilde{v}_x')$$
$$\left. + \left[\frac{(\tilde{v}_x - N_2)}{a_2^2} - \frac{(\tilde{v}_x' - N_1)}{a_1^2}\right] G_2(\tilde{v})\, g_1(\tilde{v}_x') \right\} \tag{36}$$

where

$$g_1(\tilde{v}_x) = \frac{n_1(\tilde{x})}{(2\pi)^{\frac{1}{2}} a_1} e^{-(\tilde{v}_x - N_1)^2/2a_1^2}. \tag{37}$$

In view of the presence of the factor $\mathcal{E}\mathcal{E}^*$ in the denominator, the integral on the right-hand side of (5) can generally be evaluated by numerical methods alone. However, the integral can be evaluated analytically for the case when $\zeta \ll 1$ which is satisfied for a wide spectrum of the unstable waves. In this case, $Z(\zeta)$ can be expanded as a power series and retaining the first two terms of the series, we have

$$\zeta Z(\zeta) = i\sqrt{\pi}\,\zeta - 2\zeta^2. \tag{38}$$

The square of the dielectric constant is then given by

$$\mathcal{E}\mathcal{E}^* = \left[1 + \sum_{\alpha=1,2,e} \frac{\omega^2{}_{p\alpha}}{k^2 V_\alpha{}^2} - \frac{2}{k^2} \sum_\alpha \frac{\omega^2{}_{p\alpha}\zeta_\alpha{}^2}{V_\alpha{}^2}\right]^2 + \frac{\pi}{k^4}\left[\sum_\alpha \frac{\omega^2{}_{p\alpha}\zeta_\alpha}{V_\alpha{}^2}\right]^2 \tag{39}$$

where

$$\omega^2{}_{p\alpha} = \frac{4\pi n_\alpha e^2}{m_\alpha}$$

and is a function of x inside the shock. The density varies in a thin layer of the shock, say of thickness δ, from its upstream value to its downstream value. A simple assumption useful for numerical computations is that the density varies linearly with distance inside the shock. We choose our units such that $r_0 = 1$. In this case one can write

$$n_1(\tilde{x}) = N_1; \quad n_2(\tilde{x}) = 0$$

in the range $(-\infty, 0)$

$$n_1(\tilde{x}) = N_1\left(1 - \frac{\tilde{x}}{\delta}\right)$$

and

$$n_2(\tilde{x}) = N_2 \frac{\tilde{x}}{\delta}$$

in $(0, \delta)$

$$n_1(\tilde{x}) = 0; \quad n_2(\tilde{x}) = N_2$$

in the range (δ, ∞). (40)

The kinetic equation for the ions is now given by

$$F = F_0 + F_1 = -\lambda \int_0^{\tilde{x}} \frac{1}{D^2 k_x^3 \tilde{v}_x} \left(D \frac{\partial N}{\partial \tilde{v}_x} - N \frac{\partial D}{\partial \tilde{v}_x} \right) d\tilde{x} dk_x \tag{41}$$

where N and D are given respectively by (4) and (5) of Appendix I. The integration with respect to k_x can be performed very easily and the evaluation is given in Appendix I. One finds that the distribution function within the shock front is given by

$$F = F_0 + F_1 = -\lambda \int_0^{\tilde{x}} (I_1 - I_2) \frac{d\tilde{x}}{\tilde{v}_x} \tag{42}$$

where

$$I_1 = \frac{1}{2(\beta_1 - \alpha_1^2)^{\frac{1}{2}}} \frac{\partial N}{\partial \tilde{v}_x} \tan^{-1} \frac{k_x^2 + \alpha_1}{\sqrt{\beta_1 - \alpha_1^2}} \tag{43}$$

$$I_2 = \sqrt{2} N v_0 \left(\frac{\omega^2{}_{p1} \zeta_1}{V_1^3} + \frac{\omega^2{}_{p2} \zeta_2}{V_2^3} \right) \left[\frac{1}{k_x^4 + 2\alpha_1 k_x^2 + \beta_1} + \frac{(\alpha_1 + \gamma_1)}{(\beta_1 - \alpha_1^2)^{3/2}} \right.$$

$$\left. \times \tan^{-1} \frac{(k_x^2 + \alpha_1)}{\sqrt{\beta_1 - \alpha_1^2}} + \frac{(\alpha_1 + \gamma_1)}{(\beta_1 - \alpha_1^2)} \frac{(k_x^2 + \alpha_1)}{(k_x^4 + 2\alpha_1 k_x^2 + \beta_1)} \right]. \tag{44}$$

N, D, α_1 and β_1 are given by the equations (4), (5), (7) and (8) respectively of Appendix I. The above integral gives the ion distribution function and can be evaluated numerically. The integral form is however very convenient for studying several properties of the plasma within the shock which involve a knowledge of the distribution function such as the charge separation and the electric field at the shock boundary and the coefficient of viscosity arising from the turbulent wave-particle interactions.

6. Viscosity

When the distribution function is Maxwellian, pressure is a scalar for an ordinary gas, but when collisions are taken into account, pressure has generally six components and is a tensor. By treating the collision term in the

Boltzmann equation as a perturbation and expanding the distribution function as a series by the Chapman-Enskog method, the dissipative terms in the pressure tensor involving the coefficient of viscosity make their appearance. The coefficient of viscosity is obviously dependent on the collision frequency. Another context wherein an anisotropic pressure is encountered is a collisionless plasma embedded in a strong magnetic field. Here the finite Larmor Radius corrections to the distribution function introduce a new kind of viscosity known as the gyroviscosity, but this is not dissipative. In the present problem, since the flow across the shock is reduced from supersonic to subsonic velocities, there should be a dissipative mechanism which converts the ordered kinetic energy of flow into thermal energy, and a different kind of viscosity should be anticipated which depends purely on wave-particle interactions. We shall calculate in this section the corrections to the pressure arising from the wave-particle interactions.

The components of the pressure tensor are now given by

$$
\begin{aligned}
P_{xx} &= M \int (\tilde{v}_x - N_1)^2 f\, d\tilde{v} \\
P_{xy} &= M \int (\tilde{v}_x - N_1)\, \tilde{v}_y\, f\, d\tilde{v} \\
P_{xz} &= M \int (\tilde{v}_x - N_1)\, \tilde{v}_z\, f\, d\tilde{v} \\
P_{yy} &= P_{zz} = M \int \tilde{v}_y^2\, f\, dv = M \int \tilde{}_z^2\, f\, d\tilde{v} \\
P_{yz} &= M \int \tilde{v}_y \tilde{v}_z\, f\, d\tilde{v}.
\end{aligned}
\tag{45}
$$

In the shock layer, the distribution function is given by

$$f = f_M + f_1 \tag{46}$$

where f_1 is the correction arising from the wave-particle interaction. As is well known, the contribution of the Maxwellian term in the distribution function to the pressure tensor leads to the scalar term and we shall therefore calculate the contribution arising from the correction term f_1, to the various components of the pressure.

Let us first consider the non-diagonal elements of the pressure tensor. We have now

$$P_{xy} = -\lambda M v_0^2 \int (\tilde{v}_x - N_1)\, \tilde{v}_y \left[P \frac{\partial N}{\partial \tilde{v}_x} - QN\right] d\tilde{v}_x d\tilde{v}_y d\tilde{v}_z.$$

The term involving $\partial N/\partial \tilde{v}_x$ can be integrated by parts with respect to $\tilde{v}_x$ and the integrand vanishes at the limit for $\tilde{v}_x$. Hence

$$
\begin{aligned}
P_{xy} = -\lambda M v_0^2 \int [P + Q(\tilde{v}_x - N_1)]\, \tilde{v}_y \Big[\Big\{ \tilde{v}_x \Big(\frac{1}{a_1^2} - \frac{1}{a_2^2} \Big) \\
- \Big(\frac{N_1}{a_1^2} - \frac{N_2}{a_1^2} \Big) \Big\} G_1(\tilde{v})\, g_2(\tilde{v}_x) + \Big\{ \tilde{v}_x \Big(\frac{1}{a_2^2} - \frac{1}{a_1^2} \Big) \\
- \Big(\frac{N_2}{a_2^2} - \frac{N_1}{a_1^2} \Big) \Big\} G_2(\tilde{v})\, g_1(\tilde{v}_x) \Big]\, d\tilde{v}_x d\tilde{v}_y d\tilde{v}_z. \qquad (47)
\end{aligned}
$$

The integration with respect to $\tilde{v}_y$ can be carried out immediately and one can see that it leads to zero because of the symmetry in the range $(-\infty, \infty)$. Hence

$$P_{xy} = 0. \qquad (48)$$

By the same argument, one can show that

$$P_{xz} = P_{yz} = 0. \qquad (49)$$

We shall next evaluate the diagonal elements of the pressure tensor. We have

$$P_{xx} = p + P_{xx}^{(1)}$$

where

$$P_{xx}^{(1)} = v_0^2 \int (\tilde{v}_x - N_1)^2 \, F d\tilde{v}. \qquad (50)$$

This integral has been evaluated in Appendix I and one finds that

$$P_{xx}^{(1)} = 0. \qquad (51\ a)$$

Thus, we find

$$P_{xx} = p. \qquad (51\ b)$$

By symmetry, we have

$$
\begin{aligned}
P_{yy} &= P_{zz} \\
&= p + P_{yy}^{(1)}
\end{aligned}
$$

where

$$P_{yy}^{(1)} = v_0^2 \int \tilde{v}_y^2 F d\tilde{v}$$

$$= -v_0^2 \lambda \int_0^{\tilde{x}} d\tilde{x} \int \tilde{v}_y^2 (I_1 - I_2) \frac{d\tilde{v}_x}{\tilde{v}_x} d\tilde{v}_y d\tilde{v}_z. \tag{52}$$

This integral has been evaluated in Appendix II and does not vanish. Thus we have

$$P_{zz} = P_{yy}$$

$$= p + \frac{M\lambda v_0^2}{2\pi} N_1 N_2 \int_0^{\tilde{x}} d\tilde{x} \left[\frac{n_1(\tilde{x})\, n_2(\tilde{x})}{N_1 N_2}\right] \left(\frac{a_1}{a_2} - \frac{a_2}{a_1}\right)$$

$$\times \left\{ \left(\frac{1}{a_1^2} - \frac{1}{a_2^2}\right)(Q_0 + P_0' - P_{-1}) \right.$$

$$\left. - \left(\frac{N_1}{a_1^2} - \frac{N_2}{a_2^2}\right)(Q_{-1} + P_{-1}' - P_{-2}) \right\} \tag{53}$$

where P_n, P_n' and Q_n are defined in Appendix II. Thus we find that the components of the pressure perpendicular to the direction of fluid flow is given by

$$P_{yy} = p + \nu f(\tilde{x}) \tag{54}$$

where

$$\nu = \frac{M\lambda v_0^2 N_1 N_2}{2\pi}. \tag{55}$$

Now

$$\frac{1}{a_1^2} - \frac{1}{a_2^2} = \Delta\left(\frac{1}{a_1^2}\right) \sim \delta \frac{d}{dx}\left(\frac{v_0^2}{V_1^2}\right) \tag{56}$$

and

$$\frac{N_1}{a_1^2} - \frac{N_2}{a_2^2} = \Delta\left(\frac{N_1}{a_1^2}\right) \sim \delta \frac{d}{dx}\left(\frac{v_1 v_0}{V_1^2}\right) \tag{57}$$

where $\Delta[\ldots]$ denotes the variation of the quantity inside the bracket across the shock. In our units, δ is of the order of magnitude of unity. To find out the order of magnitude of the terms involving the viscosity ν, it will be useful to consider the integrand in (53) or $dP_{yy}/d\tilde{x}$ which is given by

$$\frac{dP_{yy}}{d\tilde{x}} \quad \frac{\lambda M v_0^2 \delta N_1 N_2}{2\pi} \left(\frac{n_1(\tilde{x})\, n_2(\tilde{x})}{N_1 N_2} \right) \begin{pmatrix} V_1 & V_2 \\ V_2 & V_1 \end{pmatrix}$$

$$\left[(Q_0 \quad P_0' \quad P_{\ 1}) \frac{d}{d\tilde{x}} \left(\frac{v_0^2}{V_1^2} \right) \right.$$

$$\left. \quad (Q_{-1} \quad P'_{-1} - P_{\ 2}) \frac{d}{d\tilde{x}} \left(\frac{v_1 v_0}{V_1^2} \right) \right]. \tag{58}$$

The two terms involving the derivative of $\tilde{x}$ are proportional to $dT/d\tilde{x}$ and $dM/d\tilde{x}$, where T is the temperature and M is the Mach number of the flow, and thus the expression resembles the conventional expression for the pressure tensor involving the coefficient of thermal conductivity and viscosity. We hope to give numerical results for the ion distribution function, the shock thickness as well as the pressure tensor in a later paper. It is clear that wave-particle interaction introduces a new type of viscosity and could provide a dissipative mechanism that transforms the kinetic energy of the streaming plasma into the thermal energy of the hot ionised gas behind the shock.

7. References

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APPENDIX I

The integral (42) for the distribution function may be written as

$$F = -\lambda \int^{\tilde{x}} (I_1 - I_2) \frac{d\tilde{x}}{\tilde{v}_x} \tag{1}$$

where

$$I_1 = \int \frac{1}{D} \frac{\partial N}{\partial \tilde{v}_x} \frac{dk_x}{k_x^3} \tag{2}$$

and

$$I_2 = \int \frac{N}{D^2} \frac{\partial D}{\partial \tilde{v}_x} \frac{dk_x}{k_x^3} \tag{3}$$

where

$$N = \left\{\tilde{v}_x \left(\frac{1}{a_1^2} - \frac{1}{a_2^2}\right) - \left(\frac{N_1}{a_1^2} - \frac{N_2}{a_2^2}\right)\right\} G_1(\tilde{v})\, g_2(\tilde{v}_x)$$

$$+ \left\{\tilde{v}_x \left(\frac{1}{a_2^2} - \frac{1}{a_1^2}\right) - \left(\frac{N_2}{a_2^2} - \frac{N_1}{a_1^2}\right)\right\} G_2(\tilde{v})\, g_1(\tilde{v}_x) \tag{4}$$

and

$$D = \left\{1 + \frac{\omega^2_{p1}}{k^2 V_1^2} + \frac{\omega^2_{p2}}{k^2 V_2^2} - \frac{2}{k^2}\left(\frac{\omega^2_{p1}\zeta_1^2}{V_1^2} + \frac{\omega^2_{p2}\zeta_2^2}{V_2^2}\right)\right\}^2$$

$$+ \frac{\pi}{k^4}\left\{\frac{\omega^2_{p1}\zeta_1}{V_1^2} + \frac{\omega^2_{p2}\zeta_2}{V_2^2}\right\}^2. \tag{5}$$

We shall first evaluate I_1.

Since $\partial N / \partial \tilde{v}_x$ is independent of k_x, we have

$$I_1 = \frac{\partial N}{\partial \tilde{v}_x} \int \frac{1}{k_x^3 D}\, dk_x = \frac{\partial N}{\partial \tilde{v}_x} \int \frac{k_x dk_x}{k_x^4 + 2\alpha_1 k_x^2 + \beta_1} \tag{6}$$

where

$$\alpha_1 = \left\{\frac{\omega^2_{p1}}{V_1^2} + \frac{\omega^2_{p2}}{V_2^2} - 2\left(\frac{\omega^2_{p1}\zeta_1^2}{V_1^2} + \frac{\omega^2_{p2}\zeta_2^2}{V_2^2}\right)\right\} \tag{7}$$

$$\beta_1 = \left\{\frac{\omega^2_{p1}}{V_1^2} + \frac{\omega^2_{p2}}{V_2^2} - 2\left(\frac{\omega^2_{p1}\zeta_1^2}{V_1^2} + \frac{\omega^2_{p2}\zeta_2^2}{V_2^2}\right)\right\}^2$$

$$+ \pi\left\{\frac{\omega^2_{p1}\zeta_1}{V_1^2} + \frac{\omega^2_{p2}\zeta_2}{V_2^2}\right\}^2. \tag{8}$$

Obviously,

$$\beta_1 - \alpha_1^2 > 0.$$

The lower limit for k_x is zero in the above integral. This can be integrated immediately as it is in the standard form. One gets

$$I_1 = \frac{1}{2}\frac{\partial N}{\partial \tilde{v}_x}\frac{1}{(\beta_1 - \alpha_1^2)^{\frac{1}{2}}}\tan^{-1}\frac{k_x^2 + \alpha_1}{\sqrt{\beta_1 - \alpha_1^2}}. \tag{9}$$

Next

$$I_2 = N\int \frac{\left(\frac{\partial D}{\partial \tilde{v}_x}\right)}{D^2 k_x^3}\,dk_x$$

$$= -4\sqrt{2}\,Nv_0\left(\frac{\omega^2_{p1}\zeta_1}{V_1^3} + \frac{\omega^2_{p2}\zeta_2}{V_2^3}\right)\int \frac{dk_x k_x (k_x^2 - \gamma_1)}{(k_x^4 + 2\alpha_1 k_x^2 + \beta_1)^2} \tag{10}$$

where

$$\gamma_1 = -\alpha_1 + \frac{\pi}{4}\frac{\left(\frac{\omega^2_{p1}\zeta_1}{V_1^2} + \frac{\omega^2_{p2}\zeta_2}{V_2^2}\right)\left(\frac{\omega^2_{p1}}{V_1^3} + \frac{\omega^2_{p2}}{V_2^3}\right)}{\left(\frac{\omega^2_{p1}\zeta_1}{V_1^3} + \frac{\omega^2_{p2}\zeta_2}{V_2^3}\right)}. \tag{11}$$

After simplification, one finds that

$$I_2 = -\sqrt{2}\,Nv_0\left(\frac{\omega^2_{p1}\zeta_1}{V_1^3} + \frac{\omega^2_{p2}\zeta_2}{V_2^3}\right)\int \frac{dt\,(2t - 2\gamma_1)}{(t^2 + 2\alpha_1 t + \beta_1)^2} \tag{12}$$

where

$$t = k_x^2.$$

The above integral is again in the standard form and on integrating it, we find that

$$I_2 = \sqrt{2}\,Nv_0\left(\frac{\omega^2_{p1}\zeta_1}{V_1^3} + \frac{\omega^2_{p2}\zeta_2}{V_2^3}\right)\Bigg[\frac{1}{(k_x^4 + 2\alpha_1 k_x^2 + \beta_1)}$$

$$+ \frac{(\alpha_1 + \gamma_1)}{(\beta_1 - \alpha_1^2)^{3/2}}\tan^{-1}\frac{k_x^2 + \alpha_1}{\sqrt{\beta_1 - \alpha_1^2}}$$

$$+ \frac{(\alpha_1 + \gamma_1)(k_x^2 + \alpha_1)}{(\beta_1 - \alpha_1^2)(k_x^4 \pm 2\alpha_1 k_x^2 + \beta_1)}\Bigg]. \tag{13}$$

APPENDIX II

We have

$$P_{xx} = p + P_{xx}^{(1)} \tag{1}$$

where

$$P_{xx}^{(1)} = v_0^2 \int (\tilde{v}_x - N_1)^2 F d\tilde{v}$$

$$= -\lambda v_0^2 \int^{\tilde{x}} d\tilde{x} \int \frac{(\tilde{v}_x - N_1)^2}{\tilde{v}_x} (I_1 - I_2)\, d\tilde{v}. \tag{2}$$

Here

$$I_1 = P \frac{\partial N}{\partial \tilde{v}_x}\ ; \quad I_2 = QN \tag{3 a}$$

where

$$P = \frac{1}{2(\beta_1 - a_1^2)^{\frac{1}{2}}} \tan^{-1} \frac{(k_x^2 + a_1)}{\sqrt{\beta_1 - a_1^2}} \tag{3 b}$$

$$Q = \sqrt{2} v_0 \left(\frac{\omega^2 p_1 \zeta_1}{V_1^3} + \frac{\omega^2 p_2 \zeta_2}{V_2^3}\right) \left[\frac{1}{(k_x^4 + 2a_1 k_x^2 + \beta_1)} + \frac{(a_1 + \gamma_1)}{(\beta_1 - a_1^2)^{3/2}} \tan^{-1} \frac{(k_x^2 + a_1)}{\sqrt{\beta_1 - a_1^2}} + \frac{(a_1 + \gamma_1)(k_x^2 + a_1)}{(\beta_1 - a_1^2)(k_x^4 + 2a_1 k_x^2 + \beta_1)}\right]. \tag{3 c}$$

Substituting these in (2), we get

$$P_{xx}^{(1)} = -\lambda v_0^2 \int^{\tilde{x}} d\tilde{x} \int \frac{(\tilde{v}_x - N_1)^2}{\tilde{v}_x} \left(P \frac{\partial N}{\partial \tilde{v}_x} - QN\right) d\tilde{v} \tag{4}$$

where N is given in Appendix I.

Integrating the term involving $P\, \partial N/\partial \tilde{v}_x$ by parts in the above integral and noting that the integrand vanishes at the limits of integration for $\tilde{v}_x$ we get

$$P_{xx}^{(1)} = \lambda v_0^2 \int^{\tilde{x}} d\tilde{x} \int N \left[\left\{Q + P\left(1 - \frac{N_1^2}{\tilde{v}_x^2}\right)\right\} + \left(\tilde{v}_x - 2N_1 + \frac{N_1^2}{\tilde{v}_x}\right) \frac{\partial P}{\partial \tilde{v}_x}\right] d\tilde{v}_x d\tilde{v}_y d\tilde{v}_z$$

$$= T_1 + T_2 \text{ (say)} \tag{5}$$

where

$$T_1 = \lambda v_0^2 \int^{\tilde{x}} d\tilde{x} \int \left\{Q - P\left(1 - \frac{N_1^2}{\tilde{v}_x^2}\right)\right\} \left[\left\{\tilde{v}_x\left(\frac{1}{a_1^2} - \frac{1}{a_2^2}\right) - \left(\frac{N_1}{a_1^2} - \frac{N_2}{a_2^2}\right)\right\} G_1(\tilde{v})\, g_2(\tilde{v}_x) + a \text{ similar term}\right] d\tilde{v}_x d\tilde{v}_y d\tilde{v}_z \tag{6}$$

and T_2 is the integral with the second term involving $\partial P/\partial \tilde{v}_x$ in the integrand. Integrating with respect to the variables $\tilde{v}_y$ and $\tilde{v}_z$ we get

$$T_1 = \frac{\lambda v_0^2}{2\pi} \int^{\tilde{x}} d\tilde{x} \frac{n_1(\tilde{x})\, n_2(\tilde{x})}{a_1 a_2} \int d\tilde{v}_x \left[Q + P\left(1 - \frac{N_1^2}{\tilde{v}_x^2}\right)\right] \left\{\tilde{v}_x\left(\frac{1}{a_1^2} - \frac{1}{a_2^2}\right) - \left(\frac{N_1}{a_1^2} - \frac{N_2}{a_2^2}\right) + \tilde{v}_x\left(\frac{1}{a_2^2} - \frac{1}{a_1^2}\right) - \left(\frac{N_2}{a_2^2} - \frac{N_1}{a_1^2}\right)\right\} e^{-\frac{1}{2}(a\tilde{v}_x^2 - 2b\tilde{v}_x + c)} \tag{7}$$

where

$$a = \frac{1}{a_1^2} + \frac{1}{a_2^2}; \quad b = \frac{N_1}{a_1^2} + \frac{N_2}{a_2^2}; \quad c = \frac{N_1^2}{a_1^2} + \frac{N_2^2}{a_2^2}. \tag{8}$$

Obviously $T_1 = 0$, because the terms in the curly bracket cancel in pairs. By the same argument, we have

$$T_2 = 0$$

thus

$$P_{xx}^{(1)} = 0. \tag{9}$$

Next consider

$$\begin{aligned} P_{yy} &= p + M\tilde{v}_0^2 \int \tilde{v}_y^2 F d\tilde{v} \\ &= p + Mv_0^2 \lambda \int d\tilde{x} \int \tilde{v}_y^2 (I_1 - I_2) \frac{d\tilde{v}}{\tilde{v}_x} \\ &= p + P_{yy}^{(1)} \end{aligned} \tag{10}$$

where

$$P_{yy}^{(1)} = -\lambda v_0^2 M \int d\tilde{x} \int \frac{\tilde{v}_y^2}{\tilde{v}_x}\left(P \frac{\partial N}{\partial \tilde{v}_x} - QN\right) d\tilde{v}.$$

Integrating the first term by parts, we get

$$P_{yy}^{(1)} = M\lambda v_0^2 \int d\tilde{x} \int \frac{\tilde{v}_y^2 N}{\tilde{v}_x}\left[Q + \frac{\partial P}{\partial \tilde{v}_x} - \frac{P}{\tilde{v}_x}\right] d\tilde{v}_x d\tilde{v}_y d\tilde{v}_z. \quad (11)$$

Substituting for N and integrating with respect to $\tilde{v}_y$ and $\tilde{v}_z$ we get

$$P_{yy}^{(1)} = \frac{M\lambda v_0^2}{2\pi} \int d\tilde{x}\, [n_1(\tilde{x})\, n_2(\tilde{x})] \left(\frac{a_1}{a_2} - \frac{a_2}{a_1}\right)$$
$$\times \int \frac{d\tilde{v}_x}{\tilde{v}_x}\left(Q + \frac{\partial P}{\partial \tilde{v}_x} - \frac{P}{\tilde{v}_x}\right)\left\{\tilde{v}_x\left(\frac{1}{a_1^2} - \frac{1}{a_2^2}\right)\right.$$
$$\left. - \left(\frac{N_1}{a_1^2} - \frac{N_2}{a_2^2}\right)\right\} e^{-\frac{1}{2}(a\tilde{v}_x^2 - 2b\tilde{v}_x + c)} \quad (12)$$

where a, b, and c have already been defined.

Let

$$P_n = \int \tilde{v}_x^n P e^{-\frac{1}{2}(a\tilde{v}_x^2 - 2b\tilde{v}_x + c)}\, d\tilde{v}_x$$
$$Q_n = \int \tilde{v}_x^n Q e^{-\frac{1}{2}(a\tilde{v}_x^2 - 2b\tilde{v}_x + c)}\, d\tilde{v}_x$$
$$P_n' = \int \tilde{v}_x^n \frac{\partial P}{\partial \tilde{v}_x} e^{-\frac{1}{2}(a\tilde{v}_x^2 - 2b\tilde{v}_x + c)}\, d\tilde{v}_x. \quad (13)$$

These integrals have a singularity at $\tilde{v}_x = 0$ for negative values of n. But from equation (3) it can be seen that $S_{ef} = 0$ for $\tilde{v}_x = 0$ and a special treatment is necessary to evaluate f at this point. For negative values of n, the integrals should be evaluated excluding a small interval containing the origin. But from the physical point of view, the case $\tilde{v}_x = 0$ is not important as it occurs with negligible probability. Substituting now (13) in (12), we get

$$P_{yy}^{(1)} = \frac{M\lambda v_0^2}{2\pi} \int d\tilde{x}\, n_1(\tilde{x})\, n_2(\tilde{x}) \left(\frac{a_1}{a_2} - \frac{a_2}{a_1}\right)$$
$$\times \left[\left(\frac{1}{a_1^2} - \frac{1}{a_2^2}\right)(Q_0 + P_0' - P_{-1}) - \left(\frac{N_1}{a_1^2} - \frac{N_2}{a_2^2}\right)\right.$$
$$\left. \times (Q_{-1} + P'_{-1} - P_{-2})\right]. \quad (14)$$