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On the ratio of two blocks of consecutive integers

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Abstract. Under certain assumptions, it is shown that eq. (2) has only finitely many solutions in integers $x\geqslant 0$, $y\geqslant 0$, $k\geqslant 2$, $l\geqslant 0$. In particular, it is proved that (2) with $a=b=1$, $l=k$ implies that $x=7$, $y=0$, $k=3$.

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1.

Let a and b be relatively prime positive integers. Erdös [5] conjectured that there are only finitely many integers $x\geqslant 0$, $y\geqslant 0$, $k\geqslant 2$, $l\geqslant 0$ with $k+l\geqslant 3$ satisfying

$$x\geqslant y+l+k \tag{1}$$

and

$$a(x+1)\cdots(x+k)=b(y+1)\cdots(y+k+l). \tag{2}$$

By letters x, y, k and l, we shall always understand that $x\geqslant 0$, $y\geqslant 0$, $k\geqslant 2$, $l\geqslant 0$ with $k+l\geqslant 3$ are integers satisfying (2). For an integer v with $|v|>1$, we denote by $P(v)$ and $\omega(v)$, respectively, the greatest prime factor of v and the number of distinct prime factors of v. Further, we write $P(0)=P(1)=P(-1)=1$ and $\omega(0)=\omega(1)=\omega(-1)=0$.

In this and the next paragraph, we state some of the earlier results on (2). Mordell [7] proved that (2) with $a=b=1$, $k=2$, $l=1$ implies that either $x=1$, $y=0$ or $x=13$, $y=4$. Avanesov [1] confirmed a conjecture of Sierpinski by proving that $x=0$, $y=0$; $x=3$, $y=2$; $x=14$, $y=7$; $x=54$, $y=19$ and $x=118$, $y=33$ are the only solutions of (2) with $a=3$, $b=1$, $k=2$, $l=1$. Tzanakis and de Weger [12] determined all the solutions of (2) with $a=1$, $b=2$, $k=2$, $l=1$. Boyd and Kisilevsky [3] showed that $x=1$, $y=0$; $x=3$, $y=1$ and $x=54$, $y=18$ are the only solutions of (2) with $a=b=1$, $k=3$, $l=1$. Cohn [4] proved that (2) with $a=1$, $b=2$, $k=4$, $l=0$ is satisfied only if $x=4$, $y=3$. Further, Ponnudurai [8] showed that $x=2$, $y=1$ and $x=6$, $y=4$ are the only solutions of (2) with $a=1$, $b=3$, $k=4$, $l=0$. Shorey [10] showed that (2) with $l=0$ and (1) implies that either k is bounded by an effectively computable number depending only on a, b or $k=[\alpha+1]$ where

$$\alpha=\log\left(\frac{b}{a}\right)\Big/\log\left(\frac{x}{y}\right).$$

Further, it is proved in [10] that (2) with $l=0$ and (1) implies that k is bounded by an effectively computable number depending only on a, b, $P(x)$, $P(y)$ and also,

$$\log x \leqslant C_1 k \tag{3}$$

where C_1 is an effectively computable number depending only on a, b, $P(y)$ and $P(x-y)$. On the other hand, we see from Cramer's conjecture on distance between consecutive primes that (2) and (1) imply that

$$(\log x)^2 > C_2 k \tag{4}$$

where $C_2 > 0$ is an absolute constant. In this paper, we shall extend these results to a more general equation (2). For a given k and l, we refer to a theorem of Siegel [11] to observe that (2) has only finitely many solutions in x and y provided that the curve represented by (2) is irreducible over the field of complex numbers and has positive genus.

It has not been possible to confirm the conjecture of Erdös, stated above, even when y is bounded. Erdös [5] considered a particular case of (2) corresponding to $y=0$, namely,

$$(X+1)\cdots(X+M)=N! \tag{5}$$

where $X \geqslant 2$, $M \geqslant 2$ and $N \geqslant 2$ are integers. Erdös[5] conjectured that $8.9.10=6!$ is the only solution of (5). Erdös[5] proved that for $\varepsilon > 0$ there exists N_0 depending only on ε such that (5) with $N \geqslant N_0$ implies that

$$X \geqslant (2-\varepsilon)^N. \tag{6}$$

We re-write (5) with $M=2$ as

$$(2X+3)^2 - 1 = 4N!$$

which reminds us of the open problem on squares of the form $N!+1$.

By fixing any three of the four variables x, y, k and l in (2), the fourth one is determined uniquely, if it exists. For given x, y and l, we start with the following result that determines the exact value of k if it exists.

Theorem 1. *Let* $y' = \max(y, 1)$ *and*

$$\beta = \left(\log\left(\frac{b}{a}\right) + l \log y'\right) \Big/ \log\left(\frac{x}{y'}\right). \tag{7}$$

There exists an effectively computable number C_3 *depending only on a and b such that* (2) *with* (1), $k \geqslant C_3$ *and*

$$y > (k+l)^2 \text{ if } 12\log(l+1) \geqslant k \tag{8}$$

implies that

$$0 < k - \beta < 1. \tag{9}$$

We observe that a restriction of the type (8) is necessary for obtaining (9). For

$0<\phi<(\log 2)/12$, $l=[e^{\phi k}]$, $0<y\leqslant(k+l)^2$ and k exceeding a sufficiently large number depending only on a, b and ϕ, we see from (7), (15) and (23) that $\beta<C_4k$ where $0<C_4<1$ is a number depending only on a, b and ϕ.

Next, we turn to a more general situation than considered in Theorem 1. For given x and y, we denote by $N(x,y)$ the number of pairs (k,l) satisfying (2) and (1). For given x, y and l, there is at most one k satisfying (2). Therefore, we see from Lemma 1 that

$$N(x,y)\leqslant C_5\log x \tag{10}$$

where C_5 is an effectively computable number depending only on a and b. In the following result, we sharpen (10) whenever y is somewhat smaller than x.

Theorem 2.(a) *There exist effectively computable numbers C_6 and C_7 depending only on a and b such that for every x and y with*

$$\log y\leqslant C_6\frac{\log x}{\log\log x}, \tag{11}$$

we have

$$N(x,y)\leqslant C_7. \tag{12}$$

(*b*) *Let $\varepsilon>0$ and $y<(1-\varepsilon)x$. Then*

$$N(x,y)\leqslant C_8\log\log x$$

where C_8 is an effectively computable number depending only on a, b and ε.

A pair (k,l) in Theorem 2 may depend on x and y. Now, for $\varepsilon>0$ and $y\leqslant x^{1-\varepsilon}$, we show that $\max(k,l)$ is bounded by a number depending only on ε, a, b and $P(x)$. See Theorem 3(b) which finds an application in the proof of Theorem 5. Furthermore, we give lower bounds for $P(x)$ and $P(x-y)$ whenever y is smaller than some power of x and we apply these estimates in the proof of Corollary 1.

Theorem 3. *Suppose that* (2) *with* (1) *is satisfied. Let* $\varepsilon>0$.

(a) *If*

$$x-y\geqslant\min\left(\varepsilon x,\frac{x((\log k)(\log\log k))^2}{k}\right), \tag{13}$$

then l is bounded by an effectively computable number depending only on ε, a b and $P(x)$.
(b) *If $y\leqslant x^{1-\varepsilon}$, then* $\max(k,l)$ *is bounded by an effectively computable number depending only on ε, a, b and $P(x)$.*
(c) *There exist effectively computable numbers $\delta>0$ and C_9 depending only on a and b such that the inequalities*

$$y\leqslant x^{\delta}\quad\text{and }P(x)\leqslant l+k$$

imply that $\max(x,y,k,l)\leqslant C_9$.
(d) *There exist effectively computable numbers $\delta_1>0$ and C_{10} depending only on a, b and $P(x-y)$ such that if $y\leqslant x^{\delta_1}$, then* $\max(x,y,k,l)\leqslant C_{10}$.
(e) *There is C_{11} depending only on a, b and y such that $P(x-y)\leqslant l$ implies that* $\max(x,k,l)\leqslant C_{11}$.

The proof of Theorem 3(a) depends on the theory of linear forms in logarithms. We combine Theorem 3(c), (d) with Lemma 8. We derive that (2) with (1) and $P(x)\leqslant l+k$ implies that l/k is bounded by an effectively computable number depending only on a and b. Also, we see that (2) and (1) imply that l/k is bounded by an effectively computable number depending only on a, b and $P(x-y)$. Furthermore, if k is fixed, we may apply the theorem of Siegel, stated above, to finitely many pairs (k,l) given by the above two assertions to derive that (2) has only finitely many solutions in x and y under certain assumptions already mentioned. Next, we consider (2) with $l/k=1$ and $a=b=1$ i.e.

$$(x+1)\cdots(x+k)=(y+1)\cdots(y+2k). \tag{14}$$

MacLeod and Barrodale [6] showed that (14) with $k\in\{2,4,5\}$ has no solution in x,y and (14) with $k=3$ implies that $x=7$, $y=0$. Further, MacLeod and Barrodale [6] proved that for a given k, there are only finitely many pairs (depending on k) x, y satisfying (14). We prove

Theorem 4. *The equation* (14) *has only one solution in integers* $x\geqslant 0$, $y\geqslant 0$, $k\geqslant 2$ *and it is given by* $x=7$, $y=0$, $k=3$.

In Theorems 2, 3 and 4, we see that y is somewhat smaller than x. In the next result, we replace this by the assumption that $P(y)$ is bounded. We combine Theorem 3(b) with the theory of linear forms in logarithms to obtain the following result.

Theorem 5. *Equation* (2) *with* (1) *implies that* $\max(x,y,k,l)$ *is bounded by an effectively computable number depending only on* a, b, $P(x)$ *and* $P(y)$.

We may derive from Theorem 3(c) that (5) implies $P(X)>N$ whenever X exceeds a sufficiently large effectively computable absolute constant. In fact, we apply Theorem 3(e) to obtain a more general result on (5).

COROLLARY 1.

Let $B\geqslant 0$ *be an integer. There exists an effectively computable number* C_{12} *depending only on* B *such that* (5) *with* $N>C_{12}$ *implies that* $P(X-B)>N$.

For the proofs of our Theorems, we prove certain estimates that are of independent interest. For example, we show unconditionally that (4) is valid whenever $y<(1-\varepsilon)x$ for $\varepsilon>0$. We formulate these estimates as the following theorem.

Theorem 6. *Suppose that* (2) *with* (1) *is satisfied. Then*

(a) *There exists an effectively computable number* $C_{13}>0$ *depending only on* a *and* b *such that*

$$x\geqslant C_{13}k^3(\log k)^{-4}. \tag{15}$$

(b) *Let* $\varepsilon>0$ *and* $y<(1-\varepsilon)x$. *Then*

$$(\log x)^2>C_{14}k \tag{16}$$

where $C_{14}>0$ *is an effectively computable number depending only on* ε, a *and* b.

(c) *There exist effectively computable numbers* C_{15} *and* $C_{16}>0$ *depending only on a and b such that if* $k \geqslant C_{15}$ *and* $y \leqslant (k+l)^3(\log(k+l))^{-5}$, *we have*

$$\log\log x > C_{16}k.$$

(d) *There exists an effectively computable number* $C_{17}>0$ *depending only on a and b such that*

$$x-y \geqslant C_{17}x^{2/3}.$$

2.

This section contains a proof of Theorem 6 and lemmas for our Theorems 1, 2, 3, 5. Throughout this section, we suppose that (1) and (2) are satisfied and we shall use this assumption without reference. We put

$$U_i = ax^i - by^{i+l} \quad \text{for } 0 \leqslant i \leqslant k \tag{17}$$

and

$$f(x,y) = x-(b/a)^{1/k}y^{1+(l/k)}. \tag{18}$$

Let

$$F(z) = (z+1)\cdots(z+k) = z^k + A_1z^{k-1} + \cdots + A_k \tag{19}$$

and

$$G(z) = (z+1)\cdots(z+l) = z^l + A'_1z^{l-1} + \cdots + A'_l. \tag{20}$$

Then, we refer to [6, p. 256] to observe that

$$0 < A_i \leqslant (k+1)^{2i}/2^i i! \quad \text{for } 1 \leqslant i \leqslant k \tag{21}$$

and

$$0 < A'_j \leqslant (l+1)^{2j}/2^j j! \quad \text{for } 1 \leqslant j \leqslant l. \tag{22}$$

We start with the following result that provides an upper bound for l.

Lemma 1. *There exists an effectively computable absolute constant c such that*

$$l \leqslant c\log(a+1) + (2\log x)/\log 2. \tag{23}$$

Proof. We re-write (2) as

$$a\frac{(x+1)\cdots(x+k)}{k!} = b\frac{(y+1)\cdots(y+k+l)}{(k+l)!}\frac{(k+l)!}{k!}. \tag{24}$$

We observe that

$$\operatorname{ord}_2(\text{R.H.S. of (24)}) \geqslant (l-1)/2. \tag{25}$$

On the other hand,

$$\operatorname{ord}_2(\text{L.H.S. of (24)}) \leqslant \operatorname{ord}_2(a) + \max_{1\leqslant i\leqslant k}\operatorname{ord}_2(x+i) \leqslant (\log(a(x+k)))/\log 2. \tag{26}$$

Finally, we combine (24), (25) and (26) to derive (23). □

We give estimates for $f(x,y)$ in the next three lemmas.

Lemma 2.

$$f(x,y)>0. \tag{27}$$

Proof. We suppose that $f(x,y)\leqslant 0$. Then, we see from (17) and (18) that $U_k\leqslant 0$, which, since $x>y$, implies that

$$U_i\leqslant 0 \quad \text{for } 0\leqslant i\leqslant k. \tag{28}$$

Now, we derive from (2), (19), (20), (17) and (28) that

$$0<aF(x)-bF(y)y^l=U_k+A_1U_{k-1}+\cdots+A_kU_0\leqslant 0 \tag{29}$$

which is a contradiction. □

By (2), we observe that $ax^k<b(y+k+l)^{k+l}$ which implies that

$$x<(b/a)^{1/k}(y+k+l)^{1+(l/k)}\leqslant\theta(y+k+l)^{1+(l/k)} \tag{30}$$

where

$$\theta=\max(1,(b/a)^{1/k})\leqslant\max(1,(b/a)).$$

Now, we give an upper estimate for $f(x,y)$.

Lemma 3. *For $l\leqslant k$, we have*

$$f(x,y)\leqslant 16\theta k(\max(k,y))^{l/k}.$$

Proof. Suppose that $y>k+l$. Then, by (30) and $l\leqslant k$, we have

$$\begin{aligned} x&<(b/a)^{1/k}y^{1+(l/k)}\left(1+\frac{k+l}{y}\right)^{1+(l/k)}\\ &\leqslant(b/a)^{1/k}y^{1+(l/k)}\left(1+\frac{(k+l)^2}{ky}+\frac{l(k+l)^3}{2k^2y^2}\right)\end{aligned}$$

which implies that

$$f(x,y)\leqslant 6\theta ky^{l/k}.$$

If $y=k+l$, we see from (30) and $l\leqslant k$ that

$$f(x,y)\leqslant 3\theta y^{1+(l/k)}\leqslant 6\theta ky^{l/k}.$$

Thus, we may suppose that $y<k+l$. Then

$$\begin{aligned} x&<\theta(k+l)^{1+(l/k)}\left(1+\frac{y}{k+l}\right)^{1+(l/k)}\\ &\leqslant\theta(k+l)^{1+(l/k)}\left(1+\frac{y}{k}+\frac{ly^2}{2k^2(l+k)}\right)\end{aligned}$$

which implies that

$$f(x,y) < x \leqslant 16\theta k^{1+(l/k)}. \qquad \square$$

For applications, it is convenient to formulate a version of Lemma 3 which is valid also for $l > k$.

Lemma 4.

$$f(x,y) \leqslant 16\theta k x^{l/k}. \tag{31}$$

Proof. By (1), we observe that $x > \max(k, y)$. Now, we apply Lemma 3 to assume that $l > k$. Then, the trivial estimate $f(x,y) < x$ implies (31). $\square$

We apply our estimates on $f(x,y)$ to give bounds for $x-y$.

Lemma 5. *There exist effectively computable numbers c_1 and $c_2 > 0$ depending only on a and b such that*

$$x-y \leqslant \begin{cases} c_1\left(k+\dfrac{lx\log x}{k}\right) & \text{if } l>0 \\ c_1\left(k+\dfrac{x}{k}\right) & \text{if } l=0 \end{cases} \tag{32}$$

and

$$x-y \geqslant \begin{cases} (c_2 ly\log(y+1))/k, & \text{if } l>0 \\ c_2 y/k & \text{if } l=0. \end{cases} \tag{33}$$

Proof. We write c_3, c_4 and c_5 for effectively computable numbers depending only on a and b. We write, by (18),

$$x-y = f(x,y) + \Delta \tag{34}$$

where

$$\Delta = (b/a)^{1/k} y^{1+(l/k)} - y. \tag{35}$$

First, we prove (32). If $l=0$, the assertion follows from (34), (35) with $b > a$ and (31). Let $l > 0$. Then, we may assume that

$$l\log x < k, \tag{36}$$

otherwise (32) follows immediately. Now, we derive from (31) and (36) that

$$f(x,y) \leqslant 16\theta e k. \tag{37}$$

Further, it is easy to see from (35) and (36) that

$$\Delta \leqslant (c_3 lx\log x)/k. \tag{38}$$

Finally, we combine (34), (37) and (38) to obtain (32).

Next, we turn to the proof of (33). By (34) and (27), we observe that

$$x-y \geqslant \Delta. \tag{39}$$

If $l=0$, the assertion follows immediately from (39), (35) and $b>a$. Let $l>0$. Then, we may suppose that $y>0$ and $l\log(y+1)>c_4$ with c_4 sufficiently large. Then, it is easy to see that $\Delta \geqslant (c_5 ly \log y)/k$ which, together with (39), implies (33).

We apply an argument of Erodös and Lemmas 1, 5 to obtain a lower bound for x in the next lemma. The case $l=0$ of this lemma is proved in [10].

Lemma 6. *There exists an effectively computable number $c_6>0$ depending only on a and b such that*

$$x(\log x)^2 \geqslant \begin{cases} c_6 k^3 l^{-2} & \text{if } l>0 \\ c_6 k^3 (\log k)^2 & \text{if } l=0. \end{cases} \tag{46}$$

Proof. We denote by $c_7,\ldots,c_{13}$ effectively computable positive numbers depending only on a and b. We may assume that $k \geqslant c_7$ with c_7 sufficiently large and

$$x<k^3 \tag{41}$$

which, together with (23), implies that

$$l \leqslant c_8 \log k. \tag{42}$$

Furthermore, we derive from (2) and (1) that none of $x+1,\ldots,x+k$ is a prime number. Therefore, it follows from the well-known results on difference between consecutive primes that

$$x \geqslant k^{3/2}. \tag{43}$$

We denote by d the greatest common divisor of $(x+1)\cdots(x+k)$ and $(y+1)\cdots(y+k+l)$. Then, by (2), we see that

$$x^k<(x+1)\cdots(x+k) \leqslant bd. \tag{44}$$

Let $S=\{x+1,\ldots,x+k\}$. For a prime $p \leqslant k$, we choose an $f(p)\in S$ such that p does not appear to a higher power in the factorization of any other element of S. Let S_1 be the subset of S obtained by deleting all $f(p)$ with $p \leqslant k$. Then, by a fundamental argument of Erdös, we have

$$\prod_{s\in S_1} s \leqslant \prod_{p\leqslant k} p^{[(k/p)]+[(k/p^2)]+\cdots} = k!.$$

Therefore, the contribution d_1 in d from all primes not exceeding k is at most

$$k^k(x+k)^{\pi(k)} \leqslant (e^4 k)^k,$$

by (41). Further, the contribution d_2 in d from all primes p with $k<p\leqslant(2k+l-1)$ is less than or equal to

$$(x+k)^{\pi(2k+l-1)-\pi(k)} \leqslant e^{4k},$$

by (41) and (42). Now, we set

$$d_3=d/d_1d_2 \text{ and } \Delta_1=\left(\prod_{\mu=-(k+l-1)}^{k-1}(x-y+\mu)\right)\Big/(2k+l-1)!.$$

Notice that Δ_1 is a positive integer not exceeding

$$e^{3k}\left(\frac{x-y}{2k}+1\right)^{2k+l-1},$$

by (42). Also, we observe from (2) that $d_3|\Delta_1$. Consequently,

$$d\leqslant(c_9k)^k\left(\frac{x-y}{2k}+1\right)^{2k+l-1}$$

which, together with (44) and (42), implies that

$$x\leqslant c_{10}k\left(\frac{x-y}{2k}+1\right)^{2+c_8(\log k)/k}. \tag{45}$$

If $l>0$, we combine (45), (43) and (32) to conclude that

$$x\leqslant c_{11}k\left(\frac{lx\log x}{k^2}\right)^{2+c_8(\log k)/k}\leqslant c_{12}k\left(\frac{lx\log x}{k^2}\right)^2,$$

by (41) and (42). If $l=0$, we obtain in a similar way that $x\leqslant c_{13}k(x/k^2)^2$. The preceding two inequalities imply (40) immediately. □

Now, we are ready to prove Theorem 6(a), (b), (d).

Proof of Theorem 6(a), (b), (d). First, we observe that (15) is an immediate consequence of (40) and (23). We re-write $y<(1-\varepsilon)x$ as $x-y>\varepsilon x$ which, together with (32), (15) and (23), implies (16). Finally, we apply (33) and (40) to obtain Theorem 6(d). □

For applications, it is convenient to combine (32) and (15) to formulate the following result.

COROLLARY 2.

There exists an effectively computable number c_{14} depending only on a and b such that

$$x-y\leqslant(c_{14}(l+1)x\log x)/k. \tag{46}$$

For the proof of Theorem 6(c), we apply (15), (16) and (23) to obtain the following result which also finds an application in the proof of Theorem 1.

Lemma 7. *Let $\chi\geqslant 1$ and $\gamma>4/\log 2$. There exist effectively computable numbers c_{15} and c_{16} depending only on a, b, χ and γ such that for $k\geqslant c_{15}$ and*

$$\log(l+1)<(\gamma\chi)^{-1}k, \tag{47}$$

we have

$$y\geqslant\begin{cases}c_{16}l^{\chi} & \text{if } l>k\\ c_{16}k^3(\log k)^{-4} & \text{if } l\leqslant k.\end{cases}$$

Proof. We may assume that c_{15} is sufficiently large. Let $l>k$. Then, we may suppose that $y\leqslant l^{\chi}$. Now, we derive from (30) and (23) that

$$\log x\leqslant c_{17}+(2l\log(3l^{\chi}))/k$$

where c_{17} and the subsequent letter $c_{18}>0$ are effectively computable numbers depending only on a and b. Therefore, by (23), we derive that $k\leqslant(\gamma\chi)\log l$ which contradicts (47). Thus, we may suppose that $l\leqslant k$. Now, we may assume that $y\leqslant k^3(\log k)^{-4}$. Then, we observe from (30) that $k\geqslant x^{1/6}$ which, by Theorem 6(b) with $\varepsilon=1/2$ and (15), implies that

$$y\geqslant x/2\geqslant c_{18}k^3/(\log k)^4. \qquad \square$$

Proof of Theorem 6(c). We may suppose that C_{15} is sufficiently large. Then, we apply Lemma 7 with $\chi=3$, $\gamma=6$ and (23) to obtain

$$c_{19}\log\log x\geqslant\log(l+1)\geqslant k/18. \qquad \square$$

Next, we prove a lemma which tells that certain assumptions involving variables of (2) are equivalent.

Lemma 8(a). *Let* $0<\delta<1$. *There exist effectively computable numbers* v_1 *and* $v_2>0$ *depending only on* δ *such that* $l\geqslant v_2k$ *whenever* $x\geqslant v_1$ *and* $y\leqslant x^\delta$.
(b) *Let* $\mu>0$. *There exist effectively computable numbers* v_3 *and* v_4 *with* $0<v_4<1$ *depending only on* μ *such that* $y\leqslant x^{v_4}$ *whenever* $x\geqslant v_3$ *and* $l\geqslant\mu k$.

Proof. (a). We may assume that v_1 is sufficiently large. Let $y\leqslant x^\delta<x/2$. Then, we apply Theorem 6(b) with $\varepsilon=1/2$ to obtain (16) which, together with (30) and (23), implies that $l\geqslant v_2k$.
(b) By (2), $y<(a/b)^{1/(k+l)}(x+k)^{k/(k+l)}$ which implies the assertion. $\square$

Finally, we state an estimate of Baker [2] on linear forms in logarithms and its p-adic analogue, due to Yu [13]. Let $\alpha_1,\ldots,\alpha_n$ be non-zero rational numbers of heights* not exceeding $A_1,\ldots,A_n$, respectively. We assume that $A_j\geqslant3$ for $1\leqslant j\leqslant n$. We put

$$\Omega=\prod_{j=1}^{n}\log A_j,\quad \Omega'=\Omega/\log A_n.$$

Then we have

Lemma 9. (Baker [2]). *There exists an effectively computable number* c_{20} *depending only on n such that the inequalities*

$$0<|\alpha_1^{b_1}\cdots\alpha_n^{b_n}-1|<\exp(-c_{20}\Omega\log\Omega'\log B)$$

have no solution in rational integers $b_1,\ldots,b_n$ *of absolute values not exceeding* $B(\geqslant2)$.

Lemma 10. (Yu [13]). *Let p be a prime number. Suppose that* $b_1,\ldots,b_{n-1}$ *and* $b_n=-1$ *are rational integers of absolute values not exceeding* $B(\geqslant2)$. *There exists an effectively computable number* c_{21} *depending only on n and p such that either* $\alpha_1^{b_1}\cdots\alpha_n^{b_n}=1$ *or*

$$\operatorname{ord}_p(\alpha_1^{b_1}\cdots\alpha_n^{b_n}-1)\leqslant c_{21}\Omega\log\Omega'\log B.$$

*The height of a rational number u_1/u_2 with $gcd(u_1,u_2)=1$ is defined as $\max(|u_1|,|u_2|)$.

3.

Proof of Theorem 1. We denote by $c_{22}, c_{23}, \ldots, c_{29}$ effectively computable positive numbers depending only on a and b. We may assume that $k \geqslant c_{22}$ with c_{22} sufficiently large. Let $F(z)$, $G(z)$ and U_i with $0 \leqslant i \leqslant k$ be given by (19), (20) and (17), respectively. We apply Lemma 7 with $\chi > 2$, $4/\log 2 < \gamma < 6$ and $\gamma\chi = 12$ to derive from (8) that

$$y > (k+l)^2. \tag{48}$$

By (27), we observe that $U_k > 0$. Therefore, it suffices to show that $U_{k-1} < 0$. We assume that

$$U_{k-1} \geqslant 0 \tag{49}$$

and we shall arrive at a contradiction. By (2),

$$aF(x) - bF(y)y^l = bF(y)R(y) \tag{50}$$

where $R(y) = 0$ if $l = 0$ and for $l > 0$,

$$R(y) = G(y+k) - y^l = R_1(y) + R_2(y)$$

with

$$R_1(y) = (y+k)^l - y^l, \quad R_2(y) = A'_1(y+k)^{l-1} + \cdots + A'_l.$$

Then, it is easy to derive from the estimates (21), (22) and (48) that

$$F(y) \leqslant 3y^k, \quad R_1(y) \leqslant 2kly^{l-1}, \quad R_2(y) \leqslant 2l^2 y^{l-1}.$$

Therefore, we derive from (50) that

$$aF(x) - bF(y)y^l = bF(y)R(y) \leqslant 6bl(k+l)y^{k+l-1}. \tag{51}$$

We notice from (17) that

$$U_k - xU_{k-1} = by^{k+l-1}(x-y) > 0$$

which, by (49), (29) and (50), implies that

$$by^{k+l-1}(x-y) \leqslant U_k = bF(y)R(y) - A_1U_{k-1} - \cdots - A_kU_0. \tag{52}$$

Now, by (49), (21) and (48), we derive that

$$-A_1U_{k-1} - \cdots - A_kU_0 \leqslant -A_2U_{k-2} - \cdots - A_kU_0 \leqslant c_{23}k^4y^{k+l-2}$$

which, together with (51) and (52), implies that

$$x - y \leqslant c_{24}(kl + l^2 + k^4y^{-1}). \tag{53}$$

If $l = 0$, we see from (33) and (53) that $y \leqslant k^{11/4}$ which contradicts Lemma 7. Thus, we may assume that $l > 0$. Now, we combine again (33) and (53) to derive that

$$y \log y \leqslant c_{25}(k^2 + kl + k^5(ly)^{-1}). \tag{54}$$

Now, we combine (54) and (48) to observe that $l^2 \leqslant c_{25}(k^2+kl+k^3l^{-1})$ which implies that

$$l \leqslant c_{26}k. \tag{55}$$

Now, we see from (54), (48) and (55) that

$$y^2 \log y \leqslant c_{27}k^5l^{-1}. \tag{56}$$

Then, we apply Theorem 6 to derive that

$$(\log x)^2 \geqslant c_{28}k. \tag{57}$$

Finally, we combine (1), (30), (57), (56) and (55) to conclude that $k < x \leqslant c_{29}$. □

Proof of Theorem 2 (a). We write $c_{30}, c_{31}, \ldots, c_{37}$ for effectively computable positive numbers depending only on a and b. Suppose that the assertion (12) is not valid. For a given k, we observe that there is at most one l satisfying (2). Therefore, we may assume that (1) and (2) are satisfied with $k=k_1$, $l=l_1$ and $k=k_2$, $l=l_2$ such that $k_1<k_2$, $k_1 \geqslant c_{30}$ and $k_2-k_1 \geqslant c_{30}$ with c_{30} sufficiently large. Then, by (2) with $k=k_1$, $l=l_1$ and $k=k_2$, $l=l_2$, we observe that

$$(X+1)\cdots(X+k_2-k_1)=(Y+1)\cdots(Y+k_2-k_1+l_2-l_1) \tag{58}$$

where

$$X=x+k_1, \quad Y=y+k_1+l_1. \tag{59}$$

By (59) and (1), we see that $X>Y$ which, together with (58), implies that $l_1<l_2$. Further, by (30), (11), (23) and Theorem 6 (b), we derive that

$$k_i \log\log x \leqslant c_{31}(l_i+1) \quad (i=1,2) \tag{60}$$

which, together with (23), implies that

$$k_i \leqslant c_{32}(\log x)/\log\log x \quad (i=1,2). \tag{61}$$

Now, we see from (58), (59), (11), (61) and (23) that

$$(k_2-k_1)\log\log x \leqslant c_{33}(l_2-l_1). \tag{62}$$

By counting the power of 2 on both the sides of (24) with $a=b=1$, $x=X$, $y=Y$, $k=k_2-k_1$ and $l=l_2-l_1$ obtained from (58), we have

$$l_2-l_1 \leqslant 2 \max_{1\leqslant i\leqslant k_2-k_1} \mathrm{ord}_2(X+i)+1. \tag{63}$$

We show that

$$\text{either } l_1 \leqslant c_{34}\log k_2 \quad \text{or} \quad l_2-l_1 \leqslant c_{34}\log k_2. \tag{64}$$

If

$$\max_{1\leqslant j\leqslant k_1} \mathrm{ord}_2(x+j)=\mathrm{ord}_2(x+j_0)\leqslant 2\log k_2, \tag{65}$$

then we count the power of 2 on both the sides in (24) with $k=k_1$, $l=l_1$ to obtain

$l_1 \leqslant c_{35} \log k_2$. Therefore, we may suppose that (65) is not valid. Then, by (59), we write

$$X+i=x+j_0+k_1+i-j_0$$

to observe that

$$\mathrm{ord}_2(X+i)=\mathrm{ord}_2(k_1+i-j_0)\leqslant 2\log k_2 \tag{66}$$

for $1\leqslant i\leqslant k_2-k_1$. Then, we see from (63) and (66) that $l_2-l_1\leqslant 5\log k_2$. This proves (64). Now, we combine (60), (62), (64) and (61) to derive that either $k_1\leqslant c_{36}$ or $k_2-k_1\leqslant c_{36}$ which is not possible if $c_{30}>c_{36}$.

(b) Suppose that (1) and (2) with $k=k_1$, $l=l_1$ and $k=k_2$, $l=l_2$ are satisfied. Then, we observe that (64) is valid. Now, we apply Theorem 6(b) to derive that either $l_1\leqslant c_{37}\log\log x$ or $l_2-l_1\leqslant c_{37}\log\log x$ where c_{37} is an effectively computable number depending only on ε, a and b. Finally, we observe that for a given l there is at most one k satisfying (2) to complete the proof of Theorem 2(b). □

4.

Proof of Theorem 3(a). We denote by $d_1,\dots,d_5$ effectively computable positive numbers depending only on ε, a, b and $P(x)$. We may assume that $x>d_1$ with d_1 sufficiently large, otherwise the assertion follows from (1). By (24), (25) and (26), we see that

$$l\leqslant d_2 \max_{1\leqslant i\leqslant k} \mathrm{ord}_2(x+i)\leqslant d_2\left(\max_{1\leqslant i\leqslant k}\mathrm{ord}_2(-xi^{-1}-1)+\frac{\log k}{\log 2}\right).$$

Further, we write x as $\Pi_{j=1}^{\omega(x)} p_j^{\mathrm{ord}_{p_j}(x)}$ where $p_1,\dots,p_{\omega(x)}$ are the distinct prime factors of x. Now, we apply Lemma 10 to $\mathrm{ord}_2(-xi^{-1}-1)$ with $p=2$, $n=\omega(x)+2\leqslant P(x)+2$, $\alpha_1=-1$, $b_1=1$; $\alpha_{j+1}=p_j$, $b_{j+1}=\mathrm{ord}_{p_j}(x)$ with $1\leqslant j\leqslant \omega(x)$ and $\alpha_n=i$, $b_n=-1$, to obtain

$$l\leqslant d_3(\log\log x)\log k$$

which, together with (46) and (13), implies that

$$(\log k)(\log\log k)^2\leqslant d_4(\log x)(\log\log x).$$

Therefore, since $\omega(x)\leqslant P(x)$, we have

$$x>k^{\omega(x)}. \tag{67}$$

Consequently, there exists a prime p dividing x such that

$$p^{\mathrm{ord}_p(x)}>k. \tag{68}$$

Now, we count the power of p on both the sides of (24) to derive that

$$[l/p]-1\leqslant \mathrm{ord}_p(a) \tag{69}$$

which implies that $l\leqslant d_5$. □

(b) We apply Lemma 8(a) with $\delta=1-\varepsilon$ to conclude that $k\leqslant d_6 l$ where d_6 and the subsequent letter d_7 are effectively computable numbers depending only on ε, a, b and $P(x)$. Now, we apply Theorem 3(a) to conclude that $\max(k,l)\leqslant d_7$. □

(c) We write $d_8,\ldots,d_{13}$ for effectively computable numbers depending only on a and b. There is no loss of generality in assuming that $x>d_8$ with d_8 sufficiently large. Suppose that (2) with $P(x)\leqslant l+k$ and $y\leqslant x/2$ is satisfied. For a prime p with $k<p\leqslant k+l$ and $p|x$, we derive from (2) that $p|a$. Therefore

$$P(x)\leqslant \max(k,P(a)).$$

Consequently, we observe from prime number theory that

$$\omega(x)\leqslant d_9 k/\log k. \tag{70}$$

First, we show that

$$l\leqslant d_{10}k. \tag{71}$$

Let $x\leqslant k^{\omega(x)}$. Then, we see from (23) and (70) that $l\leqslant d_{11}k$. Thus, we may assume (67) which implies (68), (69) and hence, $l\leqslant d_{12}k$. This proves (71). Finally, we combine (30), (71) and Theorem 6(b) with $\varepsilon=1/2$ to conclude that $y>x^{d_{13}}$.

(d) Let $d_{14},\ldots,d_{18}$ be effectively computable positive numbers depending only on a, b and $P(x-y)$. We may assume that $x>d_{14}$ with d_{14} sufficiently large. Suppose that (2) with $y<x/2$ is satisfied. Then, the inequality (16) is valid. Further, we may suppose that

$$l\geqslant \max(P(x-y),2a), \tag{72}$$

otherwise, we may derive from (30) and (16) that $y\geqslant x^{d_{15}}$.

Let $F(z)$ be given by (19). We re-write (2) as

$$0\neq aF(x)-aF(y)=F(y)(b(y+k+1)\cdots(y+k+l)-a). \tag{73}$$

For a prime p dividing $x-y$, we see from (73) and (72) that

$$\mathrm{ord}_p(x-y)\leqslant \mathrm{ord}_p(F(y))+\mathrm{ord}_p(a). \tag{74}$$

Further, we observe that

$$\mathrm{ord}_p(F(y))\leqslant \max_{1\leqslant i\leqslant k}\mathrm{ord}_p(y+i)+\left[\frac{k}{p}\right]+\left[\frac{k}{p^2}\right]+\cdots \tag{75}$$

Now, we derive from $y<x/2$, (74) and (75) that

$$\log\left(\frac{x}{2}\right)\leqslant \log(x-y)=\sum_{p|(x-y)}\mathrm{ord}_p(x-y)\log p\leqslant d_{16}(\log(y+k)+k)$$

which, by (16) and (23), implies that either $y\geqslant x^{d_{17}}$ or $l\leqslant d_{17}k$. Finally, as above, we apply (30) and (16) to assume that the latter inequality is not valid.

(e) We may assume (72). Therefore, the inequality (74) is valid. Consequently, there is an effectively computable number d_{18} depending only on a, b and y such that

$$\log\left(\frac{x}{2}\right) \leqslant \log(x-y) \leqslant \log a + \pi(l)\log(y+k) + 2k\sum_{p\leqslant l}\frac{\log p}{p}$$
$$\leqslant d_{18}\left(\frac{l\log k}{\log l} + k\log l\right)$$

which, together with (23) and Theorem 6(c), completes the proof. □

Proof of Theorem 5. We denote by $d_{19},\ldots,d_{28}$ effectively computable positive numbers depending only on a, b, $P(x)$ and $P(y)$. We may assume that $x > d_{19}$ with d_{19} sufficiently large, otherwise the theorem follows from (1). We apply Lemma 9 to conclude that

$$x - y \geqslant x(\log x)^{-d_{20}}. \tag{76}$$

On the other hand, we derive from (46) and (23) that

$$x - y \leqslant d_{21}x(\log x)^2/k. \tag{77}$$

We combine (76) and (77) to derive that

$$k \leqslant (\log x)^{d_{22}}. \tag{78}$$

Now, we show that

$$y \leqslant (\log x)^{d_{23}}. \tag{79}$$

For proving (79), we refer to (23) and (78) to assume that

$$y > (k+l)^4. \tag{80}$$

Then, by (27), (2), (15) and (80), we observe that

$$0 < U_k \leqslant d_{24}((k+l)^2 y^{k+l-1} + k^2 x^{k-1}) \tag{81}$$

where U_k is given by (17). On the other hand, we apply again Lemma 9 to obtain

$$U_k \geqslant \max(x^k, y^{k+l})((k+l)\log x)^{-d_{25}}$$

which, together with (78) and (23), implies that

$$U_k \geqslant \max(x^k, y^{k+l})(\log x)^{-d_{26}}. \tag{82}$$

Finally, we combine (81), (82), (78) and (80) to obtain (79). Therefore, we conclude from Theorem 3(b) with $\varepsilon = 1/2$ to obtain that $\max(k, l) \leqslant d_{27}$ which, together with (30) and (79), implies that $x = \max(x, y, k, l) \leqslant d_{28}$. □

Proof of Corollary 1. We denote by d_{29}, d_{30} and d_{31} effectively computable positive numbers depending only on B. We may assume that $N \geqslant d_{29}$ with d_{29} sufficiently large. In (2), we put $a = 1$, $x = X$, $k = M$, $b = B!$, $y = B$ and $l = N - M - B$. By (6), we notice that $X \geqslant N$ so that (1) is satisfied. Therefore, we derive from Theorem 6(c) that

$$M \leqslant d_{30}\log\log X. \tag{83}$$

Further, we see from (5) that

$$N \geqslant d_{31} (\log X)/\log\log X. \tag{84}$$

Now, we apply Theorem 3(e), (83) and (84) to conclude that

$$P =: P(X-B) > N-M-B > M+B. \tag{85}$$

Further, we see from (5) and (85) that

$$P \left| \left(\frac{N!}{M!} - \frac{(B+1)\cdots(B+M)}{M!} \right) \right. . \tag{86}$$

Finally, we derive from (86) and (85) that $P > N$. □

5.

This section is devoted to preliminaries for the proof of Theorem 4. Let $F(z)$ be given by (19). Then

$$A_j = \sum_{i=i_0=1}^{k} \sum_{i_1=1}^{i_0-1} \cdots \sum_{i_{j-1}=1}^{i_{j-2}-1} i_0 i_1 \cdots i_{j-1} \quad \text{for } 1 \leqslant j \leqslant k.$$

We write the right hand side of (14)

$$(y+1)\cdots(y+2k) = \prod_{j=1}^{k} (u+j(2k-j+1)) \tag{87}$$

where

$$u = y(y+2k+1). \tag{88}$$

Let

$$G(z) = \prod_{j=1}^{k} (z+j(2k-j+1)) = z^k + B_1 z^{k-1} + \cdots + B_k \tag{89}$$

where

$$B_j = \sum_{i=i_0=1}^{k} \sum_{i_1=1}^{i_0-1} \cdots \sum_{i_{j-1}=1}^{i_{j-2}-1} i_0 i_1 \cdots i_{j-1} (2k-i_0+1)\cdots(2k-i_{j-1}+1) \tag{90}$$

for $1 \leqslant j \leqslant k$.

Further, we put

$$\Delta_q = \Delta_{q,k} = \sum_{i=1}^{k} i^q \quad \text{for } q = 1, 2, \ldots .$$

Then, we have $A_1 = \Delta_1$, $A_2 = (\Delta_3 - \Delta_2)/2$,

$$A_3 = (3\Delta_5 - 10\Delta_4 + 9\Delta_3 - 2\Delta_2)/24,$$

$$A_4 = (\Delta_7 - 7\Delta_6 + 17\Delta_5 - 17\Delta_4 + 6\Delta_3)/48,$$

$$B_1 = (2k+1)\Delta_1 - \Delta_2,\ 3B_2 = \Delta_5 - (5k+4)\Delta_4 + (6k^2+12k+5)\Delta_3 - (6k^2+7k+2)\Delta_2$$

and

$$3B_3=-\frac{1}{6}\Delta_8+\left(\frac{4}{3}k+\frac{22}{15}\right)\Delta_7-\left(\frac{7}{2}k^2+\frac{91}{10}k+\frac{149}{30}\right)\Delta_6$$
$$+\left(3k^3+\frac{35}{2}k^2+\frac{137}{6}k+\frac{49}{6}\right)\Delta_5-\left(10k^3+\frac{59}{2}k^2+\frac{51}{2}k+\frac{20}{3}\right)\Delta_4$$
$$+\left(9k^3+\frac{37}{2}k^2+\frac{71}{6}k+\frac{71}{30}\right)\Delta_3-\left(2k^3+3k^2+\frac{7}{5}k+\frac{1}{5}\right)\Delta_2.$$

Further, we derive from [9, p. 6] that

$$\Delta_1=k(k+1)/2,\quad \Delta_2=k(k+1)(2k+1)/6,\quad \Delta_3=k^2(k+1)^2/4,$$
$$\Delta_4=k(k+1)(6k^3+9k^2+k-1)/30,\quad \Delta_5=k^2(k+1)(2k^3+4k^2+k-1)/12,$$
$$\Delta_6=k(k+1)(6k^5+15k^4+6k^3-6k^2-k+1)/42,$$
$$\Delta_7=k^2(k+1)(3k^5+9k^4+5k^3-5k^2-2k+2)/24$$

and

$$\Delta_8=k(k+1)(10k^7+35k^6+25k^5-25k^4-17k^3+17k^2+3k-3)/90.$$

Consequently, we obtain the following expressions for A_1, A_2, A_3, A_4 and B_1, B_2, B_3.

Lemma 11. $A_1=k(k+1)/2$, $A_2=k(k+1)(k-1)(3k+2)/24$,

$$A_3=k^2(k+1)^2(k-1)(k-2)/48,$$
$$A_4=k(k+1)(k-1)(k-2)(k-3)(15k^3+15k^2-10k-8)/5760 \tag{91}$$

and

$$B_1=k(k+1)(2k+1)/3,\quad B_2=k(k+1)(20k^4+16k^3-11k^2-19k-6)/90, \tag{92}$$
$$B_3=k(k+1)(280k^7-28k^6-830k^5-745k^4$$
$$+136k^3+557k^2+486k+144)/5670.$$

Let

$$f=\left[\frac{2k^2}{3}\right]+k-\left[\frac{k}{2}\right].$$

We write

$$F(z+f)=z^k+A_1(f)z^{k-1}+\cdots+A_k(f) \tag{93}$$

where

$$A_i(f)=\binom{k}{i}f^i+\binom{k-1}{i-1}f^{i-1}A_1+\cdots+\binom{k-i+1}{1}fA_{i-1}+A_i \tag{94}$$

for $1\leqslant i\leqslant k$.

Then, we apply Lemma 11 to obtain the following result.

Lemma 12. *Let*

$$\delta=\begin{cases}0 & \text{if } k\equiv 0 \pmod 6\\ -1/6 & \text{if } k\equiv 1,5 \pmod 6\\ -2/3 & \text{if } k\equiv 2,4 \pmod 6\\ 1/2 & \text{if } k\equiv 3 \pmod 6\end{cases} \tag{95}$$

Then

$$A_1(f)-B_1=(\tfrac{1}{6}+\delta)k,$$

$$A_2(f)-B_2=\frac{2}{45}k^5+\left(\frac{1}{9}+\frac{2\delta}{3}\right)k^4+\left(-\frac{1}{24}+\frac{\delta}{3}\right)k^3+\left(-\frac{7}{72}+\frac{\delta}{2}(\delta-1)\right)k^2$$
$$+\left(-\frac{1}{60}-\frac{\delta}{2}(1+\delta)\right)k$$

and

$$A_3(f)-B_3=\frac{4}{135}k^8+\left(\frac{79}{2835}+\frac{2}{9}\delta\right)k^7-\frac{5}{36}k^6+\left(-\frac{169}{1080}-\frac{13}{18}\delta+\frac{1}{3}\delta^2\right)k^5$$
$$+\left(\frac{67}{720}-\frac{17}{24}\delta-\frac{1}{2}\delta^2\right)k^4+\left(\frac{1013}{6480}+\frac{9}{24}\delta-\frac{7}{12}\delta^2+\frac{1}{6}\delta^3\right)k^3$$
$$+\left(\frac{1}{72}+\frac{2}{3}\delta+\frac{1}{4}\delta^2-\frac{1}{2}\delta^3\right)k^2$$
$$+\left(-\frac{8}{315}+\frac{1}{6}\delta+\frac{1}{2}\delta^2+\frac{1}{3}\delta^3\right)k.$$

Proof. We write $g=\frac{2}{3}k^2+\frac{1}{2}k$ and $f-g=\delta$. Then, we check that (95) is satisfied. Further, we see from (94) that $A_1(f)=kg+A_1+\delta k$, $A_2(f)=\binom{k}{2}g^2+(k-1)A_1g+A_2+\delta\binom{k}{2}(2g+\delta)+\delta(k-1)A_1$ and $A_3(f)=\binom{k}{3}g^3+\binom{k-1}{2}A_1g^2+\binom{k-2}{1}A_2g+A_3+\delta\binom{k}{3}(3g^2+3\delta g+\delta^2)+\delta\binom{k-1}{2}A_1(2g+\delta)+\delta\binom{k-2}{1}A_2$. Now, we apply Lemma 11 to complete the proof of Lemma 12. □

Finally, we obtain estimates for all $A_i(f)$ and B_j.

Lemma 13. *For* $1\leqslant i\leqslant k$, *we have*

$$A_i(f)\leqslant(1+\phi_k)\binom{k}{i}f^i \tag{96}$$

where

$$\phi_k=\frac{3}{4}\left(1+\frac{1}{k}\right)^2\left(1-\frac{3}{8}\left(1+\frac{1}{k}\right)^2\right)^{-1}. \tag{97}$$

Furthermore,

$$B_j\leqslant k^j(k+1)^{2j}/j! \quad \textit{for } 1\leqslant j\leqslant k \tag{98}$$

and

$$B_4\leqslant(2k)^4A_4,\quad B_4\geqslant(k+1)^4A_4. \tag{99}$$

Proof. By (94), we see that

$$A_i(f)\leqslant\binom{k}{i}f^i\left(1+\frac{A_1}{f}\frac{i}{k}+\frac{A_2}{f^2}\frac{i(i-1)}{k(k-1)}+\cdots+\frac{A_i}{f^i}\frac{i!}{k(k-1)\cdots(k-i+1)}\right). \tag{100}$$

Further, by (21) and $f>(2k^2)/3$, we observe that

$$A_j f^{-j} \leqslant \frac{1}{j!}\left(\frac{3}{4}\right)^j\left(1+\frac{1}{k}\right)^{2j}$$

which, together with (100), implies (96) where ϕ_k is given by (97). By (90), we see that $B_4 \geqslant (k+1)^4 A_4$ and

$$B_j \leqslant (2k)^j A_j, \quad \text{for } 1 \leqslant j \leqslant k,$$

which, by (21), implies (98). □

6.

In this section, we shall prove Theorem 4. The computations for the proof of Theorem 4 are carried out on a pocket calculator. Throughout this section, we assume that (14) is satisfied. For a prime $p>0$, we write

$$n=b_0+b_1p+\cdots+b_\mu p^\mu$$

where $b_0,\ldots,b_\mu$ are integers satisfying $b_\mu \neq 0$ and $0 \leqslant b_i < p$ for $0 \leqslant i \leqslant \mu$. Then, we start with the following well-known result.

Lemma 14.

$$\operatorname{ord}_p(n!)=\frac{n-(b_0+\cdots+b_\mu)}{p-1}. \tag{101}$$

For the proof of Lemma 14, we observe that

$$\operatorname{ord}_p(n!)=\sum_{i=1}^{\mu}\left[\frac{n}{p^i}\right]=\sum_{i=1}^{\mu}(b_i+b_{i+1}p+\cdots+b_\mu p^{\mu-i})$$

which implies (101). We denote by τ the number of positive integers i such that

$$\left[\frac{y+2k}{2^i}\right]-\left[\frac{y}{2^i}\right]-\left[\frac{2k}{2^i}\right]>0.$$

Then, we apply Lemma 14 to obtain the following lower bound for x and this is fundamental for our argument.

Lemma 15.

$$x \geqslant 2^{k+\tau}-k. \tag{102}$$

Since $\tau \geqslant 0$, the inequality (102) *implies that*

$$x \geqslant 2^k - k. \tag{103}$$

Proof. We re-write (14) as

$$\frac{(x+1)\cdots(x+k)}{k!}=\frac{(y+1)\cdots(y+2k)}{(2k)!}\frac{(2k)!}{k!}. \tag{104}$$

By (101),

$$\mathrm{ord}_2\left(\frac{(2k)!}{k!}\right)=k. \tag{105}$$

Further, we observe that

$$\mathrm{ord}_2\left(\frac{(y+1)\cdots(y+2k)}{(2k)!}\right)=\sum_{i=1}^{\infty}\left(\left[\frac{y+2k}{2^i}\right]-\left[\frac{y}{2^i}\right]-\left[\frac{2k}{2^i}\right]\right)$$

and every summand is either 0 or 1. Therefore

$$\mathrm{ord}_2\left(\frac{(y+1)\cdots(y+2k)}{(2k)!}\right)=\tau. \tag{106}$$

On the other hand, we see that

$$\mathrm{ord}_2\left(\frac{(x+1)\cdots(x+k)}{k!}\right)\leqslant \max_{1\leqslant i\leqslant k}\mathrm{ord}_2(x+i)\leqslant(\log(x+k))/\log 2. \tag{107}$$

Finally, we combine (104), (105), (106) and (107) to obtain (102).

As an application of Lemma 15, we prove

Lemma 16. *If* (14) *holds, then*

$$(x+1)\cdots(x+k)>\left(x+\frac{k}{2}\right)^k. \tag{108}$$

Proof. It is easy to check (108) for $k\leqslant 4$. Thus, we may assume that $k\geqslant 5$. Suppose that

$$(x+1)\cdots(x+k)\leqslant\left(x+\frac{k}{2}\right)^k. \tag{109}$$

By Lemma 11, we observe that $A_1-\binom{k}{1}\left(\frac{k}{2}\right)=\frac{k}{2}$, $A_2-\binom{k}{2}\left(\frac{k}{2}\right)^2=\frac{k(k-1)(5k+2)}{24}$ and $A_3-\binom{k}{3}\left(\frac{k}{2}\right)^3=\frac{k^2(k-1)(k-2)(2k+1)}{48}$ are positive. Further, we see from (103) and $k\geqslant 5$ that $x>k^2$. Now, we derive from (109) that

$$\frac{1}{2}kx^{k-1}\leqslant\binom{k}{4}\left(\frac{k}{2}\right)^4x^{k-4}\left(1+\frac{1}{10}+\frac{1}{10^2}+\cdots\right)\leqslant 5k^8x^{k-4}/1728$$

which implies that $172x^3<k^7$. Now, since $x>k^2$, we derive that $k>172$. Then, we apply (103) to conclude that $x>k^3$ and this is a contradiction.

For small values of y and k, the next two lemmas are useful. The first may be confirmed by direct checking.

Lemma 17.

$$P((x+1)\cdots(x+5))\geqslant 31\quad \textit{for } 54\leqslant x\leqslant 600$$

and

$$P((x+1)\cdots(x+6))\geqslant 37 \quad \text{for } 120\leqslant x\leqslant 600.$$

Lemma 18. *Equation* (14) *implies that*

$$y\geqslant\begin{cases}17 & \text{if } k\in\{5,6,7,8\}\\ 28 & \text{if } k=9\\ 32 & \text{if } k=10\end{cases}$$

Proof. By the arithmetic-geometric mean, we observe that

$$((x+1)\cdots(x+k))^{1/k}<\frac{(x+1)+\cdots+(x+k)}{k}=x+\frac{k+1}{2}$$

which, together with (108), implies that

$$[((x+1)\cdots(x+k))^{1/k}]=x+\left[\frac{k}{2}\right]. \tag{110}$$

By applying again the arithmetic-geometric mean, we see from (87) and (92) that

$$[((y+1)\cdots(y+2k))^{1/k}]\leqslant u+\left[\frac{(k+1)(2k+1)}{3}\right]. \tag{111}$$

Now, we combine (14), (110) and (111) to derive that

$$0<x-u\leqslant\left[\frac{(k+1)(2k+1)}{3}\right]-\left[\frac{k}{2}\right]. \tag{112}$$

Let $k\in\{5,6,7,8\}$ and $y\leqslant 16$. Then, we see from (88) and (112) that $x\leqslant 600$. Now, we apply Lemma 17 to derive that $x<54$ if $k\in\{5,6,7\}$ and $x<120$ if $k=8$ which, together with (103), imply that $k=5$ and $x\geqslant 27$. Therefore, we see from (88) and (112) that $y(y+11)<54$ which implies that $y\leqslant 3$. Furthermore, we observe $[(y+10)/4]-[y/4]-[10/4]=1$ if $y=2,3$ to derive from Lemma 15 with $\tau\geqslant 1$ that $y\leqslant 1$. Then, since the left hand side of (14) is not divisible by 31, we see that $x\geqslant 31$. Consequently, we observe that the left hand side of (14) is greater than the right hand side of (14).

Let $k=10$ and $y\leqslant 31$. Then, we observe from (88) and (112) that $x\leqslant 1684$ which, by Lemma 15, implies that $\tau=0$. On the other hand, we observe that $[(y+20)/32]-[y/32]-[20/32]=1$ if $y\geqslant 12$. Consequently, we conclude that $y\leqslant 11$ which implies that $x\leqslant 424$ contradicting (103).

Let $k=9$ and $y\leqslant 27$. Then, by (88), (112) and (103), we observe that $x\leqslant 1301$, $x\geqslant 503$ and $y\geqslant 14$. Then, we observe $[(y+18)/32]-[y/32]-[18/32]=1$ to apply Lemma 15 with $\tau\geqslant 1$ to derive that $x\geqslant 1015$ which implies that $y>22$. If $y\in\{23,26,27\}$, we see $\tau\geqslant 2$ to obtain from (102) that $x\geqslant 2039$ which is not possible. If $y=24$, then $x\leqslant 1091$ and, by looking at the prime factors of 1086, 1091, 1093, we derive from (14) that $x\leqslant 1076$ which is not possible, since the left hand side of (14) is less than the right hand side of (14). If $y=25$, then $x\leqslant 1159$ and we argue, as above by looking at the prime factors of 1151, 1159, 1162, to derive $x\leqslant 1149$ for arriving at a contradiction. □

Now, we are ready to prove an inequality for (87) analogous to (108).

Lemma 19. *Let* $u>0$ *if* $k=3$ *and*

$$\theta=\begin{cases}1 & \text{if } k\equiv 0 \pmod 3\\ 2 & \text{if } k\equiv 1,2 \pmod 3\end{cases}$$

Then

$$\prod_{j=1}^{k}(u+j(2k-j+1))>\left(u+\frac{(k+1)(2k+1)-\theta}{3}\right)^k.$$

Proof. The Lemma can be verified for $k\leqslant 4$. By direct computations, we may assume that $u\leqslant 21$ if $k=5$, $u\leqslant 145$ if $k=6$ and $u\leqslant 125$ if $k=7$. Now, we refer to (88) and Lemma 18 to exclude these possibilities. Thus, we may suppose that $k\geqslant 8$. We set

$$h(\theta)=h(\theta,k)=((k+1)(2k+1)-\theta)/3.$$

We observe that

$$h(\theta)\leqslant k(2k+3)/3. \tag{113}$$

We suppose that

$$\prod_{j=1}^{k}(u+j(2k-j+1))\leqslant(u+h(\theta))^k. \tag{114}$$

By (113), we see that

$$(u+h(\theta))^k\leqslant u^k+kh(\theta)u^{k-1}+\binom{k}{2}\left(\frac{k(2k+3)}{3}\right)^2u^{k-2}+\cdots+\binom{k}{k}\left(\frac{k(2k+3)}{3}\right)^k. \tag{115}$$

We put

$$D_2=\frac{2}{45}k^5-\frac{2}{9}k^4-\frac{1}{6}k^3+\frac{5}{18}k^2+\frac{1}{15}k,$$

$$D_3=\frac{k(k-1)(k-2)}{5670}(168k^5+32k^4-870k^3-1120k^2-423k-72)$$

and

$$R'=\binom{k}{5}\left(\frac{k(2k+3)}{3}\right)^5u^{k-5}+\cdots+\binom{k}{k}\left(\frac{k(2k+3)}{3}\right)^k.$$

Further, we observe from (112), (103) and $k\geqslant 8$ that $u>k^2(2k+3)/9$. Therefore

$$R'\leqslant 2\binom{k}{5}\left(\frac{k(2k+3)}{3}\right)^5u^{k-5}. \tag{116}$$

Thus, we derive from (114), (89), (115), (116) and Lemma 11 that

$$\left(\frac{\theta k}{3}\right)u^{k-1}\leqslant D_2u^{k-2}+D_3u^{k-3}+\binom{k}{4}\left(\frac{k(2k+3)}{3}\right)^4u^{k-4}+2\binom{k}{5}\left(\frac{k(2k+3)}{3}\right)^5u^{k-5}. \tag{117}$$

We derive from (117) that $u \leqslant 550$ if $k=8$, $u \leqslant 1200$ if $k=9$ and $u \leqslant 1500$ whenever $k=10$. Now, we apply (88) and Lemma 18 to exclude these cases. Furthermore, the inequality (117) implies that $u < 1950$ if $k=11$, $u < 3980$ if $k=12$ and these cases are excluded by (112) and (103).

Thus, we may assume that $k \geqslant 13$. Then, we see from (112) and (103) that

$$u \geqslant \frac{7}{25} k^4. \tag{118}$$

Further, we observe that

$$D_2 < \frac{2}{45} k^5, \quad D_3 < \frac{171}{5670} k^8,$$

$$\binom{k}{4}\left(\frac{k(2k+3)}{3}\right)^4 \leqslant \frac{1}{24}\left(\frac{29}{39}\right)^4 k^{12} \leqslant \frac{13}{1000} k^{12}$$

and

$$2\binom{k}{5}\left(\frac{k(2k+3)}{3}\right)^5 \leqslant \frac{1}{60}\left(\frac{29}{39}\right)^5 k^{15} \leqslant \frac{1}{250} k^{15}.$$

Therefore, we derive from (117) and $\theta \geqslant 1$ that

$$u < \frac{2}{15} k^4 + \frac{513}{5670} \frac{k^7}{u} + \frac{39}{1000} \frac{k^{11}}{u^2} + \frac{3}{250} \frac{k^{14}}{u^3}$$

which, together with (118), implies that

$$\frac{7}{25} k^4 \leqslant u < \frac{2}{15} k^4 + k^3 + k^2.$$

Consequently, we conclude that $k \leqslant 7$ which is a contradiction. □

As an immediate consequence of Lemma 19 and (110), we obtain the following improvement of (112).

COROLLARY 3.

Suppose that (14) *is satisfied. Then, either* $x=7$, $y=0$, $k=3$ *or*

$$x - u = f. \tag{119}$$

Proof. We may assume that $u > 0$ if $k=3$. Then, we see from Lemma 19 that

$$\left[\left(\prod_{j=1}^{k} (u + j(2k - j + 1))\right)^{1/k}\right] \geqslant u + [h(\theta)] = u + \left[\frac{k(2k+3)}{3}\right]. \tag{120}$$

On the other hand, we see from the arithmetic-geometric mean and (92) that the left hand side of (120) is less than $u + [(k+1)(2k+1)/3]$. Consequently, we conclude that (120) holds with equality sign. Hence, the assertion (119) follows from (14), (87) and (110).

Proof of Theorem 4. We may assume that $u > 0$ if $k=3$. Then, we conclude from (14),

(87) and (119) that

$$(u+f+1)\cdots(u+f+k)=\prod_{j=1}^{k}(u+j(2k-j+1)). \tag{121}$$

We may verify, by direct computation, that (121) is not possible whenever $k \leqslant 7$. Thus, we may suppose that $k \geqslant 8$. Further, by (121), (93) and (89), we see that

$$(A_1(f)-B_1)u^{k-1}+(A_2(f)-B_2)u^{k-2}+\cdots+(A_k(f)-B_k)=0.$$

Thus

$$(A_1(f)-B_1)u^{k-1}+(A_2(f)-B_2)u^{k-2}+(A_3(f)-B_3)u^{k-3} \leqslant (B_4-A_4(f))u^{k-4}+R_3 \tag{122}$$

and

$$(B_1-A_1(f))u^{k-1}+(B_2-A_2(f))u^{k-2}+(B_3-A_3(f))u^{k-3} \leqslant (A_4(f)-B_4)u^{k-4}+R_4 \tag{123}$$

where

$$R_3=B_5u^{k-5}+\cdots+B_k, \quad R_4=A_5(f)u^{k-5}+\cdots+A_k(f). \tag{124}$$

By (119) and (103), we see that

$$u \geqslant \frac{22}{45}k(k+1)^2 \text{ if } k \geqslant 9 \quad \text{and } u \geqslant \frac{1}{2}fk. \tag{125}$$

Thus, we derive from (124), (96), (98) and (125) that

$$R_3 \leqslant \frac{2}{145}k^5(k+1)^{10}u^{k-5} \quad \text{if } k \geqslant 9 \quad \text{and } R_4 \leqslant \frac{17}{4}f^5\binom{k}{5}u^{k-5}. \tag{126}$$

Next, we turn to the coefficient $B_4-A_4(f)$ in (122) and (123). By (91), we calculate $A_4=22449$ if $k=8$ and $A_4=157773$ if $k=10$. Then, we derive from (94) and (99) that

$$(A_4(f)-B_4)/10^4 \leqslant \begin{cases} 30730 & \text{if } k=8 \\ 487570 & \text{if } k=10 \end{cases} \tag{127}$$

Similarly, by observing $B_4-A_4(f) \leqslant B_4-\binom{k}{4}f^4-\binom{k-1}{3}A_1f^3$, we obtain

$$(B_4-A_4(f))/10^5 \leqslant \begin{cases} 45990 & \text{if } k=9 \\ 606420 & \text{if } k=11 \end{cases} \tag{128}$$

and

$$(B_4-A_4(f))/u^2 \leqslant \begin{cases} 11450 & \text{if } k=12 \\ 7660 & \text{if } k=13 \end{cases} \tag{129}$$

since, by (119) and (103), $u \geqslant 3982$ if $k=12$ and $u \geqslant 8060$ if $k=13$. Next, we notice that

$$f \leqslant \tfrac{3}{4}k^2 \tag{130}$$

and

$$u \geqslant \tfrac{2}{5}k^4 \quad \text{if } k \geqslant 14. \tag{131}$$

First, we consider the case that $k \equiv 2, 4 \pmod 6$. By Lemma 12 with $\delta=-2/3$, we see that $A_1(f) - B_1 = -k/2$ and

$$A_2(f) - B_2 < \frac{2}{45}k^5 - \frac{1}{3}k^4, \quad A_3(f) - B_3 < \frac{4}{135}k^8 - \frac{341}{2835}k^7,$$

since $k \geqslant 8$. Therefore, we derive from (123) that

$$\tfrac{1}{2}ku^{k-1} < \left(\frac{2}{45}k^5 - \frac{1}{3}k^4\right)u^{k-2} + \left(\frac{4}{135}k^8 - \frac{341}{2835}k^7\right)u^{k-3}$$
$$+ (A_4(f) - B_4)u^{k-4} + R_4. \tag{132}$$

Then, we see from (132), (127) and (126) that $u \leqslant 530$ if $k = 8$ and $u \leqslant 1300$ if $k = 10$. Now, we apply (88) and Lemma 18 to assume that $k \geqslant 14$. Further, we see from (96), (97), (130) and (126) that

$$A_4(f) \leqslant (1 + \phi_{14})f^4k^4/24 \leqslant k^{12}/25 \tag{133}$$

and

$$R_4 \leqslant 17f^5k^5u^{k-5}/480 \leqslant 9k^{15}u^{k-5}/1000. \tag{134}$$

Thus, we combine (132), (131), (133) and (134) to derive that

$$\frac{2}{5}k^4 \leqslant u < \frac{4}{45}k^4 + \frac{8}{135}\frac{k^7}{u} + \frac{2}{25}\frac{k^{11}}{u^2} + \frac{9}{500}\frac{k^{14}}{u^3} \leqslant \frac{4}{45}k^4 + k^3 + k^2$$

which implies that $k \leqslant 4$.

We argue, as above, to derive from (122) that

$$\tfrac{2}{3}ku^{k-1} \leqslant (B_4 - A_4(f))u^{k-4} + R_3 \quad (k \equiv 3 \pmod 6), \tag{135}$$

$$\tfrac{1}{6}ku^{k-1} + \left(\frac{2}{45}k^5 + \frac{7}{72}k^4\right)u^{k-2}$$
$$\leqslant (B_4 - A_4(f))u^{k-4} + R_3 \quad (k \equiv 0 \pmod 6)) \tag{136}$$

and

$$\left(\frac{2}{45}k^5 - \frac{7}{72}k^3\right)u^{k-2} \leqslant (B_4 - A_4(f))u^{k-4} + R_3 \quad (k \equiv 1, 5 \pmod 6)). \tag{137}$$

We see from (136), (137), (129) and (126) that $u < 3982$ if $k = 12$ and $u < 8060$ if $k = 13$. On the other hand, we apply (119) and (103) to exclude these possibilities. Further, we see from (135), (137), (128) and (126) that $u \leqslant 1300$ if $k = 9$ and $u \leqslant 3950$ if $k = 11$. In view of (88) and Lemma 18, the first case is not possible. Let $k = 11$. Then $\tau = 0$ and $35 \leqslant y \leqslant 52$. Further, we observe $[(y + 22)/64] - [y/64] - [22/64] = 1$ if $y \geqslant 42$ to derive from Lemma 15 that $y \leqslant 41$. Further, corresponding to every y with $35 \leqslant y \leqslant 41$, there is precisely one value of x given by (119) and (88). Finally, we count the power of 2 on both the sides of (14) in each of these seven cases to arrive at a contradiction.

Thus, we may assume that $k \not\equiv 2, 4 \pmod 6$ and $k \geqslant 15$. Then, we apply (131) to derive from (135), (136) and (137) that

$$\left(\frac{2}{45}k^5 - \frac{7}{72}k^3\right)u^{k-2} \leqslant (B_4 - A_4(f))u^{k-4} + R_3 \quad (k \not\equiv 2, 4 \pmod 6)). \tag{138}$$

Further, by (98) and (126), we see that

$$B_4 - A_4(f) < B_4 \leqslant 3k^{12}/40, \quad R_3 \leqslant 4k^{15}u^{k-5}/145. \tag{139}$$

Now, we combine (138), (139) and (131) to obtain

$$\frac{2}{45}k^5 \leqslant \frac{3}{40}\frac{k^{12}}{u^2} + \frac{4}{145}\frac{k^{15}}{u^3} + \frac{7}{72}k^3 \leqslant \frac{15}{32}k^4 + k^3$$

which implies that $k \leqslant 12$. □

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