Proc. Indian Acad. Sci. (Math. Sci.), Vol. 93, Nos 2 & 3, December 1984, pp. 109–116

On the ratio of values of a polynomial

T N SHOREY

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400 005, India

Abstract. For given positive integers a and b, the equation $a(x+1)\ldots(x+k) = b(y+1)\ldots(y+k)$ in positive integers is considered. More general equations are also considered.

Keyword. Exponential diophantine equations.

1. Introduction

We shall always assume that a, b, x, y and k are positive integers satisfying $b > a$ and $x - y \geqslant k$. We consider the equation

$$a(x+1)\ldots(x+k) = b(y+1)\ldots(y+k). \tag{1}$$

We re-write (1) as

$$u_k + A_1 u_{k-1} + \ldots + A_k u_0 = 0 \tag{2}$$

where

$$u_n = ax^n - by^n, \quad 0 \leqslant n \leqslant k$$

and $A_1, \ldots, A_k$ are positive integers given by

$$(X+1)\ldots(X+k) = X^k + A_1 X^{k-1} + \ldots + A_k.$$

Let α be a real number given by

$$ax^\alpha = by^\alpha.$$

Thus

$$\alpha = \left(\log\frac{b}{a}\right)\left(\log\frac{x}{y}\right)^{-1} > 0.$$

We prove:

THEOREM 1. *There exist effectively computable positive numbers C_1, C_2 and C_3 depending only on a and b such that* (1) *with $k \geqslant C_1$ implies that*

$$C_2 k^2 x^{-1} < k - \alpha < C_3 \eta \tag{3}$$

where

$$\eta = \min(k^2 x^{-1}, x^{-1/3}).$$

In particular, (1) implies

$$0 < k - \alpha < 1, \quad k \geqslant C_1.$$

Thus we have:

Corollary 1. *Equation* (1) *implies that either* $k < C_1$ *or* $k = [\alpha+1]$.

For given positive integers a, b and $k \geqslant 3$ such that the binary form $aX^k - bY^k$ is irreducible over rationals, it follows from a theorem of Schinzel [3] that (1) has only finitely many solutions in x and y. We remark that this result is not effective.

We apply an estimate of Baker [1] on linear forms in logarithms to prove:

Theorem 2. *Equation* (1) *implies that* $\max(x, y, k)$ *is bounded by an effectively computable number depending only on a, b and the greatest prime factor of xy.*

It follows from theorem 1 that (1) implies

$$x \geqslant C_4 k^3$$

where $C_4 > 0$ is an effectively computable number depending only on a and b. Further it follows from Crammer's conjecture on distance between consecutive primes that (1) implies

$$(\log x)^2 \geqslant C_5 k$$

where $C_5 > 0$ is an absolute constant. We apply an inequality of van der Poorten [2] on p-adic linear forms in logarithms to obtain the following result.

Theorem 3. *Equation* (1) *implies that*

$$\log x \leqslant C_6 k$$

where $C_6 > 0$ *is an effectively computable number depending only on a, b and the greatest prime factor of* $y(x-y)$.

We state the following direct consequence of theorem 3.

Corollary 2. *Equation* (1) *implies that* $\max(x, y)$ *is bounded by an effectively computable number depending only on a, b, k and the greatest prime factor of* $y(x-y)$.

Now we apply an argument of theorem 1 to a more general equation. For positive integers m and H, denote by $S(m, H)$ the set of all polynomials

$$P(X) = B_0 X^m + B_1 X^{m-1} + \ldots + B_m$$

where $B_0 > 0$, $B_1 > 0$ and $B_2, \ldots, B_m$ are non-negative integers not exceeding H. For positive integers a_1, b_1, x_1 and y_1 with $b_1 > a_1$ and $x_1 > y_1$, let β be a positive real number given by

$$a_1 x_1^\beta = b_1 y_1^\beta.$$

Then we have:

Theorem 4. *Let* $\delta > 0$ *and* $P \in S(m, H)$. *Let* a_1, b_1, x_1 *and* y_1 *with* $b_1 > a_1$ *and* $x_1 > y_1$ *be positive integers. For* $l_1 \in \mathbf{Z}$ *with*

$$|l_1| \leqslant m^{-1} x_1^{m-1-\delta}.$$

suppose that

$$a_1 P(x_1) = b_1 P(y_1) + l_1. \tag{4}$$

There exist effectively computable positive numbers C_7, C_8 *and* C_9 *depending only on* δ, a_1,

b_1 and H such that for every $m \geqslant C_7$, we have

$$C_8 x_1^{-1} < m - \beta < \min(C_9 x_1^{-1}, 1). \tag{5}$$

Thus (4) implies

$$0 < m - \beta < 1, \quad m \geqslant C_7.$$

By taking $P(X) = X^m + X^{m-1} + \ldots + 1$, we see from theorem 4 that

$$a_1 \frac{x_1^{m+1} - 1}{x_1 - 1} = b_1 \frac{y_1^{m+1} - 1}{y_1 - 1} + l_1$$

with $m \geqslant C_7$ implies (5).

2. Linear forms in logarithms

We shall need the following results on linear forms in logarithms for the proof of theorems 2 and 3. Let $\alpha_1, \ldots, \alpha_n$ be non-zero rational numbers of heights not exceeding $A_1, \ldots, A_n$ respectively, where we assume that $A_j \geqslant 3$ for $1 \leqslant j \leqslant n$. The height of a rational number E/F with $(E, F) = 1$ is defined as $\max(|E|, |F|)$. Put

$$\Omega' = \prod_{j=1}^{n-1} \log A_j \quad \text{and} \quad \Omega = \Omega' \log A_n.$$

THEOREM A (Baker [1]). *There exist effectively computable absolute constants $C_{10} > 0$ and $C_{11} > 0$ such that the inequalities*

$$0 < |\alpha_1^{b_1} \ldots \alpha_n^{b_n} - 1| < \exp(-(C_{10} n)^{C_{11} n} \Omega \log \Omega' \log B)$$

have no solution in rational integers $b_1, \ldots, b_n$ of absolute values not exceeding $B(\geqslant 3)$.

THEOREM B (van der Poorten [2]). *Let $p > 0$ be a prime number. There exist effectively computable absolute constants $C_{12} > 0$ and $C_{13} > 0$ such that the inequalities*

$$\infty > \operatorname{ord}_p(\alpha_1^{b_1} \ldots \alpha_n^{b_n} - 1) > (C_{12} n)^{C_{13} n} \Omega (\log B)^2 \frac{p}{\log p}$$

have no solution in rational integers $b_1, \ldots, b_n$ with absolute values at most $B(\geqslant 3)$.

3. Proof of theorem 1

Denote by $c_1, c_2, \ldots, c_{18}$ effectively computable positive numbers depending only on a and b. We may assume that $k \geqslant c_1$ with c_1 sufficiently large. Then, since $x \neq y$, observe that $u_k u_{k-1} \neq 0$ (see lemma 1 of [4]). Further $c_1 \leqslant k \leqslant x - y < x$ and (1) implies that none of $(x+1), \ldots, (x+k)$ is a prime number. Therefore it follows from the well-known results on difference between consecutive primes* that

$$x \geqslant k^{3/2}. \tag{6}$$

---

* It is not necessary to use the results on difference between consecutive primes.

Now we sharpen inequality (6). Denote by d the greatest common divisor of $(x+1)\ldots(x+k)$ and $(y+1)\ldots(y+k)$. By (1), we see that $(x+1)\ldots(x+k)/d$ divides b. In particular

$$x^k < (x+1)\ldots(x+k) \leqslant bd.$$

By an argument of Erdos, the contribution in d from primes not exceeding $2k$ is at most

$$k^k(x+k)^{\pi(2k)}$$

Further, by (1), the contribution in d of primes greater than $2k$ does not exceed

$$(x-y+k)^{2k}((2k)!)^{-1}.$$

By (1), we see that $ax^k < b(y+k)^k$ which implies

$$x-y \leqslant c_2k^{-1}x+2k. \tag{7}$$

Hence

$$x^k < bd \leqslant (c_3k)^k x^{\pi(k)}\left(\max\left(\frac{x}{k^2},1\right)\right)^{2k}$$

which, together with (6), implies

$$x \geqslant c_4k^3. \tag{8}$$

Combining (7) and (8), we have

$$x-y \leqslant 2c_2k^{-1}x. \tag{9}$$

All the summands on the left side of (2) cannot be of the same sign. Therefore, since $x > y$, we see that $u_k > 0$ which, together with (9), implies

$$x-y \geqslant c_5k^{-1}x. \tag{10}$$

Suppose that $u_{k-1} > 0$. Observe that

$$u_k - xu_{k-1} = b\,y^{k-1}(x-y).$$

Therefore, by (9) and (10), we see that

$$u_k \geqslant c_6k^{-1}x^k.$$

Notice that

$$A_n \leqslant \binom{k}{n}k^n \leqslant k^{2n}, \quad 1 \leqslant n \leqslant k. \tag{11}$$

Re-write (1) as

$$u_k + A_1u_{k-1} = -A_2u_{k-2} - \ldots - A_ku_0. \tag{12}$$

The absolute value of the left side of (12) is at least

$$\max(u_k, A_1u_{k-1}) \geqslant c_6k^{-1}x^k.$$

On the other hand, it follows from (11) and (8) that the absolute value of the right side of (12) is at most

$$c_7k^4x^{k-2}.$$

Comparing these estimates, we obtain

$$x \leqslant c_8 k^{5/2}.$$

Now we apply (8) to conclude that $k \leqslant c_9$ which is not possible if $c_1 > c_9$.

Hence $u_k > 0$ and $u_{k-1} < 0$. By continuity, we see that

$$0 < k - \alpha < 1. \tag{13}$$

Re-write (1) as

$$u_k = -A_1 u_{k-1} - A_2 u_{k-2} - \dots - A_k u_0.$$

Further

$$|u_{k-1}| = -u_{k-1} = b\, y^{k-1}(1 - (x/y)^{-(\alpha - k + 1)}).$$

Now it follows from (9), (10), (13) and $A_1 = \dfrac{k(k+1)}{2}$ that

$$c_{10}^{-1}(\alpha - k + 1)kx^{k-1} \leqslant |A_1 u_{k-1}| \leqslant c_{10} kx^{k-1}.$$

Similarly

$$|A_2 u_{k-2}| \leqslant c_{11} k^3 x^{k-2}$$

which, together with (8), implies that

$$|A_2 u_{k-2}| \leqslant c_{12} x^{k-1}.$$

Further, by (11) and (8),

$$|A_3 u_{k-3} + \dots + A_k u_0| \leqslant c_{13} k^6 x^{k-3} \leqslant c_{14} x^{k-1}.$$

Therefore $u_k \leqslant 2c_{10} kx^{k-1}$. Also, by (9), (10) and (13),

$$c_{15}\left(\frac{k-\alpha}{k}\right)x^k \leqslant u_k \leqslant c_{16}\left(\frac{k-\alpha}{k}\right)x^k.$$

Consequently $k - \alpha < c_{17} k^2 x^{-1} < 1/4$. Now notice that $x^{-k} u_k \geqslant (2c_{10})^{-1} kx^{-1}$ and hence $k - \alpha > c_{18} k^2 x^{-1}$. Thus we have

$$c_{18} k^2 x^{-1} < k - \alpha < c_{17} k^2 x^{-1}$$

which, together with (8), implies (3). This completes the proof of theorem 1.

4. Proof of theorems 2 and 3

Denote by $v_1, \dots, v_7$ effectively computable positive numbers depending only on a, b and the greatest prime factor of xy. If $k = 1$, then we re-write (1) as

$$ax - by = b - a \neq 0$$

which, by theorem A, implies that $\max(x, y) \leqslant v_1$. Thus we may assume that $k > 1$. Further we apply again theorem A to obtain

$$x - y \geqslant x(\log x)^{-v_2}$$

which, together with (9), implies

$$k \leqslant (\log x)^{v_3}. \tag{14}$$

Let δ be the least non-negative integer such that $u_{k-\delta} \neq 0$. Observe that $\delta \in \{0, 1\}$, since $x \neq y$. By theorem A,

$$|u_{k-\delta}| \geqslant x^{k-\delta}(k \log x)^{-v_4}$$

On the other hand, it follows from (2), (11) and (14) that

$$|u_{k-\delta}| \leqslant x^{k-\delta-1} (\log x)^{v_5}.$$

Comparing these estimates, we obtain

$$x \leqslant k^{v_6}$$

which, together with (14), implies that $\max(x, y, k) \leqslant v_7$. This completes the proof of theorem 2.

Proof of theorem 3

Denote by $v_8, v_9, \ldots$ effectively computable positive numbers depending only on a, b and the greatest prime factor of $y(x-y)$. We may assume that $x \geqslant v_8$ with v_8 sufficiently large, otherwise the assertion follows immediately. Suppose that

$$\log x > k.$$

It follows from (1) that $x - y$ divides

$$\Delta = (b-a)(y+1)\ldots(y+k) \neq 0.$$

Thus, for a prime p dividing $x-y$, we have

$$\mathrm{ord}_p(x-y) \leqslant \mathrm{ord}_p(\Delta).$$

Let $1 \leqslant n_0 \leqslant k$ satisfy

$$\mathrm{ord}_p(y+n_0) \geqslant \mathrm{ord}_p(y+n)$$

for $n = 1, \ldots, k$. By theorem B, we have

$$\mathrm{ord}_p(y+n_0) \leqslant v_9 (\log\log x)^3.$$

Therefore

$$\mathrm{ord}_p(x-y) \leqslant \mathrm{ord}_p(\Delta) \leqslant v_9 (\log\log x)^3 + v_{10} k.$$

Hence

$$\log(x-y) \leqslant v_{11}((\log\log x)^3 + k).$$

On the other hand we see, by (10) which is also valid for $k < c_1$, that

$$\log(x-y) \geqslant \log x - \log k - v_{12}.$$

Comparing these estimates, we obtain $\log x \leqslant v_{13} k$. This completes the proof of theorem 3.

5. Proof of theorem 4

In the proof of theorem 4, we omit some details which already appear in the proof of theorem 1. Denote by $c_{19}, c_{20}, \ldots$ effectively computable positive numbers depending only on δ, a_1, b_1 and H. We may assume that $m \geqslant c_{19}$ with c_{19} sufficiently large. Let $P \in S(m, H)$ be given by

$$P(X) = B_0 X^m + B_1 X^{m-1} + \ldots + B_m.$$

Re-write (4) as

$$B_0 U_m + B_1 U_{m-1} + \ldots + B_m U_0 = l_1 \tag{15}$$

where

$$U_n = a_1 x_1^n - b_1 y_1^n; \quad 0 \leqslant n \leqslant m.$$

Observe that (4) implies

$$c_{20} m^{-1} x_1 < x_1 - y_1 < c_{21} m^{-1} x_1. \tag{16}$$

Observe that

$$U_m - x_1 U_{m-1} = b_1 y_1^{m-1}(x_1 - y_1)$$

which implies

$$\max(|U_m|, |U_{m-1}|) \geqslant c_{22} m^{-1} x_1^{m-1}.$$

Thus all the summands on the left side of (15) cannot be of the same sign. Then, since $x_1 > y_1$, we see that $U_m > 0$. Suppose $U_{m-1} > 0$. Then

$$U_m \geqslant c_{23} x_1^{m-1}.$$

Further (15) implies that $x_1 \leqslant c_{24}$ which, by (4), is not possible if c_{19} is sufficiently large. Hence $U_m > 0$ and $U_{m-1} < 0$. Thus

$$0 < m - \beta < 1. \tag{17}$$

Let $N = [(\log m)^2]$. Using (16) and (17), we have

$$c_{25}\left(\frac{m-\beta}{m}\right) x_1^m < U_m < c_{26}\left(\frac{m-\beta}{m}\right) x_1^m,$$

$$c_{27}\left(\frac{\beta - m + 1}{m}\right) x_1^{m-1} < |U_{m-1}| < c_{28} m^{-1} x_1^{m-1},$$

$$|U_{m-r}| \leqslant c_{29} r m^{-1} x_1^{m-r}, \quad 2 \leqslant r \leqslant N$$

and

$$|U_{m-r}| \leqslant c_{30} x_1^{m-r}, \quad N < r \leqslant m.$$

In view of these estimates, (15) implies (5). This completes the proof of theorem 4.

Remark

In theorem 4, the restriction on $|l_1|$ can be relaxed to $|l_1|$ does not exceed constant times $m^{-1} x_1^{m-1}$.

Acknowledgement

The author is thankful to Dr R Balasubramanian for useful comments and discussions.

References

[1] Baker A 1977 *Transcendence theory: Advances and applications* (London and New York: Academic Press) 1–27

[2] van der Poorten A J 1977 *Transcendence theory: Advances and applications* (London and New York: Academic Press) 29–57

[3] Schinzel A 1968 *Comment. Pontif Acad. Sci.* **2** pp. 9

[4] Shorey T N 1982 *Acta Arith.* **41** 255–260