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Airworthiness of aircraft. Part 1. A stochastic model

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Abstract. It is argued that a stochastic approach to airworthiness analysis is necessary because of the inherent random variations in aircraft performance. It is then shown that the history of any performance variable in fleet service is best regarded as a 'stochastic corrective process', characterised by slow deterioration punctuated by rapid improvement at overhaul. Considering specifically the engine-out take-off climb performance of an aircraft, a model is proposed by taking into account gradient changes due to the deterioration of airframe, engine and propeller (if present), the frequency and effectiveness of overhaul, and the occurrence of deviations in the gradient achieved on any flight from the true capability of the aircraft. In analysing the effect of overhaul a distinction is made between maintenance by rectification and by replacement. In the model proposed a total of 14 parameters govern the probability distribution of the achieved gradient in fleet history, and consequently also the incident rate during take-off climb.

Keywords. Airworthiness; flight safety; stochastic modelling; take-off climb.

1. Introduction

Safety in aviation depends on a number of factors such as the ability of the aircrew, the operational facilities available, environmental factors, the performance of the aircraft, and the efficiency of maintenance, inspection, certification etc. It does not seem possible to obtain absolute safety or reliability, if only because of the inherent random variations affecting system performance that occur within the system as well as in the surrounding environment. We propose in this paper a model for the effect of such variations which determine the achieved safety levels of aircraft operation.

Every aircraft is built according to certain airworthiness standards, which (among other things) specify performance levels to be achieved in various phases of flight. The objective of these standards is to ensure that variations in aircraft performance, for whatever reason (*e.g.* airplane drag rise, engine power loss, or adverse flight conditions), do not lead, except possibly on very rare occasions, to a performance below an acceptable critical level. We confine our attention here largely to the take-off climb phase partly because of recent controversies in the country regarding what constitutes safe climb performance, and partly because the detailed study of one phase of flight is enough to illustrate the principles of the method we employ.

The standards for take-off climb performance were discussed extensively more than two decades ago (Cook and Weaver 1948, ICAO 1953), but the viewpoint adopted in these attempts at the *formulation* of a satisfactory code does not suffice to *analyse*

A list of symbols appears at the end of the paper.

a given pattern of airline operation, certification, maintenance etc. The importance of the related question of performance deterioration in service has long been recognized (e.g. Hardingham 1952); nevertheless we are aware of no work that assesses quantitatively the effect of deterioration and maintenance processes on airworthiness. It is the purpose of the present paper to provide a model for such an assessment; a computer simulation of the model will be described in part 2 of the paper.

2. The need for a statistical approach

From the viewpoint of airworthiness codes, a basic fact in aeroplane performance is that no two apparently identical aeroplanes flying under apparently identical conditions develop the same performance. To quote a specific example, Tye (1952) found that, in a fleet of some 30 aeroplanes, the average rate of climb obtained was 285 ft/min; but on about 10% of the occasions, the rate of climb was below 235 ft/min. *Such variations are therefore a fact of life.* There are many reasons for this variability in performance. Even among aircraft of the same type, differences exist in drag due to dents or other damage to the airframe, inaccuracies in flap setting, differences in the state of leading edge boots, in leaks through doors and gaps; and in power due to the engine setting etc. Also, during fleet service, the drag may increase due to the aerodynamic surfaces becoming dirty, the engine power will generally go down, and the propeller if present may deteriorate. These deterioration rates will in general be different among different airframes, engines and propellers.

In fact even the same aircraft when flown by different pilots on the same day will in general achieve different climb gradients! The reasons for this are that the flying techniques will vary among the pilots, and at various times for the same pilot, in setting forward speed, in overcoming yaw due to asymmetric thrust on engine failure, in controlling the engine. Further, atmospheric conditions like wind and turbulence also affect performance.

The deterioration mentioned above is of course sought to be overcome through a series of maintenance checks and engine and propeller changes. If the duration of withdrawals of the aircraft for such checks or changes is short compared to the interval between them, we may think of the aircraft as returning to service almost immediately, usually with an improved performance and possibly new deterioration rates, and the cycle continues again. Although the maximum time intervals permitted between such checks and changes are specified for each type of aircraft, a rigid schedule can rarely be maintained in actual practice. Furthermore, the improvements effected at the checks and changes also depend on a variety of factors, and cannot always be identical.

It is this basic variability in aircraft performance, arising from a large number of factors, that forces on us a statistical approach to the formulation of airworthiness codes as well as to the analysis of airworthiness problems.

3. Basic concepts underlying climb requirements

We illustrate our approach by a detailed analysis of engine-out climb performance of a twin-engined aircraft. The basic variable here is the climb gradient G' (usually

expressed in per cent units) achieved by the aircraft during take-off when one of its power units fails or is shut down for some reason. (In the rest of this paper we refer in general to 'engine failure', with the understanding that the words include even those shut-downs not caused by a genuine failure, but due to such reasons as a false fire alarm.)

The basic design problem here is to reconcile the safety requirement for ability to climb on only one engine with the economic requirement for avoiding over-power during cruise. The problem is usually solved by introducing special take-off power ratings, but the climb gradients obtained (of the order of a few per cent) depend on a small difference between two large quantities (namely thrust and drag; see section 5). Small fluctuations in engine and airframe parameters therefore play a critical role in designing the safety of the overall system.

We begin by discussing certain basic concepts underlying the formulation of a suitable statistical model for this problem.

3.1. *Datum surface*

The datum performance of an aircraft is that of an imaginary aeroplane which would just use up the distances for take-off and landing, and fly over the terrain enroute with minimum allowable clearance. The datum surface is defined (Cook and Weaver 1948) as the surface below which flight implies conditions predisposing to an accident. Whenever the flight path of an aircraft lies below this datum surface, an 'incident' may be said to have occurred. The datum surface has to be fixed *a priori* from operational considerations and experience.

3.2. *Acceptable level of safety*

The inability to achieve at least the datum or critical performance is here called a performance failure. If G'_{cr} represents this performance, and $p(G')$ is the probability density* of the achieved climb gradient G', then the probability of performance failure is

$$\text{pr (perf flr)} = P(G'_{cr}) = \int_{-\infty}^{G'_{cr}} p(G')\, dG'. \tag{1}$$

If this and engine failure probabilities are assumed to be statistically independent, the incident probability—which would be a measure of the risk—is

$$\epsilon \equiv \text{pr (inc)} = \text{pr (eng flr)} \cdot \text{pr (perf flr)}. \tag{2}$$

To arrive at a quantitative value for the acceptable risk the Standing Committee on Performance of the International Civil Aviation Organization (ICAO 1953) argued that the earlier codes having excluded single-engined transport aeroplanes, on which each time an engine failure occurs an incident would also necessarily occur, an incident rate equal to the engine failure rate cannot be tolerated; and hence the power unit failure rate (approximately 10^{-4} per hour at the time of the ICAO report),

*We use the symbols pr, p and P to denote any probability, density and cumulative distribution respectively; the particular variables they concern are shown as their arguments.

was taken as the upper limit to the acceptable incident rate. Further, since the codes had not excluded twin-engined aeroplanes, the probability of the simultaneous failure of both engines on such a plane (about 10^{-8} per hr) would set a lower limit for the acceptable incident rate. With some further detailed analysis, the Committee concluded that an acceptable 'design incident rate' would be in the range 2×10^{-6} to 7×10^{-6} per flight.

As Cook and Weaver (1948) point out, all incidents do not automatically lead to accidents; apart from performance shortfalls, occurrence of accidents depends on the terrain, the airplane's structural and ditching characteristics, fire-proofing arrangements, etc. Perhaps from considerations such as these, the British Civil Airworthiness Regulations (ARB 1966) have prescribed the total probability of a catastrophe (defined as an accident 'which results in the loss of the aeroplane and/or in fatalities') to be less than 10^{-7} per flight or per hour as appropriate. For comparison one may note that for a life expectancy of about 60 years the corresponding average risk is about 2×10^{-6} per hour.

3.3. *Standards and performance margin in airworthiness codes*

In the ICAO Airworthiness Technical Manual (ICAO 1974) and the BCAR the standards are specified in terms of so-called 'gross' and 'net' performances. BCAR defines the 'gross' as the quantity 'as likely to be exceeded as not' by aeroplanes of a given type. In other words, the gross performance is the statistical median, although for all practical purposes it may be taken to be equal to the mean (Narasimha and Ramani 1975, unpublished). In the above codes the value of G'_{cr} in (1) is specified as the minimum required 'net' gradient. The difference between G'_{cr} and the gross performance is called the 'margin'; a minimum value for this margin (and hence for the minimum gross performance) is specified to ensure that the incident rate in (2) is sufficiently low.

4. Nature of aircraft performance history in service

4.1. *General remarks*

Any performance variable of an aircraft, e.g. cruise speed, climb gradient, radius of turn, may be expected to vary with time in the manner shown schematically in figure 1. Let at time $t = 0$ the first aircraft enter fleet service with a value u^* for the performance variable under discussion. With aging, this performance usually declines for a variety of reasons as mentioned in section 2. Such deterioration may be thought to occur along the relatively smooth curves shown in the main figure, or, as is much more likely to be found in reality, along the rather zig-zag path that is shown in the inset. At some time say $t = t_1$, the aircraft is withdrawn for a maintenance check or (possibly partial) overhaul of some kind, and returns to fleet service generally with a better performance. As the aircraft continues in service its performance generally deteriorates again till the next overhaul effects another improvement, and so on; the cycle repeats over the life of the aircraft. The performance history of the aircraft is therefore characterized by a slow random drift due to deterioration, punctuated at intervals by relatively sudden jumps at overhauls. Such histories are best regarded

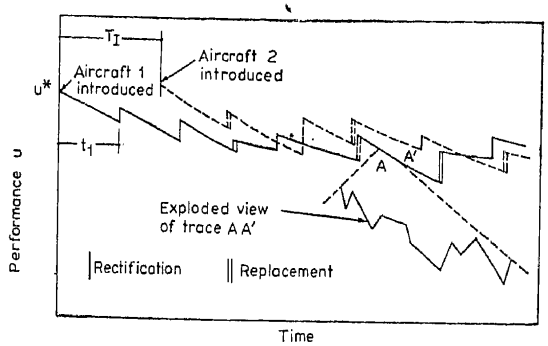

Figure 1. Typical performance history of aircraft in service.

as examples of what has been called a 'stochastic corrective process' (Narasimha 1975).

Further aircraft added to the fleet at various times undergo similar histories, and it becomes necessary to describe the performance of the fleet by an appropriate probability distribution.

Before a detailed model for such processes is described, it is useful to recognize that maintenance operations can be of two distinct kinds, depending on whether they involve rectification or replacement of components (Narasimha 1975). The first seeks to improve performance essentially by 'mending' things: e.g. readjustment of control rigging, removal of airframe dents, cropping propeller blades. In maintenance by replacement, on the other hand, a whole component—e.g. the engine or a control unit—may be replaced by another that is often either new or as good as new. In practice, both forms of maintenance may be used, either simultaneously or at separate times, or possibly on different components. For the sake of clarity, we will in the following use the word 'overhaul' for a maintenance operation of either kind, 'check' for maintenance by rectification and 'change' for maintenance by replacement.

The above considerations are very general and should be widely applicable, but are used here to analyse the aircraft take-off climb problem.

5. Engine-out climb gradient

The true climb gradient capability of an aircraft (that, for greater generality*, is assumed driven by a propeller) may be defined as

$$G = \frac{\eta P}{WV} - \frac{D}{L} \tag{3}$$

where η is the propeller efficiency, P the power delivered by engine, W the aircraft weight, V the flight speed and D and L are the aircraft drag and lift. Usually, the flight speed and aircraft attitude are prescribed for each segment of the take-off flight path. The weight being a well-defined and (in principle) easily measured quantity,

*Analysis of jet aircraft should be simpler, and would proceed by replacing $\eta P/V$ in (3) by the engine thrust.

it is convenient to consider the gradient values reduced to the manufacturer's recommended maximum weight for the altitude and temperature at take-off.

In general, the gradient actually achieved in any given flight can be written as

$$G' = G + g_S \tag{4}$$

where g_S is what we shall call ' flight scatter '. Since the average value of the achieved gradient over a large number of flights may be taken (generally) as G itself, we have (bars denoting such averages)

$$\overline{G'} = \overline{G}\,; \;\; \bar{g}_S = 0. \tag{5}$$

The number of flight tests to be conducted to attain a specified accuracy in determining G depends on the flight scatter; if the scatter is large, then more tests will have to be conducted and *vice versa*.

When an engine failure occurs in flight it is often without any previous warning, so the flight scatter tends to be larger than in test flights. If g_S is determined by deliberately cutting an engine during *flight tests*, we need to introduce a ' surprise factor ' k (>1), so that the achieved gradient after an *unforeseen* engine failure is

$$G' = G + k\, g_S. \tag{6}$$

All gradients we discuss below are those achieved following engine failure, and hence the associated probability distributions are all conditional on engine failure. For brevity, however, we will rarely state this explicitly in what follows.

5.1. *Gradient history in service*

We may now summarise the following sequence of events in the ' gradient history ' of an aircraft fleet.

(a) A new aircraft enters service after production flight tests in which the achieved gradient may have to satisfy some criterion for certification. Let the true climb gradient capability of this aircraft be G^*.

(b) In service, the aerodynamic characteristics of the airframe deteriorate with time, leading to a loss of climb gradient.

(c) Also, due to engine and propeller deterioration there is a further loss of climb gradient.

(d) After having been in service for a time T_A, the aircraft is withdrawn for an airframe maintenance check, either due to unsatisfactory performance or to pilot-reported defects, or because the scheduled airframe overhaul time has been completed. After such a check the drag standard will generally improve, and the aircraft returns to fleet service with a better gradient.

(e) Similarly, after a time T_E, the aircraft is withdrawn for an engine change. The engine is replaced and the aircraft returns to fleet service, in general with a different gradient.

(f) Similarly with propellers, which are changed at times T_P, say..

(g) Steps (b) to (f) will repeat for each aircraft introduced into the fleet throughout its service life.

(h) At intervals of time T_I, new aircraft are introduced into the fleet, and these go through the steps (b) to (g).

5.2. *The components of the gradient*

If variations of the different parameters are small, we can write (3) as

$$G = G_0 + \frac{\partial G}{\partial D} \triangle D + \frac{\partial G}{\partial P} \triangle P + \frac{\partial G}{\partial \eta} \triangle \eta + \text{higher order terms}, \tag{7}$$

where G_0 is the basic (design or average) value of the gradient for new aircraft of the type being considered; $\triangle D$, $\triangle P$ and $\triangle \eta$ are variations from similar basic values of D, P and η respectively, and the partial derivatives are all evaluated at the basic design point.

If the higher order terms in (7) can be neglected, we can adopt linearized aircraft performance theory and write

$$G = G_0 + g_A + g_E + g_P \tag{8}$$

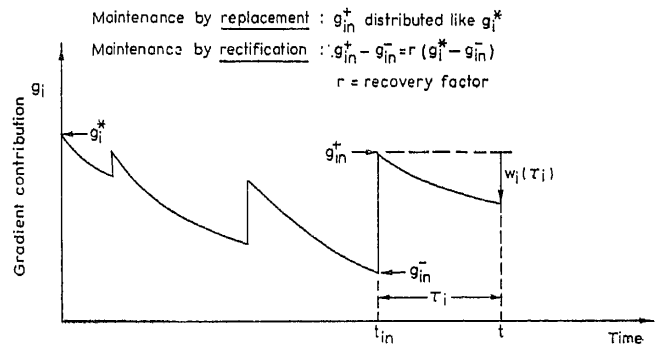

Figure 2. Variation with time of gradient contribution from component *i*.

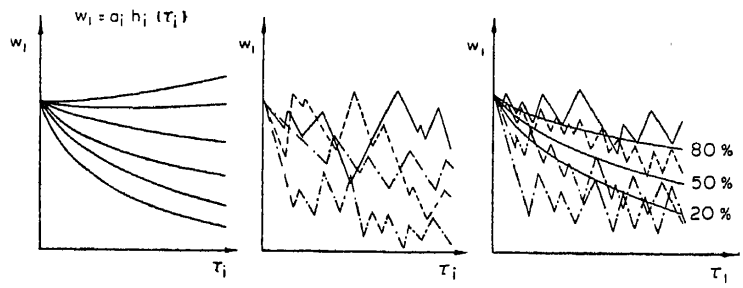

Figure 3. Realizations of stochastic processes illustrating different deterioration models. (a) Random slide model. (b) Random walk model. (c) Equivalent random slide from a random walk model. The probability of loss in performance less than that represented by the curve is indicated in percentage units on each curve.

where g_A, g_E and g_P are the corresponding first order contributions to G from variations in airframe drag, engine power and propeller efficiency respectively. Such variations exist even in new aircraft, for which we can write (denoting conditions when new by a star).

$$G = G_0 + g^*_A + g^*_E + g^*_P. \tag{9}$$

6. Models for deterioration

Consider any one of the components g_i ($i = A$, E or P). If the most recent overhaul occurred at time t_{in} for component i, the age of that component at time t is defined as

$$\tau_i = t - t_{in}. \tag{10}$$

If further the performance just after the last overhaul is

$$g_i\,(t^+_{in}) \equiv g^+_{in} \tag{11}$$

say, then we may assume

$$g^+_{in} - g_i(t) = w_i(\tau_i) \tag{12}$$

where w_i is the performance loss at age τ_i. Figure 2 illustrates all these quantities.

The loss function $w_i(\tau_i)$ is usually itself a stochastic process, and may be modelled in one of two ways, both illustrated in figure 3 and already mentioned in section 4. The first is what may be called a 'random slide' model, in which

$$w_i\,(\tau_i) = a_i\, h_i\,(\tau_i), \tag{13}$$

where the a_i are random variables but the $h_i\,(\tau_i)$ are ordinary (deterministic) functions. In this case, the nature of variation in time is fixed between any two overhauls, but from one overhaul to the next the factor a_i varies randomly. For example, if $h_i\,(\tau_i) = \tau_i$, the loss is always linear in time but the rate is a random variable.

In the second, which may be called a 'random walk' model, the loss $w_i\,(\tau_i)$ can vary zig-zag, although a mean drift downwards in performance should generally be observed. This is perhaps a more realistic model especially for intricate systems whose performance may fluctuate depending on environmental conditions or on a variety of minor checks whose effect is not worth taking into account individually. For example, the power developed by an engine exhibits effectively random fluctuations because of changes in outside air temperature and control system noise.

For certain purposes at least, it may be possible and convenient to replace a random walk model by an equivalent random slide model. Thus, if we are interested in fleet probability distributions rather than individual aircraft histories, we can always construct a random slide process in which the probability of failing to achieve any given performance in the fleet is the same as in a specified random walk model. This is done quite simply by averaging over a large ensemble of individual histories

and drawing iso-probability lines as illustrated in figure 3. These lines provide the paths in an equivalent random slide model. The main advantage of the random slide model is that it is simpler, and quicker to execute in a computer simulation, as we shall discuss elsewhere.

7. Models for the effect of overhauls

When maintenance is by rectification at a check, we may assume that the performance after the check will be better than before, but not better than when new. We may therefore introduce a 'recovery factor' defined by

$$r_i = (g_{in}^{+} - g_{in}^{-})/(g_i^{*} - g_{in}^{-}); \tag{14}$$

this represents the fraction of the total shortfall in performance from the new condition that is made up at the check. r_i may itself be a random variable, obeying an appropriate distribution in the range $0 \leqslant r_i \leqslant 1$.

If maintenance is by replacement, the performance of the new unit is clearly independent of the state of the old, replaced unit, and may be considered drawn from its own distribution.

In the take-off climb problem, airframe checks are clearly of the rectification type, whereas engine and propeller changes are of the replacement type. This does not mean that engines and propellers are never subject to any corrective action other than replacement, but rather that the effects of such relatively minor checks (e.g. compressor cleaning on a turbine engine) are most conveniently incorporated into a suitably modified effective deterioration model.

8. Parameters governing the climb problem

From (6), (8) and (12), we can now write the achieved gradient at any time t as

$$G'(t) = G_0 + \sum_i \left[g_i(t_{in}^{+}) - w_i(\tau_i)\right] + kg_S \tag{15}$$

where the summation is taken over $i = A$, E and P.

We may summarize our discussion by listing all the parameters that enter into the present model for the take-off climb problem.

(a) Time intervals between: (i) introduction of new aircraft (T_I), (ii) airframe maintenance checks (T_A), (iii) engine changes (T_E), (iv) propeller changes (T_P).

(b) Climb performance of new aircraft, resolved into the following components: (i) basic design value (G_0), (ii) variations due to airframe ($g_A{}^*$), (iii) variations due to engine ($g_E{}^*$), (iv) variations due to propeller ($g_P{}^*$).

(c) Deterioration parameters: the loss functions for (i) the airframe ($w_A(\tau_A)$), (ii) the engine ($w_E(\tau_E)$), (iii) the propeller ($w_P(\tau_P)$).

(d) Maintenance parameter: (i) recovery factor at airframe check (r_A).

(e) Flight scatter: (i) in preplanned test flights (g_S), (ii) surprise factor k.

This makes 14 parameters in all.

Two comments need to be made on the above list. Maintenance parameters for the engine and propeller do not appear explicitly, because it will be assumed that these are replaced by components which are effectively new, with performance described therefore by g_E^* and g_P^* respectively. Secondly, excepting G_0' all the parameters listed above are in general statistical, and so have to be specified by appropriate probability distributions. In any specific application, however, the variation in some of the parameters may well be considered deterministic (i.e., their probability density is a delta function).

9. Fleet distributions

The final goal of the model is to obtain for the fleet the distribution that at any time gives the probability

$$\text{pr (gradient capability} < G') = \mathrm{P}(G', t) \tag{16}$$

say, and the corresponding probability density

$$\mathrm{p}\,(G', t) = \partial \mathrm{P}(G', t)/\partial G'. \tag{17}$$

A fleet of aircraft will in general eventually attain a state of equilibrium, in which the performance distribution is independent of time. Before this asymptotic state is reached, however, there is a 'transient' during which the distribution 'relaxes' from the state characteristic of new aircraft to one characteristic of equilibrium.

Another quantity of interest is the total number of incidents during the service life of a fleet. This may be obtained from the ratio

$$\frac{\text{total number of take-offs with sub-critical performance}}{\text{total number of take-offs in service life of fleet}}.$$

If the number of take-offs is proportional to the number of aircraft and their time in service, the above ratio can be represented as an average of $p(G', t)$ over time, weighted by the fleet strength $n(t)$ at time t. Thus we have:

$$\text{overall performance failure probability } \bar{P}\,(G', t)$$

$$= \left[\int_0^t dt' \int_{-\infty}^{G'_{\mathrm{cr}}} dG'\ \mathrm{p}\,(G', t')\, n\,(t')\right] \Big/ \int_0^t n\,(t')\, dt'. \tag{18}$$

The total number of incidents can now be obtained by multiplying the above probability at the end of the service life of the fleet by the engine failure rate and the total number of take-offs.

10. Conclusions

We have formulated a stochastic model that describes the performance history of an aircraft fleet. The model can be simulated on a digital computer, as we shall show subsequently. Although, to be specific, the take-off climb problem only has been analysed, it is believed that the model is quite general, and should be valid not only for all other aircraft performance variables but also for any high-maintenance system, such as power stations.

Acknowledgements

The work reported here was inspired by an aircraft evaluation project conducted by Professor S Dhawan; we are grateful to him and to many colleagues who assisted on this project for giving us the benefit of their criticism.

List of symbols

a	random variable of the slide model in (13)
D	aircraft drag
G	true climb gradient
G_0	the basic value of the gradient for new aircraft
G'	achieved climb gradient
G'_{cr}	minimum required ' net ' gradient
G^*	climb gradient at production
g	gradient contribution to G in service
g_S	flight scatter
g^*	gradient contribution to G^* when new
g^-_{in}	gradient contribution from component i just before the nth check
g^+_{in}	gradient contribution from component i just after the nth check
h_i	deterministic function of time τ_i of the slide model in (13)
k	surprise factor
L	aircraft lift
$n(t)$	fleet strength at time t
$\mathrm{p}(x)$	probability density function
$\mathrm{pr}(X)$	probability of X
r_A	recovery factor at an airframe check
r_{in}	recovery factor for the ith component at the nth check
T	time interval between overhauls
T_I	time interval between introduction of aircraft into fleet service
t	time
t_H	duration of fleet history
t_{in}	time when the ith component is withdrawn for the nth check

u	any performance variable of the aircraft
V	flight speed
W	aircraft weight
w_i	performance loss at age τ_i
δ	Dirac delta function
Δ	small change of quantities
ϵ	incident probability
η	propeller efficiency in (3)
τ	age after overhaul

Superscripts

$'$	achieved gradient
$*$	value for new aircraft

Crown

—	steady state values

Subscripts

A	airframe
E	engine
P	propeller
i	summation index over A, E and P or otherwise

Abbreviations

ARB	Air Registration Board
AWTM	Airworthiness Technical Manual
BCAR	British Civil Airworthiness Requirements
ICAO	International Civil Aviation Organization

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