ORDER AND CHAOS IN FLUID FLOWS*

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INTRODUCTION

It comes as a great shock to many people, especially those who think of *all* technology as flowing from science (erroneously, in my view), to realize that the ancient problem of conveying water from point A to point B—a problem essentially solved thousands of years ago by experience, and one solved early in this century in terms of codified information for engineering—remains to this day an unsolved problem in physics. By this I mean that it is still not possible to predict, based *solely* on first principles such as for example Newton's Laws, how much water can be pushed through a given pipe with a given loss of pressure. The answer is of course known (figure 1), but cannot yet be

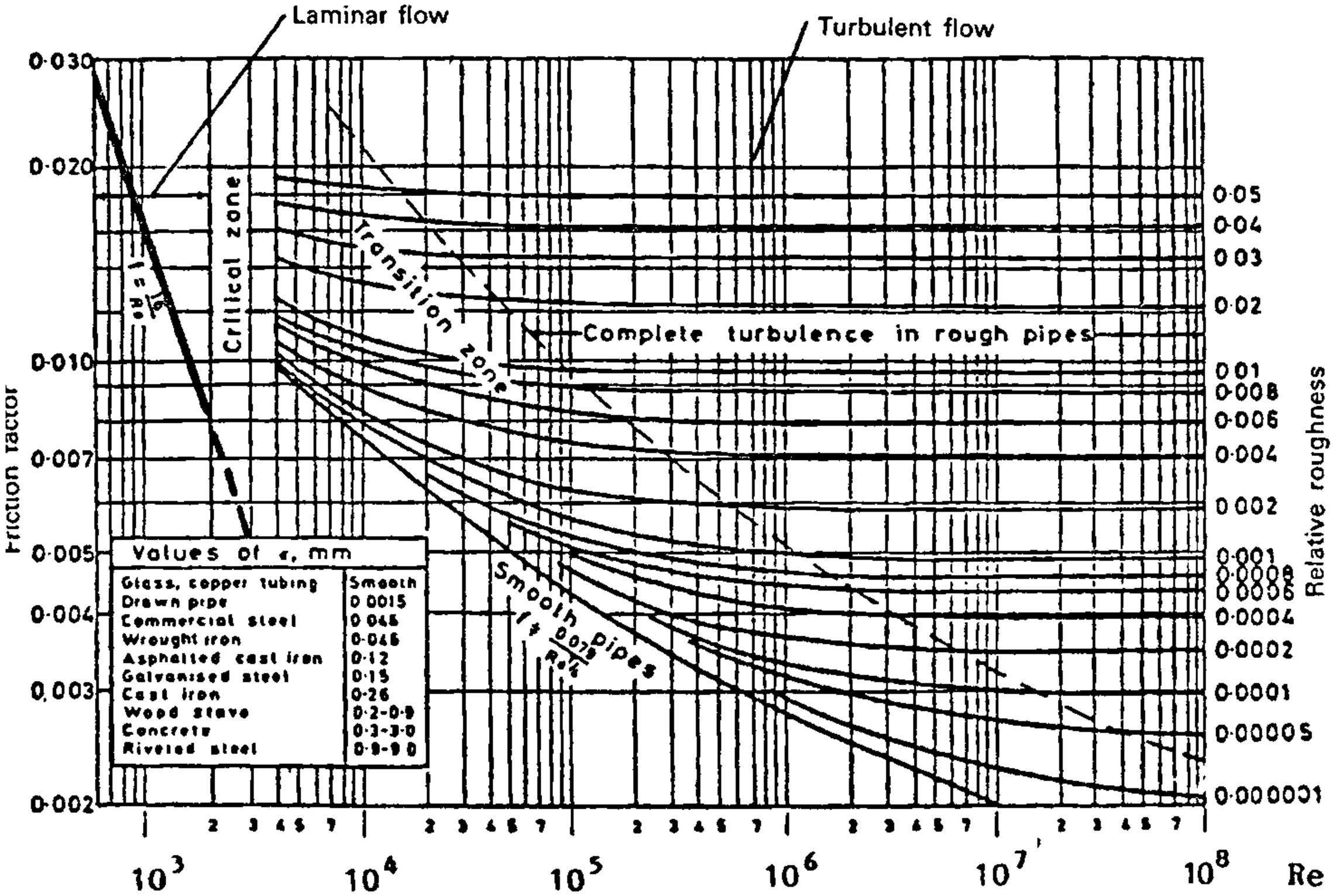

Figure 1. The 'Moody diagram', used widely by engineers for estimating the pressure differential required to push fluid through a pipe. The diagram, taken from a standard engineering handbook, presents friction losses per unit length of pipe (ordinate) at different Reynolds numbers (abscissa), for various values of surface roughness. The thick line at the top left, marked 'laminar flow', is the only information on this diagram that comes from a proper theory; all the other curves depend on experimental data in some form. Note the transition from laminar to turbulent flow at Reynolds numbers around 2000, in what is labelled the 'critical zone' Owing to an unfortunate quirk in terminology, the area marked 'transition zone' in the diagram refers to a change in regime in turbulent flow, and *not* to laminar-turbulent transition.

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predicted without at some stage having had recourse to data from testing. Some clever analysis reduces the amount of testing required to provide engineering estimates, but the need for use of test data cannot be eliminated yet.

The plumbers are therefore ahead of the scientists, by several thousand years at least.

Why is it that such ancient problems still remain unsolved in spite of all the spectacular advances that have been made in a variety of branches of science and of technology (includ ing computers)? There are several reasons, but the most basic is that the equations governing the flow of such fluids—discovered more than 150 years ago and named after Navier and Stokes—are nonlinear.

Our understanding of the behaviour of even the simplest nonlinear systems is still rudimentary. One of the striking characteristics of the particular nonlinear system describing the flow of fluids is that its behaviour is in general chaotic. We could in fact justifiably say (with apologies to an ancient scholar) *sarvam turbulence-mayam jagat*: "the whole universe is full of turbulence". However, there is reason to believe that hidden in this chaos is a considerable degree of order. What we mean by these terms will I hope become clearer as we go along, but to this day there is no satisfactory method of handling systems where order and chaos are so inextricably mixed. Some features of this complex combination of order and chaos are indeed visible to all of us. For example we all know how one can argue endlessly about the shape of clouds (figure 2). Kalidasa called them *kāmarūpa*; are they totally chaotic or is there a hidden order in the shapes we see? This question, which is something all of us ask at some early stage in our lives, is actually at the very heart, I believe, of

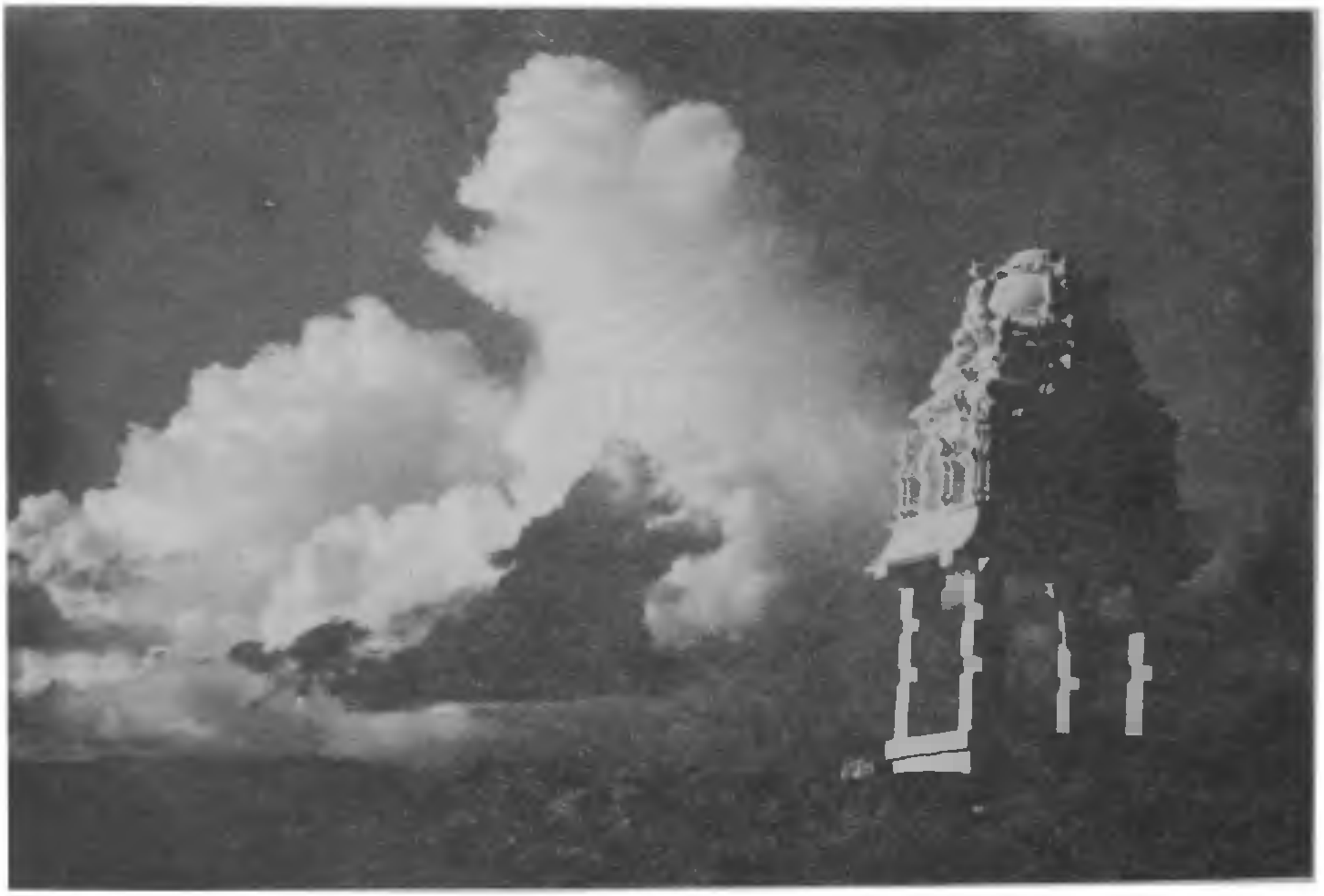

Figure 2. A photograph of monsoon clouds. What shapes do we see here? Is there any order, or are we cheating ourselves? (courtesy: Mr C. Rajagopal, NAL).

problems in fluid flow. Using the most sophisticated instrumentation available today in the controlled conditions of a laboratory, and investigating flows far simpler than a cloud, we still have to face exactly the same basic question.

FROM ORDER TO CHAOS

Before we proceed to see the implications of this combination let us first look at the simpler limits of the problem and begin with flows which exhibit considerable order.

Figure 3a shows a famous example[1]. When a fluid layer confined between two horizontal plates is warmed sufficiently (but not too much) at its lower boundary, the bottom gets lighter and the fluid tends to overturn; it does this in a beautifully ordered way, and the resulting pattern could justly adorn a saree-border. As the temperature of the lower plate is raised, the flow eventually looks chaotic: the neat rolls are replaced by irregular tongues of rising hot fluid (figure 3b). (Incidentally, this picture, taken in 1932 by two of Raman's students, Ramdas and Malurkar[2], is probably the first flow visualization published in India.)

Investigating the flow of water in a pipe, Reynolds[3] demonstrated, more than a hundred years ago, that there are two kinds of motion

Figure 3a. Convection rolls in a horizontal layer of fluid when the lower plate is warmer than the upper plate. The curves in the diagram are density contours: In the absence of convection they would have been a series of parallel lines[1].

possible—called by him direct and sinuous, now more familiar as laminar and turbulent—a distinction that is very easy to make when you open any water tap. If the opening is very small we know that water usually comes out as a smooth glassy jet. If the opening is increased, the surface loses its smoothness and the water begins to move in a very irregular "turbulent" way (now more fashionably called chaotic). Reynolds[3] showed that whether the motion is laminar or turbulent does not depend individually on the fluid or on the size of the pipe or on the velocity, but on a combination of properties which has since come to be known as the Reynolds Number (Re), defined as

$$\mathrm{Re} = V\ D/\nu,$$

where V is the fluid velocity, D is the diameter of the pipe and ν is the kinematic viscosity of the fluid. Reynolds found that if this number exceeded a critical value (something like 2300) he flow could become turbulent, whereas

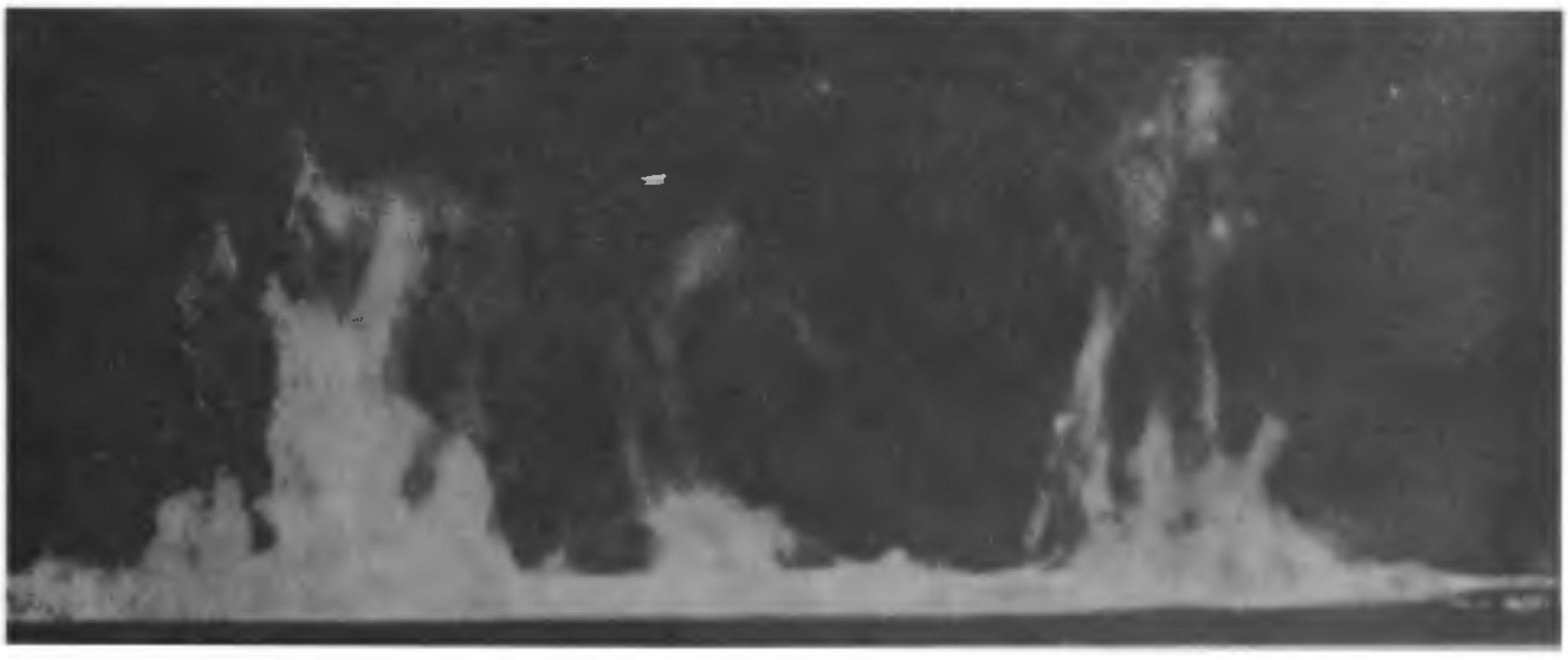

Figure 3b. Flow over a very hot surface, showing convective motion that is highly turbulent or chaotic[2].

Figure 4a. A jet of dye issuing into a tank of water quickly becomes turbulent, when it is such a good mixer that the whole tank gets coloured in a matter of seconds[15].

below this value it remained laminar. (The pressure required to push water through a pipe in laminar flow is known exactly by theory: it is turbulent flow that poses problems.) If the Reynolds number is sufficiently low the problem is linear; but as it increases the severity of the nonlinearity also increases.

Going back to the question of ordered motion, the fluid dynamicist's answer proceeds on the following lines. If the Reynolds number is extremely low, then the fluid motion is smooth, regular, and steady if boundary conditions are steady; there is no particular associated pattern of motion. As the Reynolds number increases, however, there is a stage at which the flow becomes unstable and spontaneously there is a generation of certain ordered patterns in the flow, as in the example shown in figure 3a.

As the Reynolds number increases further, these patterns break down and the flow eventually becomes irregular and turbulent. Some examples of the final state of turbulent motion are shown in figures 4a, b. Figure 4a shows how the efficient mixing that is so characterisitc of turbulent flow takes place[4]. Measurements in such flows reveal irregular fluctuations of every flow quantity, of which a typical example from the atmosphere is shown in Figure 4b.

Just exactly how does the motion break down from a highly ordered to a chaotic state? Figure 5 shows the progression from stability through order to chaos in one instance[5]. Such a

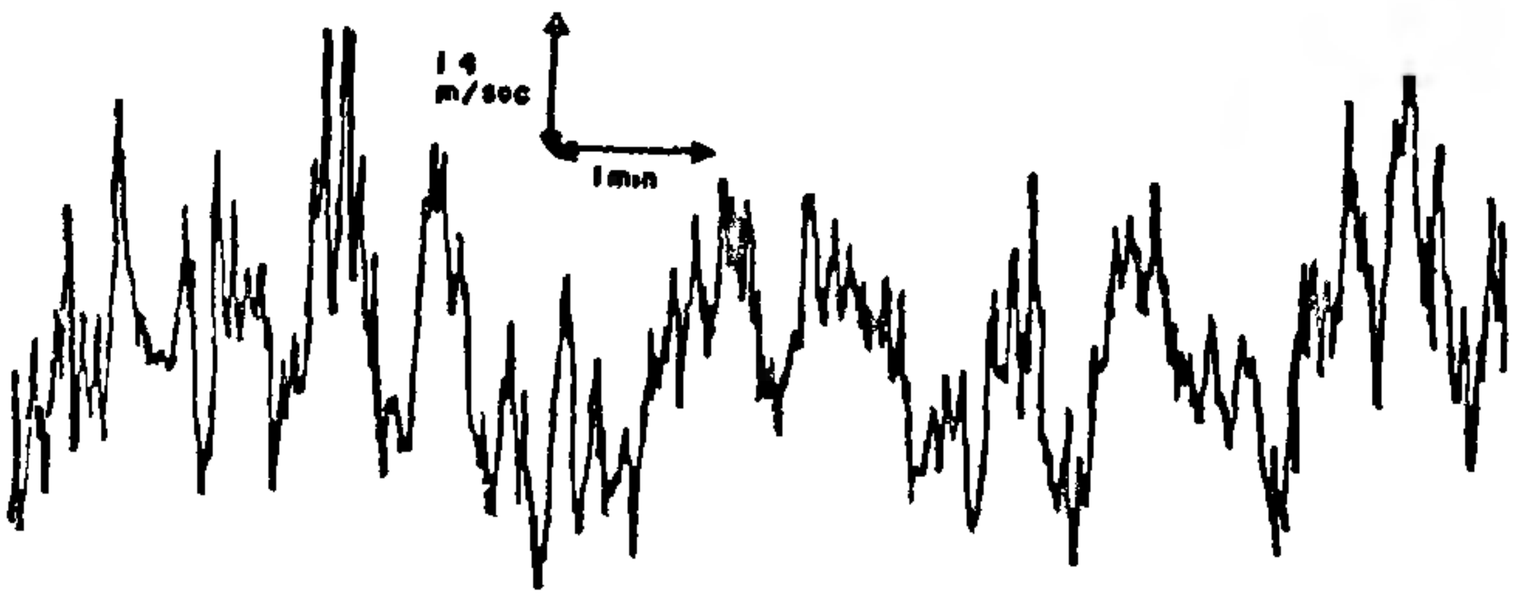

Figure 4b. A typical record of wind speed fluctuations, from a cup anemometer mounted on a short meteorological mast at the Institute[15]. The motion seems completely disordered.

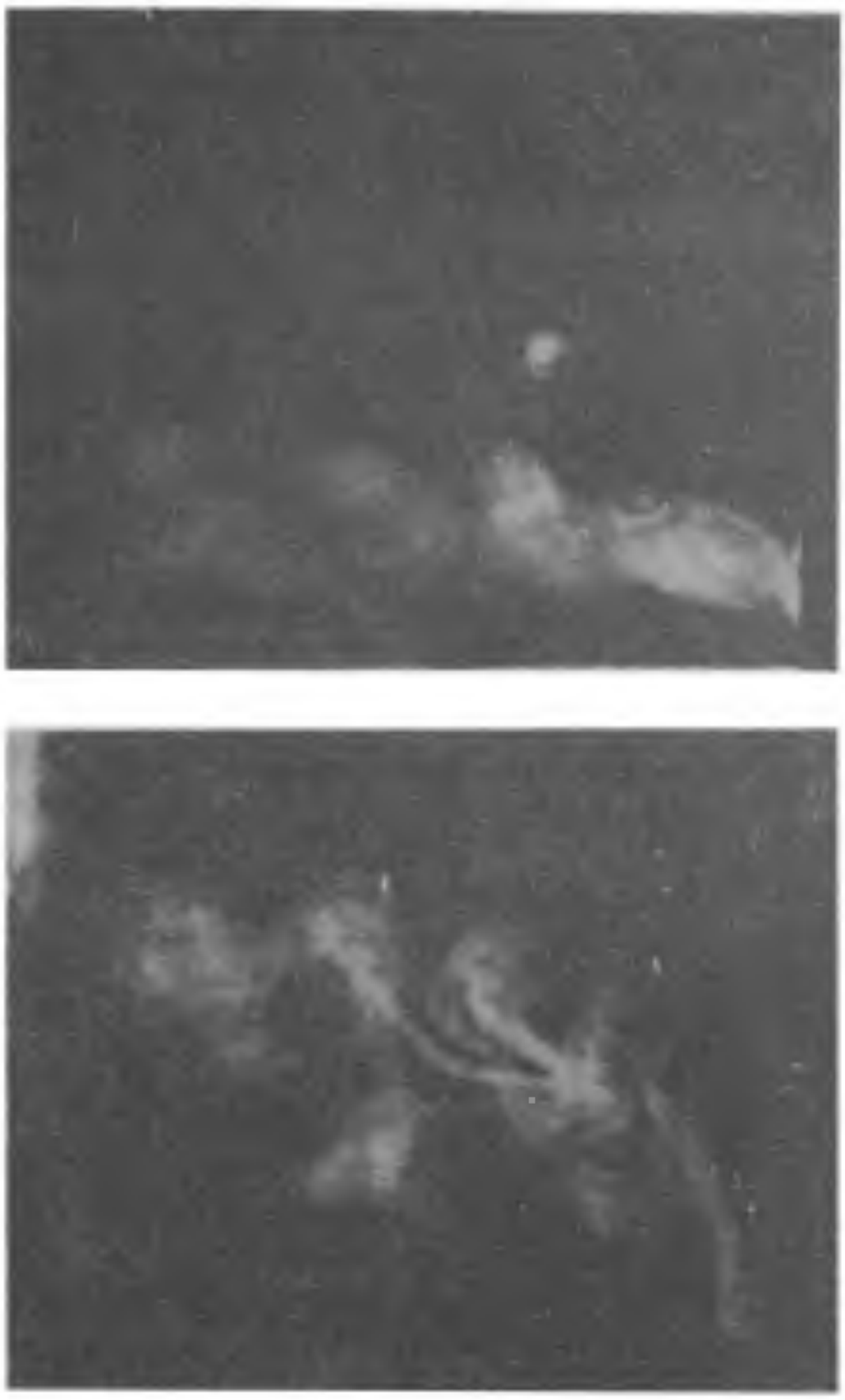

Figure 5. From stability through order to chaos. These pictures, taken in a small wind tunnel at IISc, show a jet of smoke issuing from the floor of the tunnel into wind from right. Near the floor the jet is nearly uniform and looks like a jet of water from a slightly open tap. The jet, which is basically unstable, then orders itself into a remarkable pattern of vortices, which later break down into chaotic motion. This sequence from stability to order to chaos is typical of a wide class of fluid flows.

sequence is typical, but by no means universal. In some cases, the basic waves characterizing instabilty seem to produce new instabilities. One conjecture, due originally to Landau[6], is that there is a succession of instabilities; each time the flow goes unstable, it leads to the possibility of new modes of instability and as this keeps on *ad infinitum* we eventually get choatic motion. No such infinite sequence has ever been actually observed; however there are cases in which the primary instability leads to a secondary instability, and the secondary to a tertiary. Ironically, the pipe flow that started off all this work is not unstable to small perturbations at all—but this is the kind of pathological behaviour that one usually accepts with resignation in nonlinear systems!

Is the transition from laminar to turbulent flow abrupt or gradual? In 1935, Prandtl[7], the greatest fluid dynamicist of this century, said that transition in flow past a flat plate, for example (- we can think of such a plate as an idealization of an aircraft wing or a fan blade), was "accomplished in a region of appreciable length" implying that it was gradual, but others emphasized the suddenness of the phenomenon. The truth turned out to be revealing and neatly reconciles these conflicting views. The point at which chaos is first seen is relatively sharply defined, but at this point and for an appreciable distance downstream the chaos is not present full-time. Figure 6a shows how this happens[9,10]; the sudden appearance of chaos at the point x_t is the result of the birth of what are known as turbulent spots, which are islands of chaos in a laminar sea—islands that are minute at birth, but grow as they move with the flow (eventually covering all of it). These spots make chaos intermittent; and the "intermittency", i.e. the fraction of time during which the flow is turbulent, varies from zero upstream of x_t to unity sufficiently far downstream, over a distance that can often be a substantial part of the surface[11] (figure 6b).

ORDER IN CHAOS

Paradoxically, the flows that have thus (eventually) become turbulent or chaotic full-time (intermittency = 1) have in recent decades been found to contain much ordered motion[12]! The degree of order present varies appreciably from one flow to another. For example, the velocity fluctuations shown in figure 7a (characteristic of atmospheric wind) seem completely chaotic, but in fact conceal an ordered event[13] which can be extracted only

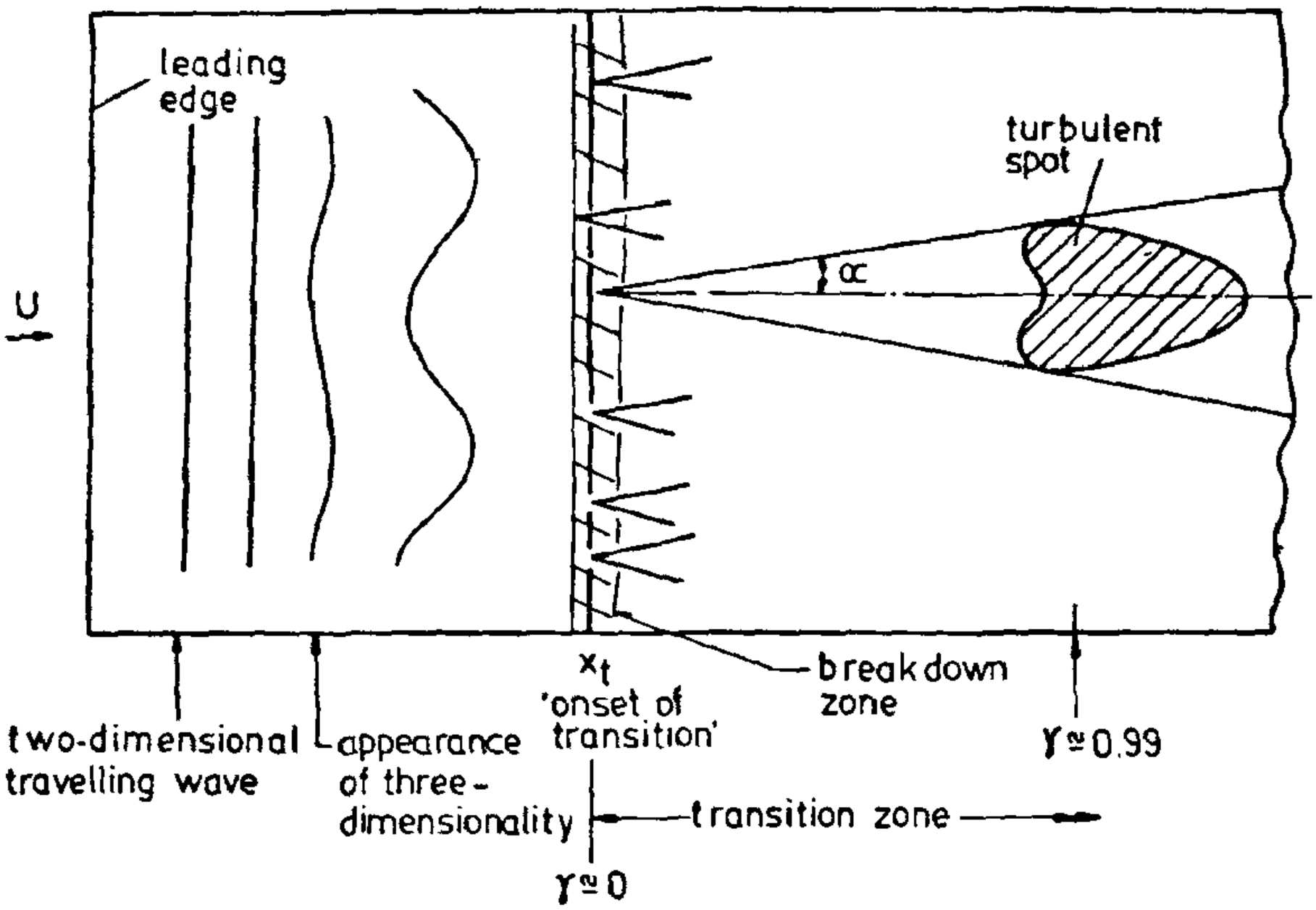

Figure 6a. A schematic diagram illustrating transition to turbulence in flow past a flat plate, which may be considered an idealization of e.g. an aircraft wing. Close to the leading edge the flow is stable; at a certain station downstream it becomes unstable, leading to the appearance of well-defined waves. These waves, which are at first two-dimensional, go through further instabilities and become three-dimensional, culminating in the appearance of turbulent 'spots' at a fairly well-defined location x_t. These spots grow in size (along the wedges shown) as they move downstream, till they cover the whole flow making it 'fully' turbulent, the intermittency γ varying from zero at x_t to unity far downstream[11].

after some rather intricate data-processing that barely escapes being dubious! The chaos here clearly masks the order. Furthermore, even the special event so detected does not repeat at regular intervals or have the same intensity when it does occur again, so that the succession of events is more like a wedding procession than a march-past; it has its own element of chaos. On the other hand, the trace of figure 7b seems highly ordered, and is evidently dominated by a purish wave; nevertheless it is technically chaotic[14] because the ("random") modulation of the wave seen in the figure actually spreads energy over *all* possible frequencies, giving its spectrum the broad band that in practical terms is the hall-mark of chaotic behaviour.

Separation of the ordered or coherent motion from the rest of chaos is, as I have already implied, a rather tricky operation and has been plagued by much scientific controversy[13]. But the presence of significant order in turbulent motion is a fact of extraordinary importance: it offers hope of greater skill at predicting turbulence as well as in managing it. So the order-*wallah* among fluid-dynamicists sings (with Robert Frost).

Let chaos storm!
Let cloud shapes swarm!
I wait for form.

FROM CHAOS TO ORDER

To add further colour to all these complex phenomena, and in partial contradicition of what we said about the universe being full of turbulence, there are a variety of ways in which chaotic motion can actually be suppressed, i.e.

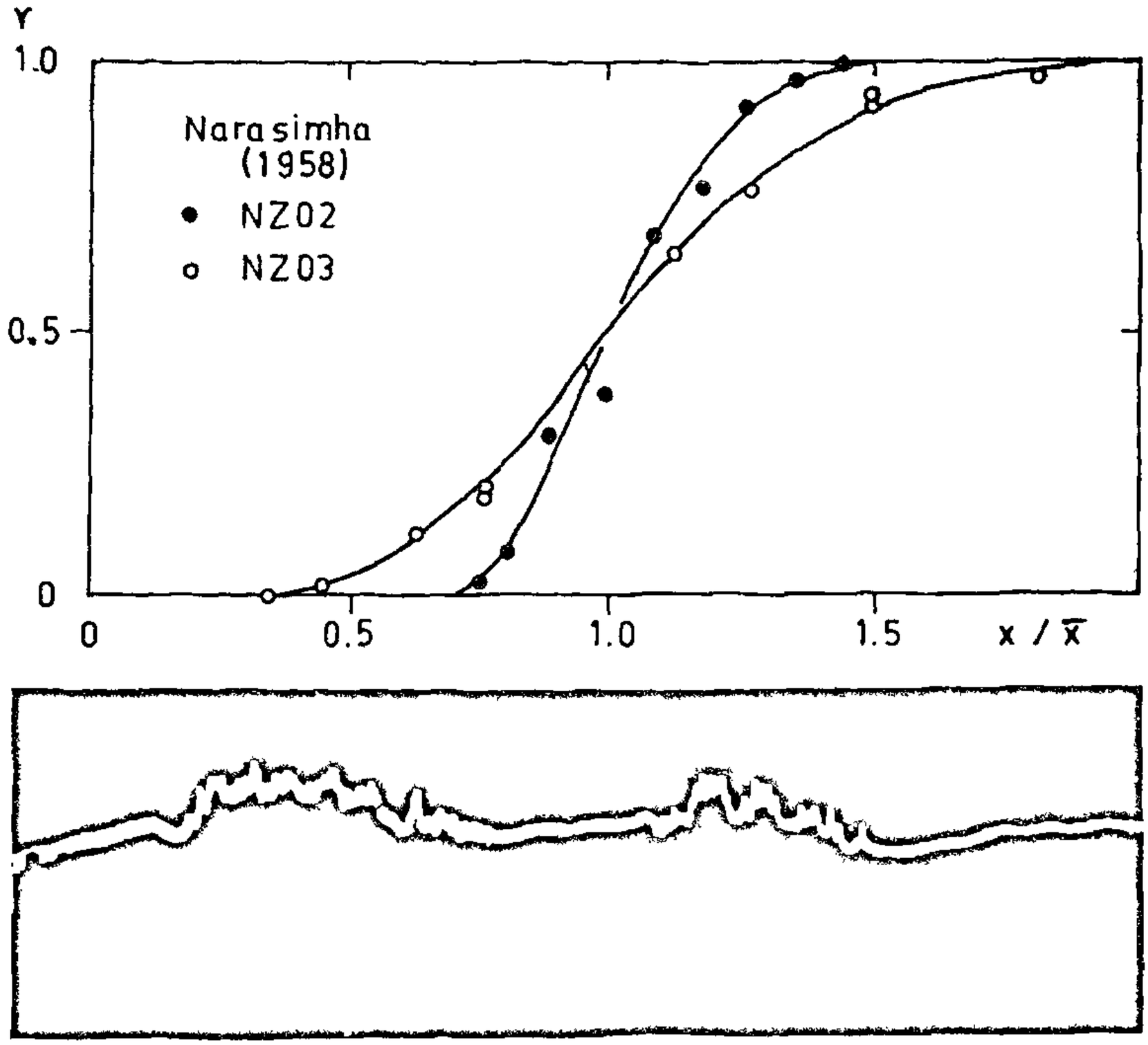

Figure 6b. The 'intermittency', which is the fraction of time that the flow is turbulent, increases *gradually* from zero to unity. The extent of the transition zone is generally comparable to that of laminar flow, but is relatively larger at lower Reynolds numbers[11]. The velocity trace below shows the intermittent behaviour characteristic of a transitional boundary layer.

a turbulent flow can under certain conditions be "relaminarized"[15]. This may strike one at first as thermodynamically impossible, but we must remember that the flow systems we consider are not thermodynamically closed; relaminarization may be seen as the analogue of crystallization. Some striking instances of this phenomenon[4] are shown in figures 8a, b. The mixing that was shown in figure 4a can be completely suppressed by heating the top of the tank, which stabilises the flow (figure 8a). In the second example (figure 8b), a flow that is already turbulent in a tube (at a Reynolds number well above the value of 2300 quoted earlier) can be made to revert to a laminar state by the simple stratagem of winding the tube round in a coil[4]; indeed, as the picture demonstrates, the curvature of the coil does not have to be particularly sharp to achieve the effect (in fact we have here another instance of nonlinear behaviour: small causes, in the form of a mild curvature of the tube, yielding large effects, namely relaminarization). As a final example, consider the jogger who is breathing hard after his exertions. The air he sucks through his wind pipe is in turbulent motion, but when it reaches the minute bronchioles deep in his lungs turbulence has been completely suppressed.

In 1949, the well-known inventor Savonius, famous for the vertical axis windmills of which several examples have been seen on this

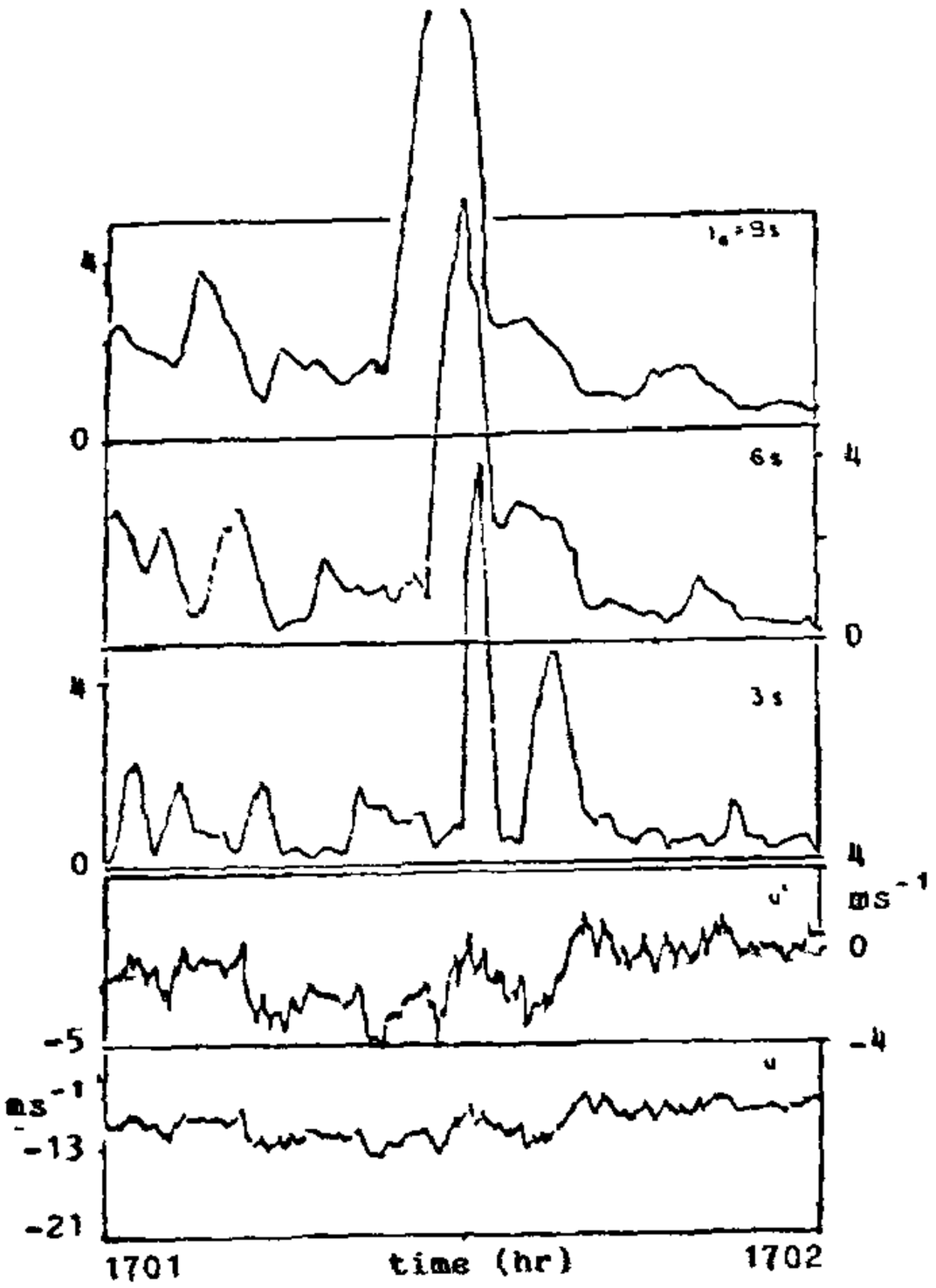

Figure 7a. How much order and how much chaos? Wind velocity in the atmosphere (lowest trace), seemingly with no order in it (second trace from bottom shows the fluctuation about the mean). A special form of data processing, called variable interval time averaging, reveals however an intense event, indicated by the spikes as seen in the upper traces corresponding to longer averaging times t_a[13].

campus, wrote how it is impossible to control the wind, i.e. to suppress chaos. As he picturesquely put it, neither Stalin nor Morrison—a well-known Labour leader then in power in Britain—could "socialize the wind". Neither capitalists nor communists can quite socialize the atmospheric wind even now, but there are a variety of other flows that can be; indeed we have just seen that it has been happening right under our noses all along.

CHAOS IN DYNAMICAL SYSTEMS

Now the fluid dynamicist believes that all these intriguing phenomena are completely contained in the classical Navier-Stokes equations. From one standpoint—very shallow in his view—it may therefore be thought that there is no "new physics" to be discovered. But in actual fact quite the contrary is true; the new physics is very much there in the nonlinearity of the equations, waiting to be discovered. But unfortunately these equations are so complicated that there is no case, not even the most idealised one we can think of, where all the stages of the transitions from order to chaos and back, or their complex combinations, can be quantitatively described. Even numerical simulations, using the biggest computers and the cleverest algorithms currently available, have not been able to achieve this sort of description. There have therefore been several attempts to look at the behaviour of equations which mimic fluid flow, although they may not

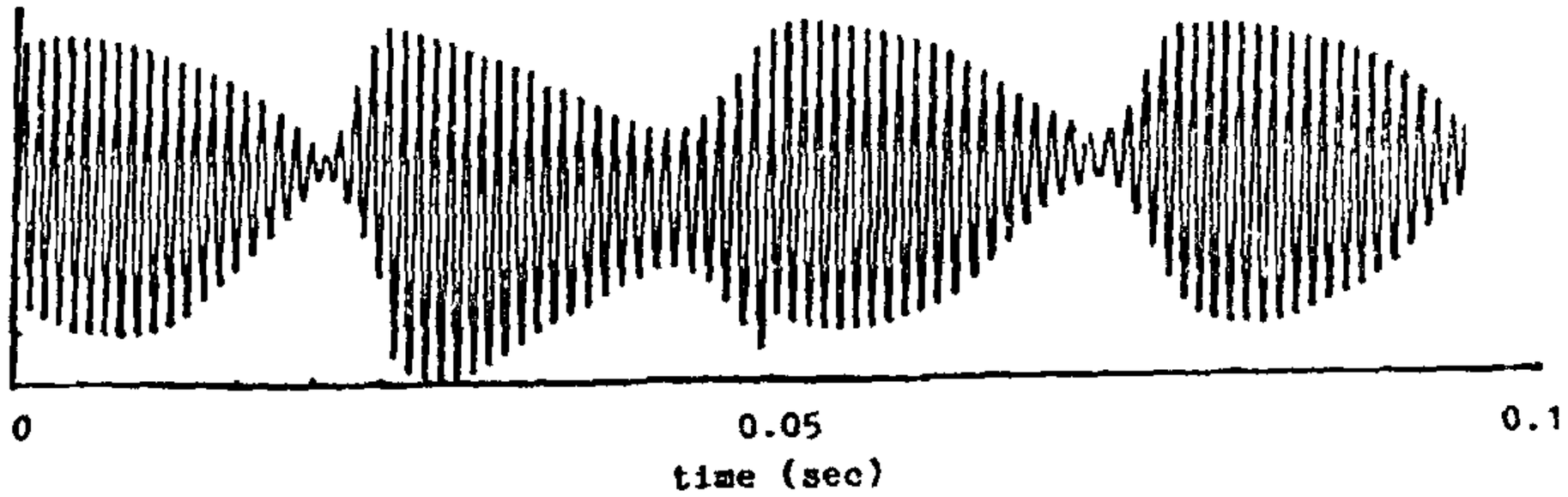

Figure 7b. Fluid flow velocity fluctuations at a point in the wake behind a circular cylinder. There is clearly a dominant wave, but the modulation seen in the record spreads energy over a wide frequency range, making the signal technically chaotic[14].

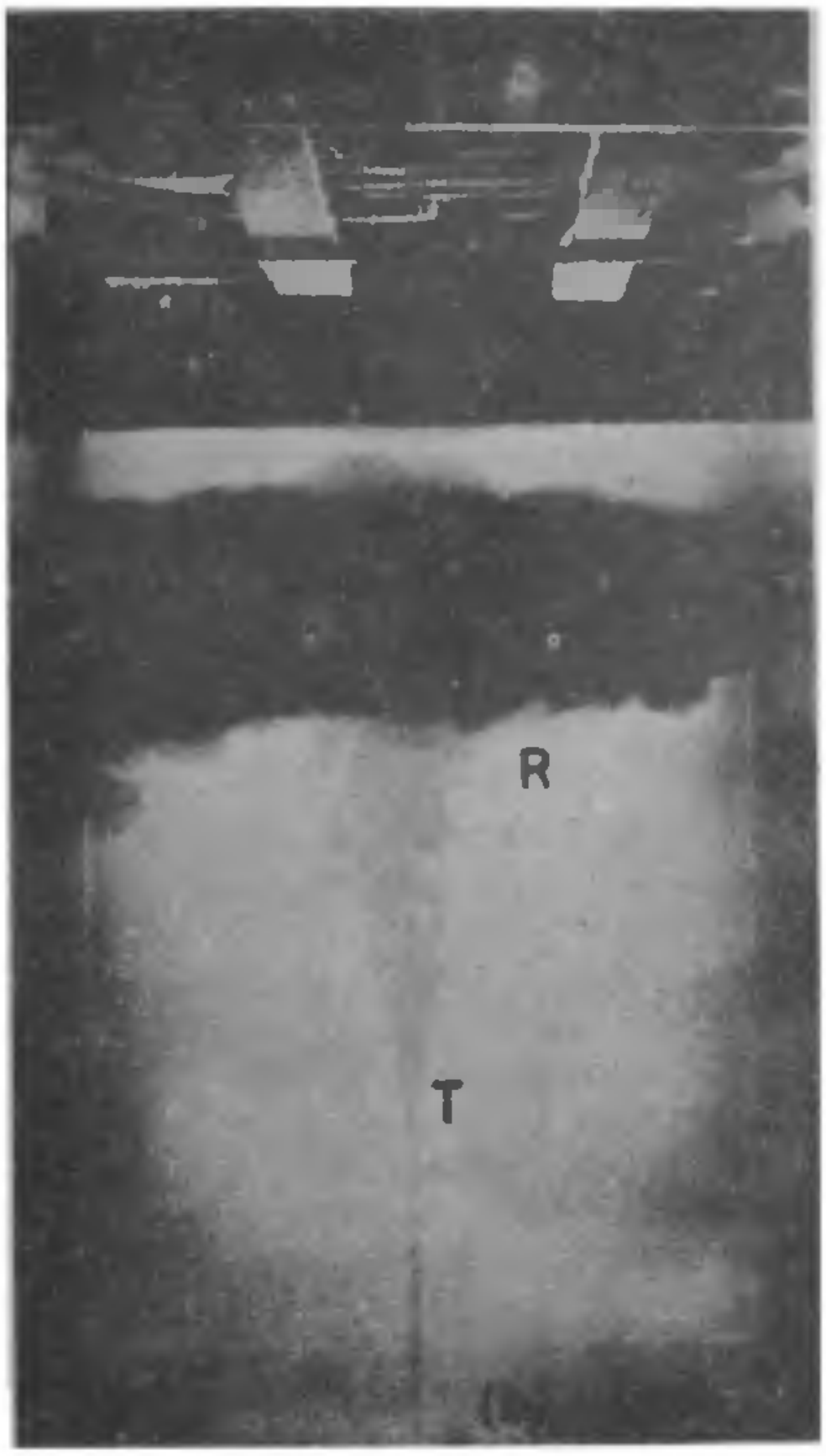

Figure 8a. Flow in the tank of figure 5b, when the top of the tank is heated. Such heating stabilizes the flow and completely suppresses chaos; the dye now collects as a cloud and does not mix with the rest of the water[15].

represent it accurately. The construction of such 'models' is by no means easy. For example, a very interesting attempt made by Burgers[16] many years ago turned out on close examination to be more relevant for shock waves[17] (another interesting non-linear phenomenon); in other words a model devised for turbulence turned out to be one for a rather vicious type of noise.

One of the most influential of such models was formulated by Lorenz[18] in 1962. Lorenz was concerned with the problem of predicting weather—and, as a problem in physics, that is basically the same as the one faced by plum-

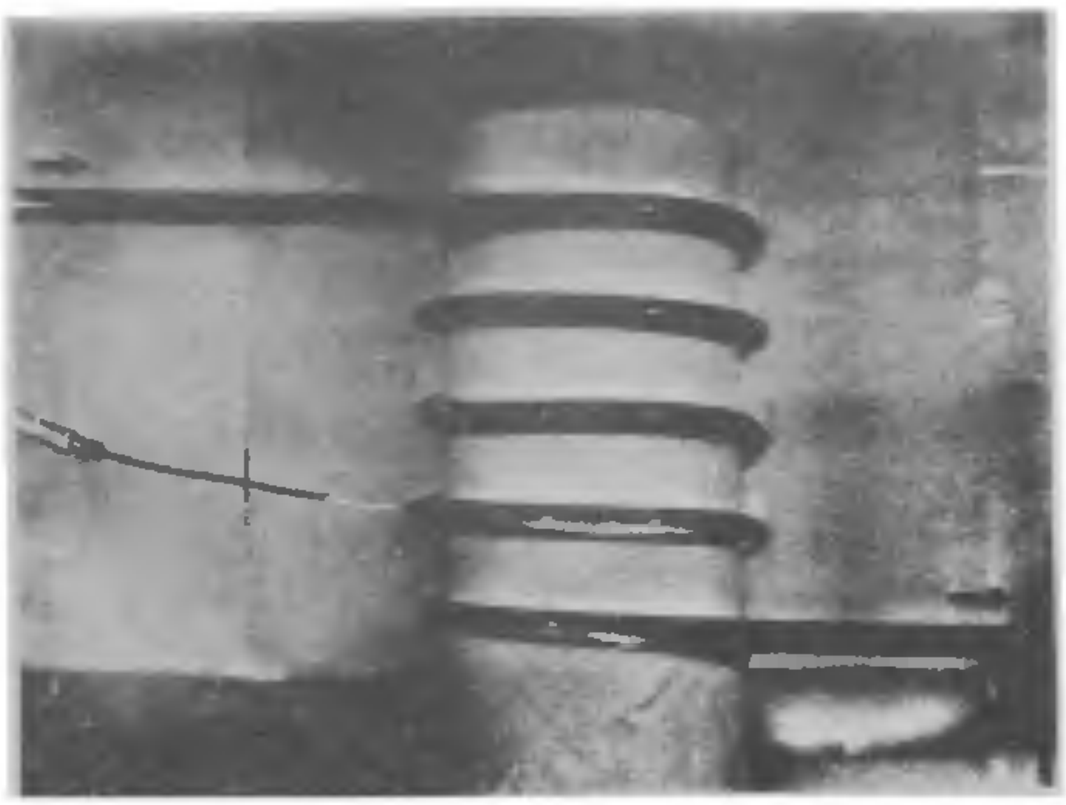

Figure 8b. This is another flow in which there is a 'reverse transition' from chaos to order. Flow coming in through the tube at the top is turbulent, as can be inferred from the rapid dispersion of dye. Flow going out of the coils is laminar: a filament of dye injected after a few turns in the coil does not spread[15].

bers, only it is even more complex—and proceeded to make a highly idealized model for the kind of convective flow that is so common in the atmosphere. In the process drastic simplifications were made (as Lorenz was fully aware), but nevertheless the behaviour of the solutions of the model equations which are nonlinear can be so strange that it repays attention. The equations are reproduced below:

$$\dot{X} = 10Y - 10X,$$
$$\dot{Y} = Y - XZ + rX,$$
$$\dot{Z} = XY - (8/3)Z;$$

here a dot indicates a time-derivative. The three unknowns in these equations, X, Y and Z, represent in some sense the state of the convective motion in the saree-border pattern of figure 3: in the model the *pattern* always remains the same (in space), but its intensity varies with time. The numbers in the model represent the conditions of the flow; the most important of these is the parameter r, which stands for the Rayleigh number. This number plays the same role in the convection problem as the Reynolds number does in the pipe flow problem. (In fact, it is a Reynolds number

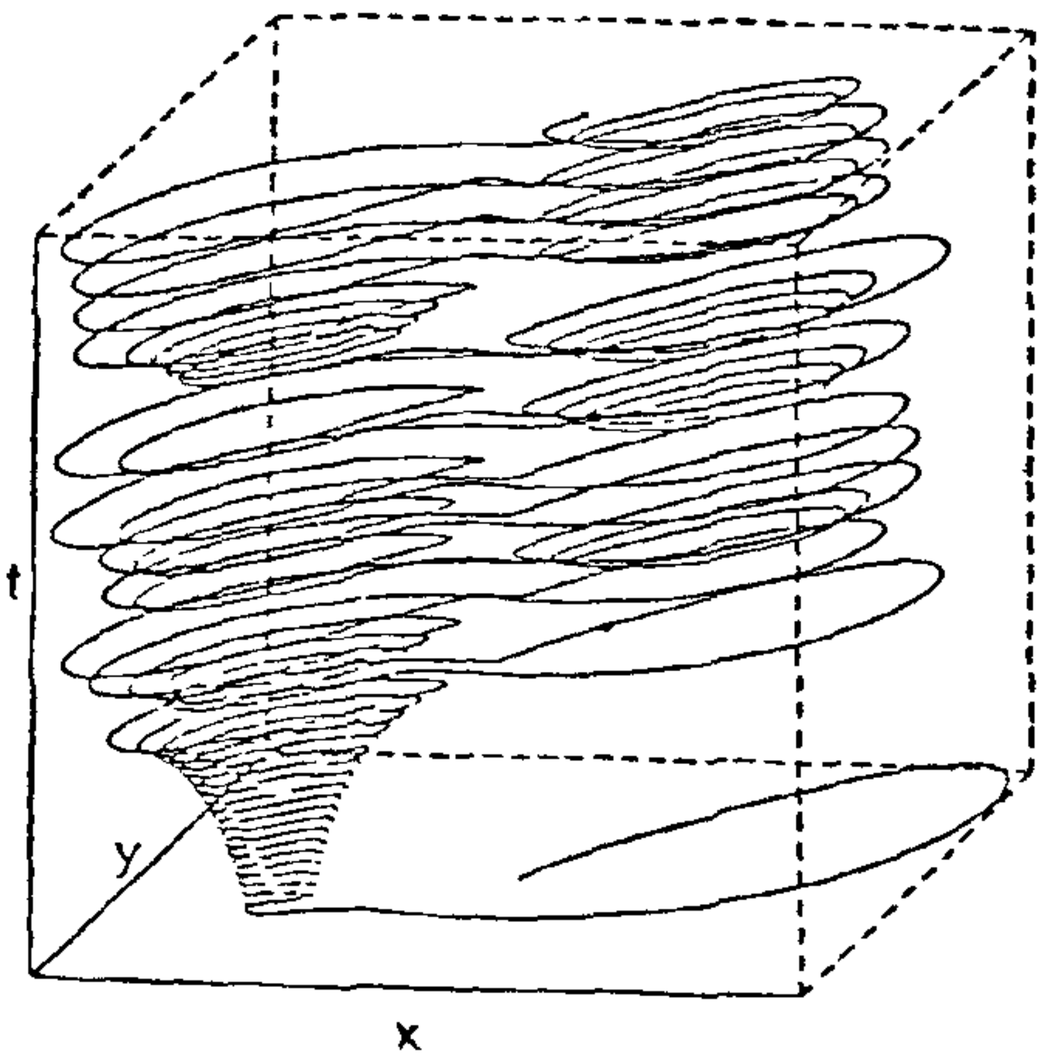

Figure 9a. Typical solution of the Lorenz equations, showing the 'chaotic' behaviour of the amplitudes X and Y as functions of time. Note the tendency of the solution to keep flipping, at irregular intervals, between two oscillatory-type states of motion[1].

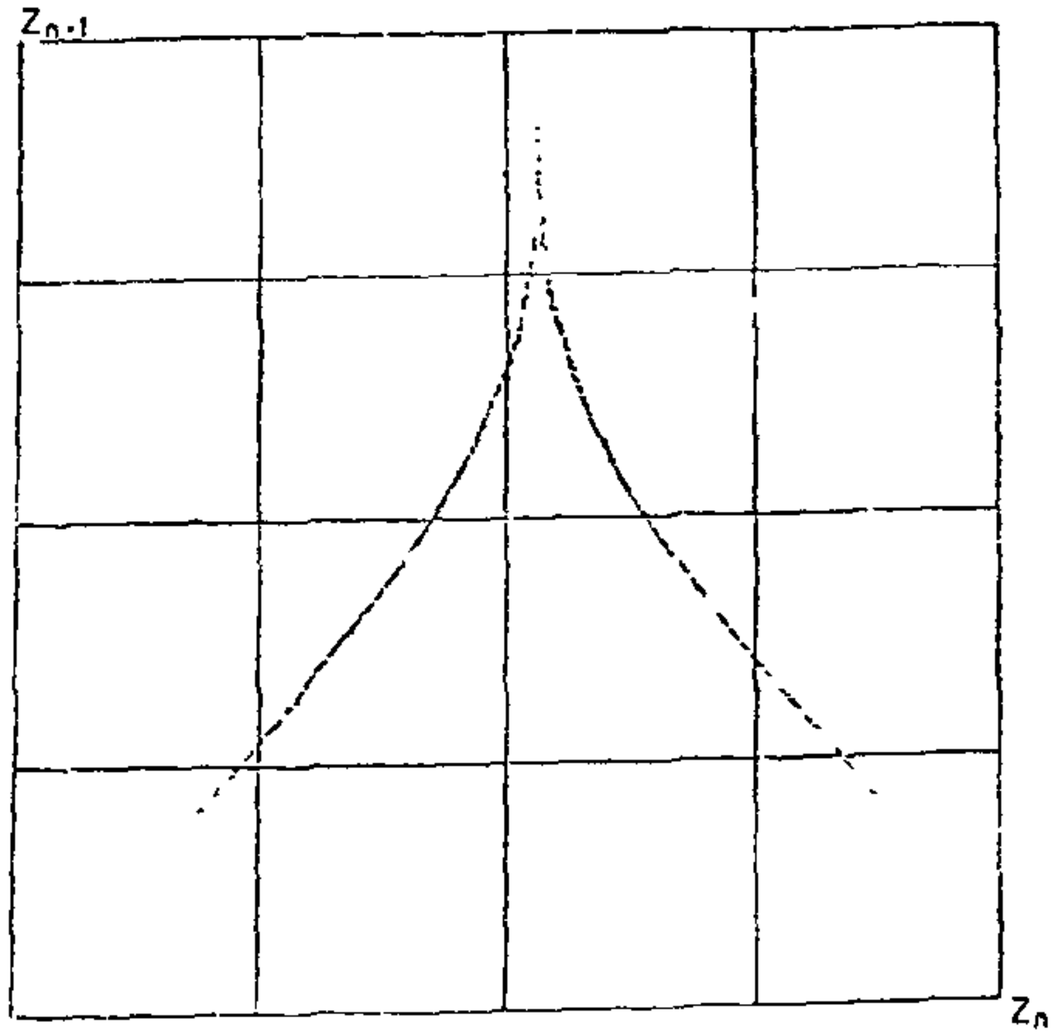

Figure 9b. Relation between successive maxima in the variable Z in the Lorenz equations. If you enter on the horizontal axis with any particular maximum, the value of the next maximum in the solution is given uniquely by the folded curve; but the previous maximum could have either of two possible values, the one before that any of four possible values, and so on. (Computation by G. S. Bhat.)

based on the velocity that a blob of rising hot fluid acquires before it cools to the surrounding temperature by thermal conduction; a critical value must be exceeded before the rolls of figure 3a appear at all.) Solutions at $r = 28$, i.e. at a Rayleigh number 28 times the critical value, appear erratic or chaotic as shown in figure 9a. It is once again ironic that the original 'exact' equations, to which the above set was devised as an approximation, have been found not to exhibit any chaotic behaviour[19] at all! Nevertheless the results for the Lorenz system have in recent years profoundly influenced the way we look at the possible mechanisms by which an ordered motion becomes chaotic.

Even the solutions of these 'model' equations are so hard to analyse that a further simplification has been worthwhile. This may be obtained by looking at the peaks in the solution; and it led to the discovery that each peak determines the next one uniquely (or nearly so), but not the previous one: see figure 9b. This interesting observation has provided the key to an enormous amount of research in recent years. For, here we have a connection made between the original partial differential equations governing the problem, through the approximate nonlinear ordinary differential equations (providing a "dynamical model") that were constructed—albeit rather dubiously—out of them, to a simple kind of "mapping" that we have discovered between the peaks. (The mapping is described by equations of the type $X_{n+1} = f(X_n)$, $n = 0, 1, 2 \ldots$, where $f(X)$ is a prescribed function of X.) Once these connections are seen, it has been realized that it would be worthwhile to look at just the maps themselves: there is the fantastic possibility that hidden in the behaviour of such simple maps may lie clues to the complex behaviour of fluid flows (see e.g. the collection of papers in Bai-Lin[20]).

It is in fact astonishing how much a study of such maps has been able to teach us about

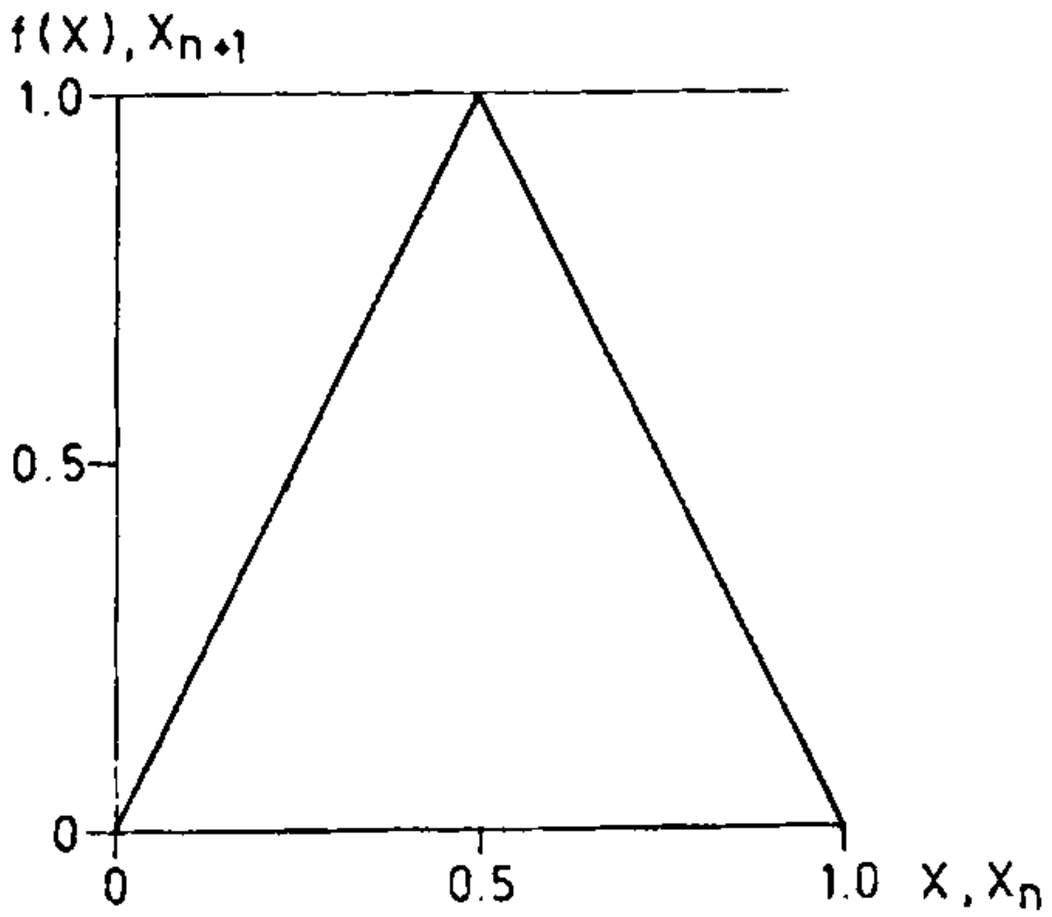

Figure 10. The simple 'tent' map, which may be considered an idealization of figure 9b. For details see text.

some possible origins of chaos[21]. To illustrate let us further idealize figure 9b, and look at the particularly simple 'tent' map (figure 10), for which

$$f(X) = 2X \text{ if } 0 \leqslant X \leqslant 1/2,$$
$$= 2 - 2X \text{ if } 1/2 < X \leqslant 1.$$

This map cuts out some of the frills of figure 9b by constraining X to the range 0 to 1, straightening out the curves and imposing symmetry about the midpoint. But it retains the crucial fold at the top: crucial because the value of X at any given stage *cannot* uniquely determine the sequence of numbers leading to it, but *does* so determine the subsequent numbers. In other words the present determines the future uniquely, but leaves the past vague.

It is not very difficult to prove that the *correlation* between two successive numbers in the sequence generated by this completely *deterministic* map is zero[22]; and that any number is just as probable as any other in such a sequence (all members of which will of course lie between 0 and 1). What we mean when we say the correlation is zero is that we would be unable to predict, by purely statistical procedures, any entry knowing any (or all) of the previous entries in the sequence. To make this more specific we can agree to denote all numbers in the sequence which fall between 0 and 0.5 by H, and those that fall between 0.5 and 1.0 by T. And we order the sequence in terms of Hs and Ts (see for example table 1). We will then find that this sequence of Hs and Ts from the map is statistically indistinguishable from the kind of sequence of heads and tails obtained by tossing a coin (also shown in table 1). In other words, we have here a completely deterministic system whose behaviour is apparently stochastic; i.e. the results appear "random".

Table 1 *Comparison of a sequence from the tent map and from coin-tossing*

A. Tent map

0.89442719100 H	0.21114561800 T	0.42220123600 T
0.84458247200 H	0.31083505600 T	0.62167011200 H
0.75665977601 H	0.48668044799 T	0.97336089598 H
0.05327820804 T	0.10655641609 T	0.21311283217 T
0.42622566434 T	0.85245132869 H	0.29509734262 T
0.59019468524 H	0.81961062951 H	0.36077874097 T
0.72155748195*H	0.55688503610 H	0.88622992779 H
0.22774014442*T	0.45508028883 T	0.91016057767 H
0.17967884467*T	0.35935768933 T	0.71871537867*H
0.56256924267*H	0.87486151466 H	0.25027697068*T
0.50055394135 H	0.99889211729 H	0.00221576542*T
0.00443153083 T	0.00886306167 T	0.01772612333*T
0.03545224667 T	0.07090449333 T	0.14180898666 T
0.28361797333 T	0.56723594666 H	0.86552810669 H
0.26894378662 T	0.53788757324 H	0.92422485252 H

B. Coin-tossing

H	H	T	H	T	T	T	T	H	T	T	T	T	H	H
H	T	H	H	H	H	T	H	T	T	T	H	H	T	H
T	T	H	H	T	H	T	T	H	H	T	T	H	H	T

Values of X_n from the tent map
$X_{n+1} = 2X_n$ $(0 \leqslant X_n \leqslant 1/2)$
$= 2 - 2X_n$ $(1/2 \leqslant X_n \leqslant 1)$,

starting with $X_0 = 1/\sqrt{5}$. Compare the two starred sequences which start very close to each other (less than 0.003) but are far apart three steps later. If we label entries less than 0.5 with a *T*, and those greater than 0.5 by an *H*, there are 21*H* and 24*T* in the tent map. These 45 entries from the tent map may be compared with results of actual trials with a coin; note that in this sequence there are 22 (*H*) out of a total of 45 trials.

Note that there are special initial values X_0 for which the sequence would either terminate (e.g. $X_0 = 1/2$, $X_1 = 1$, $X_2 = 0$) or repeat in a so called 'limit-cycle' (e.g. $X_0 = 1/3$, $X_1 = 2/3$, $X_2 = 1/3$, etc.) In general, however, the sequence is non-periodic and infinite, as in the table above.

There are two other points to be made from the entries in table 1. The first concerns the definition of chaos: note that a number in the sequence can sometimes approach very closely some previous value (e.g. compare the two starred subsequences), but eventually the later numbers depart substantially from each other. In fact X approaches its previous values arbitrarily closely, but never *stays* close. The reason is that the later values in the sequence depend sensitively on initial conditions. And this leads us to the second point. Any (arbitrarily) small uncertainty in the present state of the system (i.e. current entry in the sequence) grows eventually to a finite or substantial error in the future—thus limiting predictability. At the same time, prediction of the state of the system at any given future time is in principle possible *to any prescribed accuracy, provided the present* is known sufficiently accurately (although what is 'sufficient' can turn out to be so extreme as to be impractical). The implications for numerical weather prediction are clear: it has been picturesquely stated that the flap of a butterfly's wings could eventually change the weather. (Mercifully it won't in general change the climate, though.)

This kind of behaviour has been called "deterministic chaos", which is apt precisely *because* of the seeming contradiction in terms. An equally apt Indian word would perhaps be *chanchal*, which embodies a concept that seems already neatly to reconcile unpredictability and determinism.

Einstein is supposed to have said that he did not believe in a God who played dice. But if God is nonlinear—how can He be otherwise?—it is clear that He need not play dice to appear probabilistic or random.

The kind of example we have described above is, I feel sure, going to affect profoundly our view of statistics, let alone fluid behaviour, because the familiar distinction between deterministic and stochastic processes can no longer be sustained. Indeed a more basic question can be raised: is it possible that the processes that we classify as stochastic are in fact also basically deterministic, but only governed by the kind of nonlinear mechanism which is caricatured in this simple example? Should not the distinction rather be between regular and chaotic behaviour?

IS TURBULENCE DYNAMICAL CHAOS?

We will not pursue the implications for statistics here, but come back to fluid flows. It is interesting now to speculate that the transition to turbulence that we have described earlier is in fact nothing other than the kind of chaotic behaviour that the simple model exhibits. Spurred by this possibility, a series of results have been proved in recent years about such maps, and more often demonstrated on the computer, by methods that may well be called 'applied computer technology', i.e. computers applied to understand nature! One such striking result is due to Feigenbaum, who demonstrated that such maps had certain universal characteristics, summarized in two numbers which basically describe the relation between the parameter values at the appearance of successive instabilities (bifurcations) characterizing the solutions of the map. These numbers can be computed to extraordinary accuracy.

Thus if we consider the quadratic map $f(X) = RX(1-X)$ (which is actually a transformation of the tent map but contains the free parameter R that serves as an analogue of the Reynolds number), then as R increases the steady solutions of the map go through a sequence of instabilities, leading at each stage to the appearance of a new limit cycle with twice the old period. A limit cycle (see footnote to table 1) is a steady oscillation of fixed amplitude; a wire galloping in wind, or a water pipe that exhibits the jerky, noisy oscillation known as 'hunting', are examples of limit cycles in fluid flows. This sequence of instabilities is illustrated in a bifurcation diagram[22] of the kind shown in figure 11. If R_n is the value of R at the nth bifurcation, the first Feigenbaum number is

$$\lim_{n\to\infty} \frac{R_n - R_{n-1}}{R_{n+1} - R_n} = 4.669\,201\,6\ldots$$

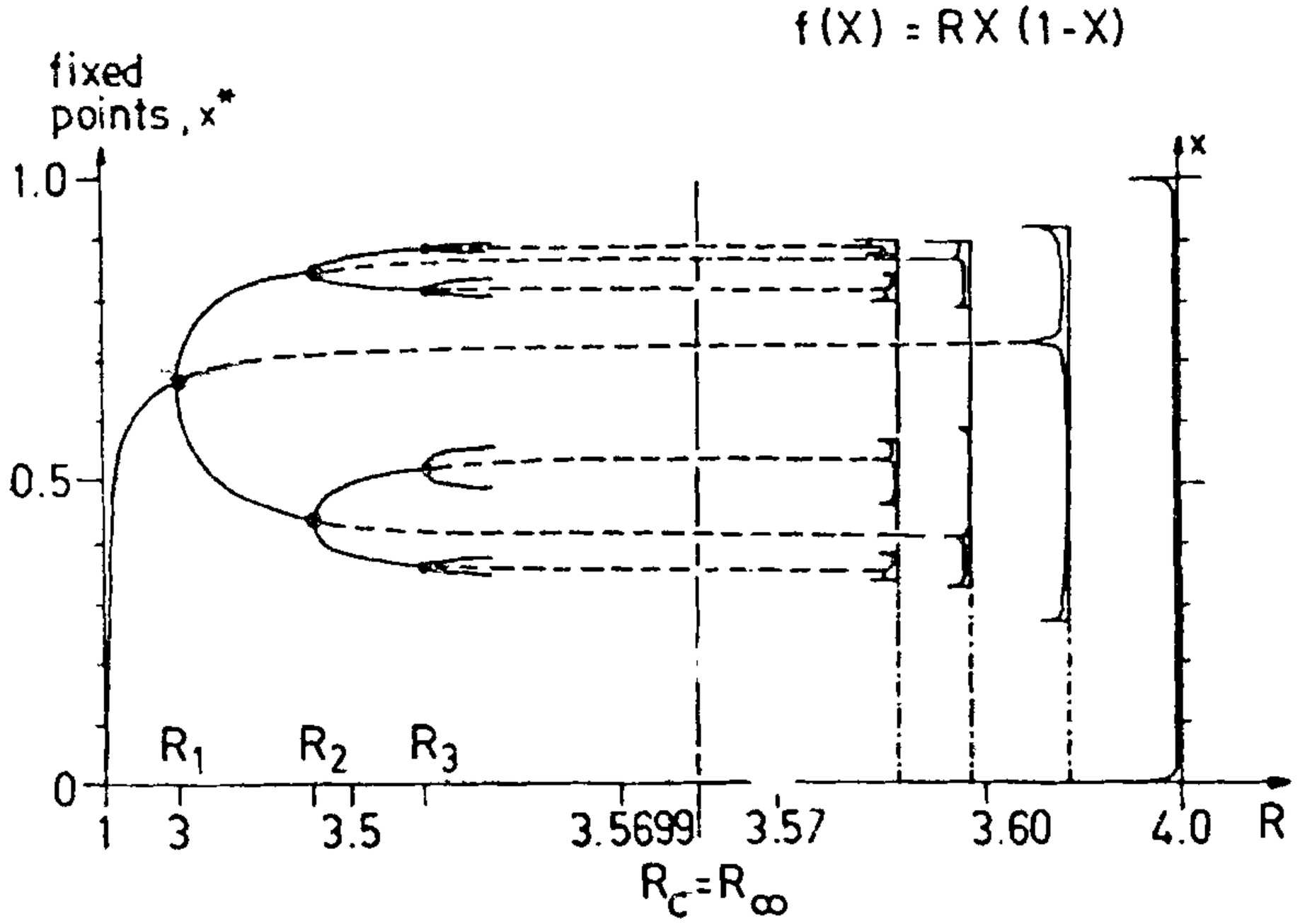

Figure 11. Bifurcation diagram for the quadratic map $f(X) = RX(1 - X)$, showing a sequence of period-doubling limit cycles as R increases (period n appears at R_n). For $R < 3$, X_n tends to a fixed point X^* which depends on R as shown. At $R = R_1 = 3$, there is a bifurcation into two (unstable) fixed points, and X_n tends to a limit cycle of period Z, i.e. every second number repeats, for $R_1 < R < R_2$. For $R_2 < R < R_3$ there is a limit cycle of period 4 (every fourth number repeats); and so on. The end of the sequence, at $R_c = R_\infty$, marks the onset of chaotic solutions. At four values of $R > R_c$ are shown the distribution of X in the solution. At $R = 4$ the distribution is given by the arc-sine law, which is infinite at $X = 0$, 1 and a minimum at $X = ½$. Note the distorted scale on the abscissa, designed to separate the values of R at which successive bifurcations occur (after Grossman and Thomae[22]).

The behaviour of X begins to show chaos at the limit point R_∞ of the sequence R_n: as seen from figure 11, $R_\infty = 3.5699...$

The fascinating question is whether we can now make the connection backwards from the map to the fluid flow and expect to find the same numbers operating in the flow problems. For example, can the Rayleigh number at which the flow becomes chaotic in the convection problem be related to the Rayleigh numbers at which the first instabilities appear, in the same way that the value of the parameter R in the map, heralding chaos, is related to the primary instabilities revealed in the map? The suggestion that this might be so—that indeed the route to chaos may be universal—caused understandable excitement in the scientific world some years ago. For it is the traditional wisdom in the study of nonlinear systems that they are idiosyncratic: it has been considered a useful exaggeration to say that no nonlinear system is like any other. Is it then nevertheless possible that all those nonlinear systems that do exhibit chaos reach that state in the *same* way?

The experimental evidence here is still somewhat ambiguous. It is true that the sequence of numbers observed in certain experiments closely mimic the Feigenbaum period-doubling sequence (figure 12)[23]. It is still however not possible to assert categorically that it is in fact the way that chaos appears in either the convection flow or any other problem. The universality of this route to turbulence appears to have been exaggerated. We cannot, for example, see its relevance to the phenomenon

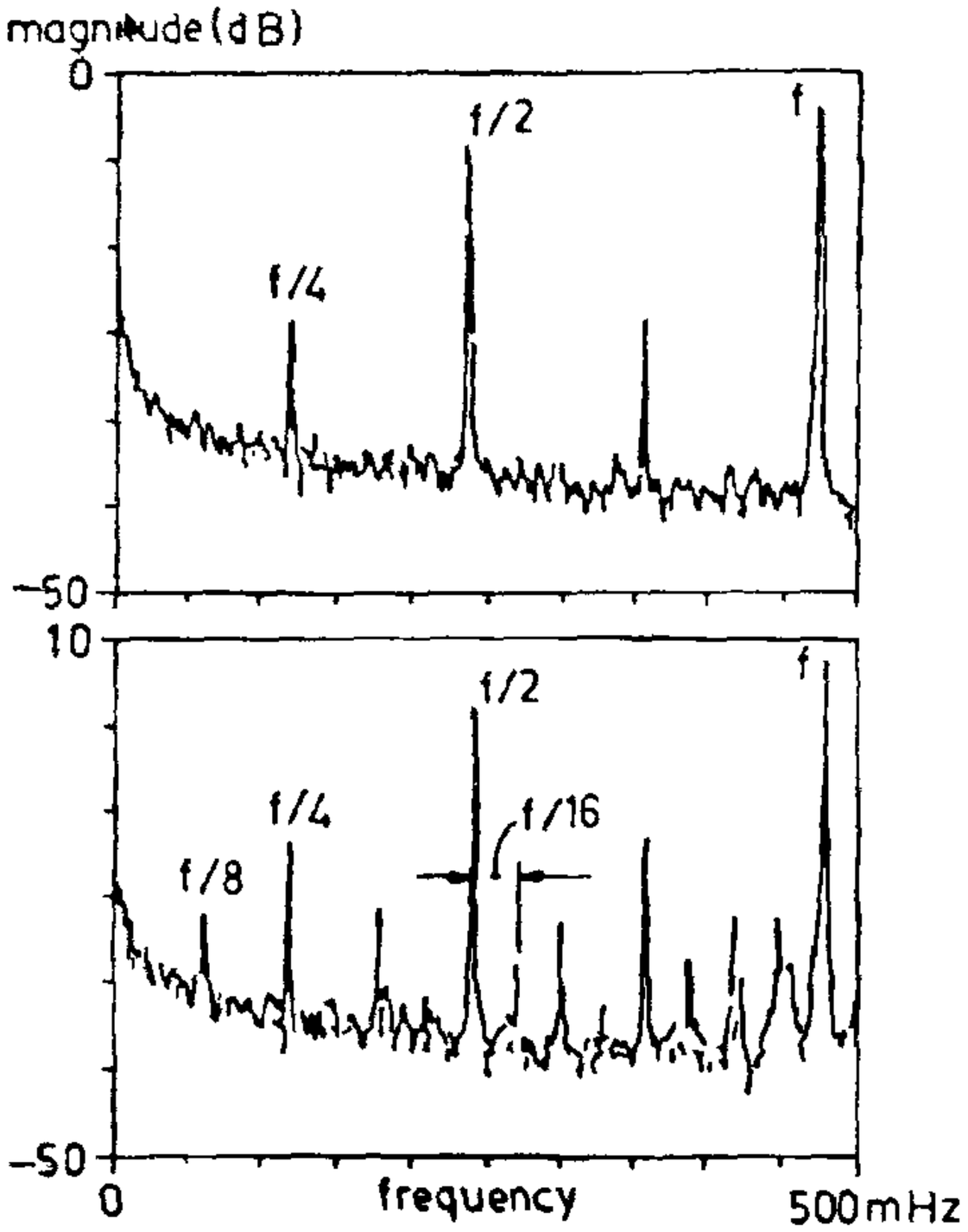

Figure 12. Spectrum of temperature fluctuations in convective flow of liquid helium in a small rectangular box. The conditions correspond to the last observable stages in a period-doubling sequence at a Rayleigh number about 40 times the critical value at which rolls appear. In the upper figure we observe two sub-harmonics (peaks at intervals of a fourth of the basic natural frequency), and in the lower four sub-harmonics (after Libchaber and Maurer[23]).

of transition through the turbulent spots we described earlier.

A different suggestion has been made by Ruelle and Takens[24]. They propose that in a variety of systems chaos does not appear after the infinite sequence of bifurcations that Feigenbaum has described or the infinite sequence of instabilities that Landau originally postulated. They show that in a wide class of systems, chaos can appear at the end of only *three* bifurcations. This they do by proving that in such systems what is known as a strange attractor can appear at this relatively early stage. The nature of such attractors is illustrated in figure 13, in particular for the Lorenz system[25]. The idea of a strange attractor, can

Figure 13. A sketch of the Lorenz strange attractor. The state of the system wanders endlessly on the attractor, getting arbitrarily close to previous positions but never *staying* close. Once the state moves away from rest (origin), it is sucked towards one of the two fixed points but is eventually flung away from it, to be suckled by the other fixed point and so on (compare figure 9a). Near the origin the attractor resembles a book-like object with infintely many sheets of zero thickness, constituting a Cantor set. If the space shown is chopped into cells of given size, and the number of cells containing the attractor counted up, it is found that as the cells get smaller the sum increases more rapidly than the inverse square of the (linear) size of each piece, and (of course) less rapidly than the inverse cube. The attractor therefore occupies more 'room' than a surface but less than a solid in three-dimensional XYZ-space: it is in fact an object with a fractional dimension (approximately 2.06 in the Lorenz case)[25]. [○ : State of rest; ● : Fixed points].

be described in simplified terms as follows:— Suppose we take a cup of coffee, stir it and let go, then usually the flow comes to rest after a while. That is, the stable state of motion for the conditions of this familiar early-morning fluid-dynamical problem is one of rest. To put this result in a little bit of jargon the state of rest is an "attractor" for this problem, in the sense that all neighbouring states tend towards one of rest eventually. (Compare the behaviour of the quadratic map in figure 11 for $R < 3$.) There are other conditions, or nonlinear systems, in which the solution eventually is not one of rest but rather one of steady oscillation with a fixed amplitude, which we have already introduced as a limit cycle. Limit cycles have the property that no matter where you start your motion, within a certain range of initial conditions, the system eventually settles down to a state of steady oscillation. Both of these are attractors; the state of rest is an attractor of dimension 0 (point), limit cycles are attractors of dimension 1 (curve, in a suitable space of states). We can also have an attractor of

dimension 2. In state space, this would correspond to a torus i.e. an object shaped like a *vadé* or donut: Here the state of the system is capable of two different kinds of closed trajectories, going around the torus either along its major circumference, so to speak, or around the smaller one across it. What Ruelle and Takens showed was that beyond these three possibilities there is a fourth, in which the point denoting the state of the system wanders forever without lying on any particular surface, getting quite close to previous positions at various times but never *staying* close to any point or cycle (compare our results with the tent map, table 1). It was suggested that turbulent fluid motion in fact represents such a strange attractor of the Navier-Stokes equations. The Lorenz system showed one of the first such strange attractors, although it was not so called at the time of the discovery; the nature of the strange attractor in the space of the variables X, Y and Z in the equations is shown in figure 13. (One may wonder whether the strange attractor idea would have been so popular if it had been called something else, e.g. singular fractal sink!)

THE TROUBLE WITH DYNAMICAL CHAOS

Although these new viewpoints are exciting and promise fresh insights into the problem of the development of chaos in fluid flows[26], we must remember that there are many difficulties—some of them not even faced yet by the new approaches. Let me give just three examples of the serious inadequacy of all currently known models. The first is that in all of these, chaos develops at low frequencies, beginning by the appearance of energy at subharmonics of a basic fundamental frequency. This is illustrated e.g. by the spectrum of $X(t)$ in the Lorenz model (figure 14). But in all fluid flows, a characteristic feature of the final transition to turbulence is the appearance of *high frequency* chaos following what is known as the cascade process. The filling of the spectrum at long waves, from slow or 'infra-red' chaos, does not explain how the short waves (high frequency oscillations) that are the true signature of turbulent flow arise. A second problem is that in most of these theories the critical value of

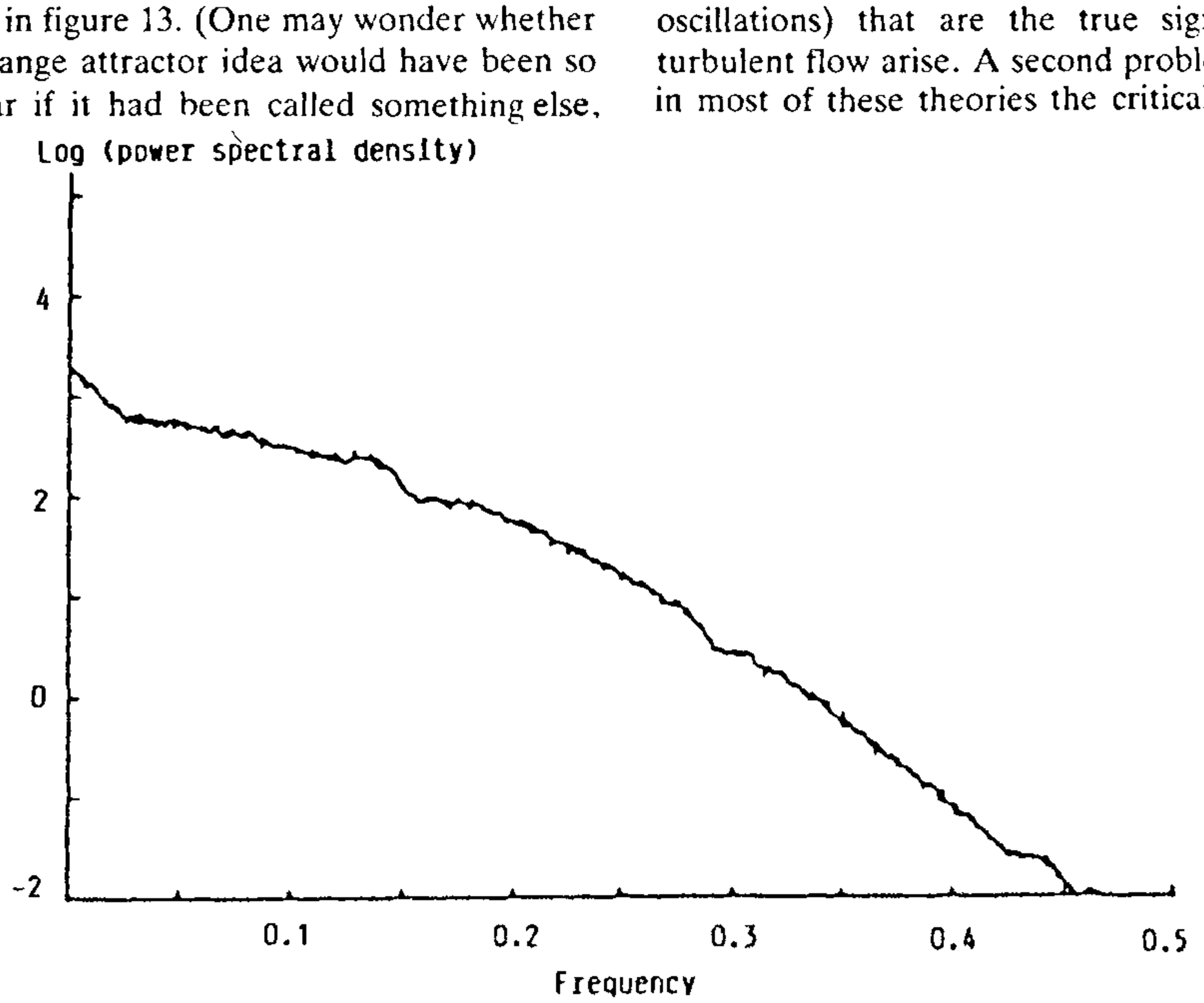

Figure 14. The power spectral density of $X(t)$ in the Lorenz system (from Farmer *et al*[27]).

the parameter, like the Reynolds or Rayleigh number at which the onset of chaos is predicted, is related directly to the values at which the initial instabilities appear. On the other hand, we know from observations extending over many decades now that there is no unique critical value for the parameter, but rather that its value depends strongly on the external disturbances in the flow. For example in transition on a flat plate, the critical Reynolds number can vary from less than 10^5 to at least something like 5×10^6, if not even higher, depending on the disturbance level (figure 15).

Finally, and to me most disconcertingly, these models that behave in such strangely attractive ways at moderately large values of the parameter do not in general preserve that trait for higher values. Thus the Lorenz attractor loses its strangeness at higher Rayleigh numbers: beyond $r = 50$ chaos disappears, and the solution is a simple and rather tame limit cycle once again (Sparrow[29])! And that seems totally unphysical.

These difficulties are so serious that one is tempted to wonder whether dynamical chaos is related to flow turbulence at all. The only way that the current position can be viewed, it seems to me, is that it has raised the significant question whether there are only a finite number of ways in which chaos can arise. Are there only certain "fundamental modes" of transition, irrespective of the situation in which chaos occurs, whether in fluid mechanics, or chemical kinetics, or aircraft motion at high angles of attack or (if we dare) social systems? Is there an analogy here to the relation between all matter and the elementary particles (albeit the latter keep proliferating)?—i.e. are all transitions to chaos made up of combinations of such fundamental modes? Or is the scenario different each time?

What we can say with confidence is that *if* there are only so many possible routes to chaos, they have certainly not all been discovered yet. If there are as many different routes as there are problems in which chaos manifests itself, it would of course be disappointing. At the present stage it appears as if the only way to find out is to observe more closely how chaos arises in each situation and construct the simplest possible dynamical models in every case. We are right now in the process of formulating models which we hope will include those basic features of turbulence that appear to us crucial in fluid flows. How far such models can go and what light they will eventually throw on the problem is still a very open question, but it is certainly something that should be very exciting to pursue.

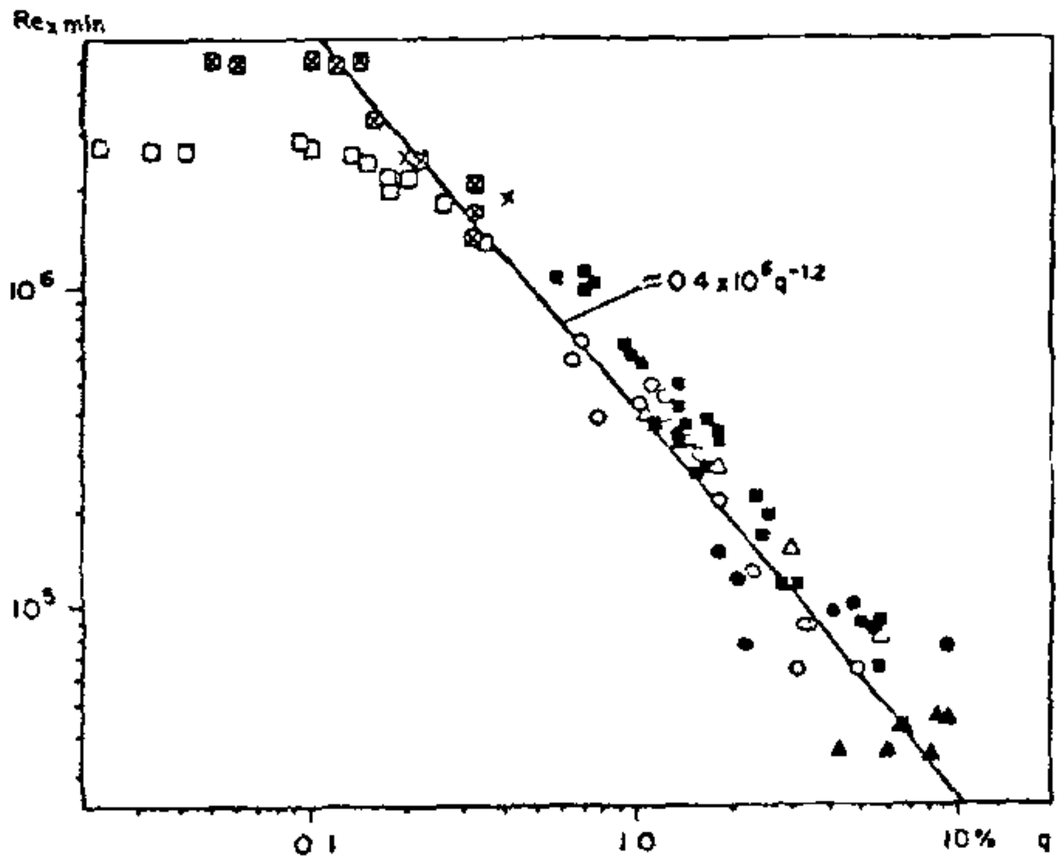

Figure 15. Dependence of the 'critical' Reynolds number, marking the birth of turbulent spots, on the disturbance level in the wind tunnels in which the experiments are conducted[28].

10 March 1987

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SCIENCE NEWS

DST WORKSHOP ON 'BIOSYSTEMATICS OF INSECTS OF IMPORTANCE IN AGRICULTURE, MEDICINE AND FORESTRY'

A DST-sponsored workshop on the above theme was conducted from 27–30th April with participation by senior entomologists from nearly 20 Universities and an equal number of young scientists. Inaugurating the workshop, Prof. S. Krishnaswamy, Vice-Chancellor, Madurai Kamaraj University, exhorted the participants to profitably use the emerging techniques in biosystematic studies so as to have a better and proper understanding of the species. The twenty-five papers presented related to tne role of ultrastructure, karyology, biochemical parameters, ecobehaviour and biogeography which sufficiently emphasised the need for such an integrated approach in order to be able to meaningfully assess the increasing variations in the natural population of insects of agricultural, medical and forestry importance, more noticeably in such pests species or vector species tending to exhibit what has come to be known as 'biotypes' 'siblings' etc. Of particular interest were the special lectures on 'Molecular biology and biosystematics of insects' by Prof. Kunthala Jayaraman of the Anna University; 'LDH system as a tool in biochemical systematics' by Prof. Kamalakar Rao of the Pachaiyappa's College, Madras and 'Raciation in *Drosophila* as demonstrated by laboratory experiments' by Dr Ranganath of the Mysore University, which discussed the emerging trends in the field of biosystematics. The plenary lecture by Prof. T. N. Ananthakrishnan of the Entomology Research Institute on 'The dimensions of species' highlighted the need for indepth investigations on various aspects involving diverse methodologies, to have a meaningful understanding of the concept of speciation, more particularly in view of the dynamics of the species.

Demonstration sessions on methodologies involving ultrastructure study, electrophoretic studies for LDH and proteins, karyology etc were also included.

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