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Cosmological models consistent with supersymmetric compactification of superstring theories

SANJAY KUMAR, S MAHAJAN*, A MUKHERJEE, N PANCHAPAKESAN and R P SAXENA

Department of Physics and Astrophysics, University of Delhi, Delhi 110 007, India
*St. Stephen's College, University of Delhi, Delhi 110 007, India

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Abstract. The compactification of 10-dimensional supergravity, coupled to super Yang–Mills theory, to curved 4-dimensional spacetimes is investigated. The requirement of unbroken supersymmetry leads to a set of consistency conditions. These are fairly restrictive, but nevertheless permit some nontrivial solutions, including the Milne universe. More general time-dependent metrics are also not ruled out.

Keywords. Superstrings; compactification; unbroken supersymmetry; spherical symmetry.

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1. Introduction

In recent times, superstring theories (Green *et al* 1987) have raised hopes of providing a unification of all forces in nature. They do also seem to offer, for the first time, a consistent way of quantizing gravity. In the limit of infinite string tension, these theories correspond to 10-dimensional supergravity coupled to super Yang–Mills theory. To make contact with 4-dimensional physics, the extra six dimensions have to be compactified. The paper of Candelas *et al* (1985) is an important landmark in this direction. It is an open question whether the sequence

$$\text{String theory} \rightarrow \text{10-dimensional field theory} \rightarrow \text{4-dimensional field theory} \tag{1}$$

is actually the correct and unique description of the process by which string theories make contact with low-energy physics. However, in spite of recent progress in formulating string theories directly in four dimensions (Antoniadis 1988), the above sequence remains interesting since it leads to definite predictions for the low-energy theory. Within this approach, Candelas *et al* (1985) derive restrictions on the compact 6-manifold by assuming that (i) the 10-dimensional geometry is of the form $M \times K$, where M is maximally symmetric and K is compact, and (ii) there is unbroken $N = 1$ supersymmetry in four dimensions. In this paper, we investigate the consequences of relaxing assumption (i), while maintaining (ii).

We seek vacuum states of the 10-dimensional theory such that the 4-dimensional metric has a nonvanishing background value

$$\langle g_{\mu\nu} \rangle = g^{cl}_{\mu\nu}(x). \tag{2}$$

We assume $g^{cl}_{\mu\nu}(x)$ to be of a general spherically symmetric form, given by the line element

$$ds^2 = -R_1(r,t)dt^2 + R_2(r,t)dr^2 + R_3(r,t)d\Omega^2, \tag{3}$$

where

$$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2.$$

The requirement of unbroken supersymmetry implies certain restrictions on the metric coefficients R_i. We attempt to satisfy these requirements under various ansatze. However, it turns out that the compactness of the 6-manifold K severely restricts the 4-dimensional background geometry. In almost all cases, we find that the simultaneous requirements of supersymmetry and compactness of K leaves us with only flat 4-geometry. We have an interesting case of a flat geometry in a nonflat coordinatization (§4), as well as a class of solutions which does not appear to reduce to flat spacetime (§5). We are yet to work out the physical implications of such background configurations.

In §2 we consider the conditions for unbroken supersymmetry and obtain the coupled set of differential equations satisfied by R_i. In the subsequent sections, we look for solutions to these equations under various ansatze. In §3, we consider time-independent metrics. In §4, we consider metrics of the Friedmann–Robertson–Walker type. In §5 a more general class of time-dependent metrics is considered, where the R_i's are products of functions of r and t. Section 6 contains some concluding remarks.

2. The consistency conditions

Following Candelas *et al* (1985), we assume vanishing background values for all fermion fields in the ground state configuration. For unbroken supersymmetry in four dimensions we require (Candelas *et al* 1985; Gates and Nishino 1986)*

$$0 = \delta\psi_\mu = \nabla_\mu\varepsilon + \frac{\sqrt{2}}{32}\exp(2\phi)(\gamma_\mu\gamma_5 \otimes H)\varepsilon \tag{4}$$

$$0 = \delta\psi_m = \nabla_m\varepsilon + \frac{\sqrt{2}}{32}\exp(2\phi)(\gamma_m H - 12H_m)\varepsilon \tag{5}$$

$$0 = \delta\lambda = \sqrt{2}(\gamma^m\nabla_m\phi)\varepsilon + \tfrac{1}{8}\exp(2\phi)H\varepsilon \tag{6}$$

$$0 = \delta\chi^\alpha = -\tfrac{1}{4}\exp(\phi)F^\alpha_{mn}\gamma^{mn}\varepsilon. \tag{7}$$

We assume (3) for the vacuum expectation value of the metric of 4-dimensional spacetime. The following special cases may be noted:

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*As a matter of fact, these equations are not exact (Gates and Nishino 1986), and there are additional terms on the r.h.s. of eqs (4)–(7). We ignore them here, since within the framework of (1) there is a regime where the relevant length scales are large compared to the Planck length, but small compared to the compactification length, so that (curvature)^2 terms can be neglected. As pointed out in the first paragraph, this is really an open question, and interesting physics may emerge from the consideration of these terms. This is currently under investigation.

i) Flat spacetime: $R_1 = R_2 = 1$, $R_3 = r^2$, ii) Schwarzschild spacetime: $\dot{R}_1 = (1 - 2M/r) = R_2^{-1}$, $R_3 = r^2$, iii) Robertson–Walker universe: $R_1 = 1$, $R_3 = R^2(t)r^2$, $R_2 = R^2(t)/(1 - kr^2)$, and iv) static de Sitter universe:

$$R_1 = (1 - kr^2) = R_2^{-1}, \quad R_3 = r^2.$$

As in Candelas *et al* (1985), (4) implies

$$[\nabla_\mu, \nabla_\nu]\varepsilon = -\frac{1}{(16)^2}\exp(4\phi)(\gamma_{\mu\nu} \otimes H^2)\varepsilon$$

i.e.

$$\frac{1}{4}R_{\mu\nu\lambda\sigma}(\gamma^{\lambda\sigma} \otimes \mathbb{1})\varepsilon = -\frac{1}{(16)^2}\exp(4\phi)(\gamma_{\mu\nu} \otimes H^2)\varepsilon \tag{8}$$

where $R_{\mu\nu\lambda\sigma}$ is the Riemann curvature tensor and

$$\gamma^{\mu\nu} = \gamma^{[\mu}\gamma^{\nu]}.$$

For a diagonal metric tensor, $R_{\mu\nu\lambda\sigma}$ is diagonal in pairs of indices, so that the product $R_{\mu\nu\lambda\sigma}\gamma^{\lambda\sigma}$ is proportional to $\gamma_{\mu\nu}$ for each μ and ν. Equations (5)–(7) do not explicitly involve $g_{\mu\nu}$; thus they can be handled as in Birrell and Davies (1982).

For the general metric of (3) we have, using the notation

$$R_i = \exp(f_i(r, t)), \quad i = 1, 2, 3, \tag{9}$$

the following expressions for $1/4R_{\mu\nu\lambda\sigma}\gamma^{\lambda\sigma}$ (here primes and dots refer to differentiation with respect to r and t respectively):

$$\mu = t, \nu = r: \{[1/2f_1'f_2' - 1/2f_1'^2 - f_1'']4R_2 + [\ddot{f}_2 + 1/2\dot{f}_2^2 - 1/2\dot{f}_1\dot{f}_2]/4R_1\}\gamma_{tr} \tag{10}$$

$$\mu = t, \nu = \theta: \{[\ddot{f}_3 + 1/2\dot{f}_3^2 - 1/2\dot{f}_3\dot{f}_1]/4R_1 - f_3'f_1'/8R_2\}\gamma_{t\theta} \tag{11}$$

$$\mu = r, \nu = \theta: \{[1/2f_2'f_3' - f_3'' - 1/2f_3'^2]/4R_2 + \dot{f}_2\dot{f}_3/8R_1\}\gamma_{r\theta} \tag{12}$$

$$\mu = \theta, \nu = \phi: \{1/2R_3 - f_3'^2/8R_2 + \dot{f}_3^2/8R_1\}\gamma_{\theta\phi} \tag{13}$$

while the other components vanish or are proportional to these. In order that factoring out $\gamma_{\mu\nu}$ from (8) leave an eigenvalue equation for the 8×8 matrix H^2, all the expressions in braces in (10) must be equal, although they may depend upon r and t. This leads to a set of three consistency conditions which can be written in the following form

$$[(\ddot{f}_2 - \ddot{f}_3) + 1/2(\dot{f}_2^2 - \dot{f}_3^2) + 1/2\dot{f}_1(\dot{f}_2 - \dot{f}_3)]/4R_1 = [f_1'' - 1/2f_1'^2 - 1/2f_1'(f_2' + f_3')]/4R_2 \tag{14}$$

$$[\ddot{f}_3 + 1/2\dot{f}_1\dot{f}_3]/4R_1 = 1/2R_3 - [f_3'^2 + f_1'f_3']/8R_2. \tag{15}$$

$$[\dot{f}_3(\dot{f}_2 - \dot{f}_3)]/8R_1 = [f_3'' - 1/2f_2'f_3']/4R_2 + 1/2R_3. \tag{16}$$

The possibility of a consistent string compactification with unbroken supersymmetry in nonflat background spacetimes depends on finding a nontrivial set of f_i (or R_i)

satisfying the consistency conditions of (14)–(16) together with the requirement that K be compact. A priori, there seems to be no argument to suggest that this is impossible. However, the technical difficulty of the problem (three coupled nonlinear partial differential equations) makes it unlikely that the most general solution to (14–16) can be found analytically. In the following sections, we subject (14)–(16) to a series of simplifying ansatze and look for solutions.

3. Time-independent geometries

When f_i's are independent of t, (14)–(16) become

$$f_1'' + 1/2 f_1'^2 - 1/2 f_1'(f_2' + f_3') = 0 \tag{17}$$

$$f_3'' + 1/2 f_3'^2 - 1/2 f_3'(f_1' + f_2') = 0 \tag{18}$$

$$1/2R_3 + (f_3'' - 1/2 f_2' f_3')/4R_2 = 0. \tag{19}$$

These equations are still too complicated to permit an analytical solution, so we make the following simplifying ansatz:

$$f_1 + f_2 = 0. \tag{20}$$

Note that this condition is satisfied by the Schwarzschild and static de Sitter geometries.
On substituting (20) into (18), we obtain

$$f_3'' + 1/2 f_3'^2 = 0$$

with the general solution

$$f_3 = 2\ln[A(r+B)],$$

i.e.

$$R_3 = A(r+B)^2. \tag{21}$$

Substituting (21) into (17), we obtain

$$f_1'' + f_1'^2 - f_1'/(r+B) = 0,$$

with the general solution

$$R_i = \exp(f_1) = \lambda(1/2r^2 + Br) + \mu. \tag{22}$$

Equation (19) is now simply a consistency condition on the solution in (21) and (22), yielding

$$[1/2\lambda AB^2 - A\mu + 1]/2A(r+B)^2 = 0.$$

Therefore the complete solution to (17)–(19) and (20) is given by (21) and (22), together with

$$1/2\lambda AB^2 - A\mu + 1 = 0. \tag{23}$$

This condition can be satisfied in infinitely many ways; thus, within these ansatze, there are infinitely many solutions to (14)–(16).

Substituting the above general solution into the expression in braces in (13), we obtain

$$1/4R_{\mu\nu\lambda\sigma}\gamma^{\lambda\sigma} = -\lambda/4\gamma_{\mu\nu}. \tag{24}$$

Since the coefficient of $\gamma_{\mu\nu}$ in (24) is a constant, the compactness of K implies, by the argument of Candelas *et al* (1985), exploiting (4)–(6), that it must be zero, corresponding to flat spacetime. It should be noted that the general solution above is related to the de Sitter metric (corresponding to $A=1$, $B=0$, $\mu=1$) by a redefinition of r.

It is amusing to note that (17)–(19) can also be solved exactly by means of the ansatz $f_2+f_3=0$, leading to the general solution

$$R_1 = A(r+B)^2$$
$$R_3 = R_2^{-1} = \lambda(1/2r^2 + Br) + \mu \tag{25}$$

with the relation

$$1 + 1/2\mu\lambda - 1/4\lambda^2 B^2 = 0. \tag{26}$$

However, substituting (25) and (26) in the expression in braces in (13), we again obtain (24)!

4. Friedmann–Robertson–Walker (FRW) metric

The FRW geometries correspond to the following special case of (3):

$$R_1 = 1, \quad R_2 = R^2(t)/(1-kr^2), \quad R_3 = R^2(t)r^2. \tag{27}$$

Equations (14) and (16) turn out to be identically satisfied under this ansatz. The remaining consistency condition, (15), reduces to

$$R\ddot{R} = \dot{R}^2 + k. \tag{28}$$

This equation can be integrated once, to obtain

$$(\mathrm{d}R/\mathrm{d}t)^2 = AR^2 - k \tag{29}$$

where A is a constant of integration. The final form of the solution depends on the values of A and k. There are three kinds of solution for $R(t)$: (a) exponentially growing (for $A>0$), (b) linear including a constant (for $A=0$), and (c) oscillatory (for $A<0$). Solutions of type (a) exists for all values of k, while those of types (b) and (c) exists only for $k<0$. In all cases it is trivial to show, from (28) and (29), that

$$\ddot{R}/R = A. \tag{30}$$

Further, for the FRW metric, the expression in braces in (10)–(13) all reduce to $\ddot{R}/2R$, so that (8) takes the form of an eigenvalue equation for H^2, with eigenvalue proportional to the constant A. Since H is anti-Hermitian, the solutions with $A>0$ are ruled out. Finally, since the eigenvalue is a constant, by the argument of §3 it must be zero. Then by (29) the only surviving solution for $R(t)$ is)

$$R(t)=(-k)^{1/2}t+B \tag{31}$$

where B is a constant. This is the well-known Milne universe (Milne 1932), which is a solution to the vacuum Einstein equations. It is a linearly growing, spherically symmetric universe which is empty of matter. Note that we have not used the Einstein equations anywhere.

The emergence of a truly cosmologically interesting solution from our consistency requirements [eqs (14)–(17)] can also be understood as follows. The Einstein equations under the FRW ansatz, with homogeneous matter density ρ and pressure p, are (Gibbons *et al* 1983)

$$\ddot{R}/R=-4\pi G(\rho+3p)/3 \tag{32}$$

$$(\dot{R}^2+k)/R=8\pi G\rho/3. \tag{33}$$

Because of the consistency requirement (28), these can be satisfied only if

$$p=-\rho.$$

A negative pressure is characteristic of an inflationary universe (Brandenberger 1985). Unfortunately, this possibility does not survive the compactification requirement, which by (31) yields $\rho=0$. Thus we are left with the vacuum Einstein equations.

As is clear from (31), the Milne universe is a linearly growing, spherically symmetric universe which is empty of matter. It is well known (Birrell and Davies 1982) that by choosing a suitable frame of reference the Milne universe can be seen to be flat. Classically, therefore, the solution (31) is nothing but Minkowski space in disguise. It must be remembered, however, that our arguments are semiclassical in nature: we are dealing with quantum fields in a background gravitational metric. In order to define particle states, one has to consider positive frequency solutions defined with respect to a particular choice of the time coordinate t. Since a choice of t corresponds to a choice of the vacuum, the latter is not invariant under general coordinate transformations. Equation (2) presupposes a choice of the vacuum. (More technically, the 4-dimensional vacuum of the comoving Milne observer is inequivalent to the Minkowski vacuum (Birrell and Davies 1982, p. 121).) For this reason, the solution (31) must be regarded as nontrivial. Moreover, note that the Einstein equations with a 'small' matter term admit a solution of the FRW type that has $k=-1$, is nonflat, and goes over to (31) in the limit $p\to 0$, $\rho\to 0$. Such a term will arise as the first quantum correction to (2).

5. Separable metrics

In this section, we consider geometries where the r and t dependences of the R_i's are factorizable. Thus

$$R_i(t)=\exp(g_i(t))\exp(h_i(r)) \tag{34}$$

or

$$f_i(r,t) = g_i(t) + h_i(r). \tag{35}$$

Notice that the FRW metric, considered in the previous section, is a special case of this class. Under this general ansatz, (14)–(16) take the form

$$\begin{aligned}&[\ddot{g}_2 - \ddot{g}_3 - 1/2\dot{g}_1(\dot{g}_2 - \dot{g}_3) + 1/2(\dot{g}_2^2 + \dot{g}_3^2)]\exp(g_2 - g_1)\\ &= [h_1'' + 1/2(h_1'^2 - h_1'h_2' - h_1'h_3')]\exp(h_1 - h_2) = C_1\end{aligned} \tag{36}$$

$$\begin{aligned}&[\ddot{g}_3 - 1/2(\dot{g}_3^2 + \dot{g}_1\dot{g}_3 + \dot{g}_2\dot{g}_3)]\exp(g_2 - g_1)\\ &\quad = -[h_3'' + 1/2(h_3'^2 - h_2'h_3' - h_1'h_3')]\exp(h_1 - h_2) = C_2\end{aligned} \tag{37}$$

$$\begin{aligned}1/2[\dot{g}_2\dot{g}_3 - \dot{g}_3^2]\exp(g_2 - g_1) &= 2\exp(g_2 - g_3)\exp(h_1 - h_3)\\ &\quad + [h_3'' - 1/2h_2'h_3']\exp(h_1 - h_2)\end{aligned} \tag{38}$$

where C_1 and C_2 are separation constants. Unlike (36) and (37), (38) does not separate into a t-dependent and an r-dependent part. This separation can however be achieved by imposing the additional condition

$$g_2 = g_3. \tag{39}$$

This is assumed in the subsequent discussion.

On substituting (39) in (36, 37) we obtain, for the time dependent parts,

$$\dot{g}_2^2\exp(g_2 - g_1) = C_1 \tag{40}$$

$$(\ddot{g}_2 - \dot{g}_2^2 - 1/2\dot{g}_1\dot{g}_2)\exp(g_2 - g_1) = C_2. \tag{41}$$

The solution of (40) can be written in the implicit form

$$\exp(g_2/2) = 1/2(C_1)^{1/2}\int \exp(g_1/2)\,dt + C_3. \tag{42}$$

Note that the allowed FRW solution of (31) is a special case of (42). On substituting (42) in (41), the latter is identically satisfied provided $-3C_1 = 2C_2$.

The space dependent parts now satisfy

$$h_1'' + 1/2(h_1'^2 - h_1'(h_2' + h_3')) = C_1\exp(h_2 - h_1) \tag{43}$$

$$h_3'' + 1/2(h_3'^2 - h_3'(h_1' + h_2')) = 3/2C_1\exp(h_2 - h_1) \tag{44}$$

$$h_3'' - 1/2h_2'h_3' + 2\exp(h_2 - h_3) = 0. \tag{45}$$

While (43)–(45) are simpler than (14)–(16), they are still too complex to permit a general analytical solution. From the result of §4, we know that (43)–(45) do possess a nontrivial solution. Presumably there may be other exact nontrivial solutions, but we have been unable to find any.

A physically reasonable special case occurs when $h_3 = r^2$. In that case (43)–(45) can be solved exactly, leading to

$$\exp(h_1) = -3C_1/4\alpha - (1 - \beta r^2) \tag{46}$$

$$\exp(-h_2) = (1 - \alpha r^2) \tag{47}$$

where α and β are constants. Clearly flat space, de Sitter space as well as the FRW spacetime are all special cases of (46)–(47). Of course, on imposing the requirements of (5)–(7), de Sitter space reduces to flat space and the FRW geometries reduce to the solution of (31).

6. Conclusions

We have considered the compactification of 10-dimensional supergravity in a class of background 4-geometries. We have imposed the requirement of unbroken supersymmetry, and found some solutions to the resulting consistency conditions. In all these cases, the geometry of the 6-manifold, and hence the field content of the 4-dimensional theory, is identical to that found by Candelas *et al* (1985); thus there are no new consequences for low energy physics. However, the vacuum of the Milne universe is different, in a semiclassical sense, from the Minkowski vacuum. The fact that the Milne universe is also a solution of the consistency conditions is encouraging for applications to cosmology using FRW metrics.

Our work can be extended and generalized in several ways. The coupled eqs (43)–(45) may be solved numerically; work on this is in progress. Of course, the separable solutions of §5 are not the most general solutions of (14)–(16), though it is not easy to see how to look after nonseparable solutions. Generalizing even further, (3) could be relaxed. Indeed, there are respectable cosmological models that do not fall in this class.

It should be emphasized that the Einstein equations have not been used anywhere as inputs. Indeed they emerge as solutions to the conditions imposed in §4. Thus the geometries found here are true vacuum solutions to the classical equations of motion (e.g. the de Sitter and Milne solutions). Since superstring theory is a "theory of everything", it should explain things like galaxy formation. Presumably, quantum fluctuations about these vacuum solutions are responsible for the creation of matter.

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