HOMOGENEOUS TUPLES OF OPERATORS
AND REPRESENTATIONS OF SOME CLASSICAL GROUPS

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ABSTRACT

Let $T = (T_1, \ldots, T_n)$ be a $n$-tuple of bounded linear operators on a fixed Hilbert space $\mathcal{H}$ and let $\varphi$ be a biholomorphic automorphism of $\Omega$, the joint spectrum of $T$. In this paper, we consider those $n$-tuples $T$ for which the joint spectrum $\Omega$ is of the form $G/K$, a bounded symmetric domain. Let $\varphi$ be any biholomorphic automorphism of the domain $\Omega$. Define, $\varphi(T)$ via a suitable functional calculus and call a $n$-tuple of operators $T$ homogeneous if $\varphi(T)$ is simultaneously unitarily equivalent to $T$ for every automorphism $\varphi$ of $\Omega$. For each homogeneous operator $T$, let $U_\varphi$ be a unitary operator implementing this equivalence. We obtain a characterisation of all the homogeneous operators Cowen-Douglas class and show that it is possible to choose the unitary $U_\varphi$ in such a way that the map $\varphi \to U_{\varphi}^{-1}$ is a unitary representation of the group of biholomorphic automorphisms of $\Omega$.

1. INTRODUCTION

Let $\Omega$ be a domain in $\mathbb{C}^n$ of the form $G/K$ where $G$ is a semisimple complex Lie group and $K$ is a maximal compact subgroup of $G$ so that $G$ operates holomorphically on $\Omega$. These domains were classified by Cartan into four domains of classical type and two exceptional ones. In this paper by a bounded symmetric domain, we will always mean one of the first four domains of classical type. For details we refer the reader to [6].

Let $T = (T_1, \ldots, T_n)$ be a pairwise commuting $n$-tuple of operators acting on a fixed Hilbert space $\mathcal{H}$. Assume that $T$ admits the closure $\text{cl} \Omega$ as a spectral set, that is, the map $\rho_T : \mathcal{P}(\Omega) \to \mathcal{L}(\mathcal{H})$ defined by

$$\rho_T(p) = p(T_1, \ldots, T_n)$$

is contractive, where $\mathcal{P}(\Omega)$ is the Banach algebra of all polynomials in $n$ variables.
with supremum norm over $\Omega$. Thus we can define $\varphi(T)$ for $\varphi$ in the closure $\mathcal{A}(\Omega)$ of $\mathcal{A}(\Omega)$ with respect to the supremum norm. For a biholomorphic automorphism in $G$ with coordinate functions $(\varphi^1, \ldots, \varphi^n)$, set $\varphi(T) = (\varphi^1(T), \ldots, \varphi^n(T))$.

1.1. **Definition.** Any $n$-tuple of pairwise commuting operators $T = (T_1, \ldots, T_n)$ admitting $c\Omega$ as a spectral set will be called homogeneous if $T$ is unitarily equivalent to $\varphi(T)$ for all $\varphi$ in $G$, that is there exists a unitary operator $U$ on $\mathcal{H}$ such that

$$U\varphi(T)U^* = (U\varphi^1(T)U^*, \ldots, U\varphi^n(T)U^*) = T.$$

1.2. **Question.** Given a domain $\Omega = G/K$ in $C^n$, characterise the homogeneous $n$-tuples of operators.

This question is of course interesting on its own right. In addition, the following proposition which is modelled after [1] [Arveson et al. [1, Proposition 1.3]] shows that each homogeneous $n$-tuple gives rise to a projective representation of the group $G$. The case of SU(1,1) was considered in [11]. We begin with some definitions.

1.3. **Definition.** A Polish space is a topological space, which is homeomorphic to a separable complete metric space.

Let $G$ be a second countable locally compact group. Recall that a projective representation of $G$ is a mapping $\varphi \rightarrow U_\varphi$ of $G$ into the group $U(\mathcal{H})$ of unitary operators on a fixed Hilbert space $\mathcal{H}$ such that

(i) $U_e = I_\mathcal{H}$ where $e$ is the identity in $G$.

(ii) $U_\varphi U_\psi = c(\varphi, \psi) U_{\varphi \psi}$, where $|c(\varphi, \psi)| = 1$,

and

(iii) $\varphi \rightarrow \langle U_\varphi \xi, \eta \rangle$ is a Borel function for each $\xi, \eta$ in $\mathcal{H}$.

The function $c: G \times G \rightarrow T$ is called a multiplier of $U$. It is uniquely determined by $U$ and is a Borel function on $G \times G$.

Also recall that a set $\mathcal{S}$ of operators on a Hilbert space is irreducible if there is no common reducing subspace $\mathcal{M}$ for all of the $T_1, \ldots, T_n$.

For any two Polish spaces $X$ and $Y$, let

(i) $E$ be any subset of $X \times Y$,

(ii) $\pi: X \times Y \rightarrow X$, $\pi(x, y) = x$ be the projection onto the first coordinate,

(iii) $E_x = \{y \in Y: (x, y) \in E\}$ be the section at $x$,

and

(iv) $D_E = \{x \in X: E_x \neq 0\}$ be the domain of $E$.

1.4. **Definition.** A selection for $E$ is a function $\varphi: D_E \rightarrow E$ which is contained in $E_x$, that is for all $x$ in $D$, $\varphi(x) \in E_x$.

The following is a powerful selection theorem due to Kenugi-Novikov [10, p. 471].

1.5. **Theorem.** If $X$ and $Y$ are Polish spaces and $E \subset X \times Y$ is a Borel set with $E_x$ compact for each $x$ in $X$ then $E$ admits a Borel selection.
We now have all the tools to prove the following

1.6. Theorem. Any irreducible $n$-tuple of operators $T$ admitting $c|\Omega$ as a spectral set is homogeneous if and only if there is a projective representation $\varphi \to U_\varphi$ of $G$ satisfying

$$U_\varphi T U_\varphi^* = \varphi(T).$$

Proof. The if part is trivial. To prove the converse, note that the set

$$E = \{ (\varphi, U) \in G \times U(H) : UTU^* = \varphi(T) \}$$

is a Borel subset of $G \times U(H)$. Each section $E_x$ is compact since $T$ is irreducible. Thus the Kenugi-Novikov theorem guarantees the existence of a Borel map

$$\varphi \to U_\varphi.$$ 

Observe that

$$U_\varphi U_\psi TU_\varphi^* U_\psi^* = U_\varphi(T)U_\psi^* =$$

$$= U_\varphi \lim_{n \to \infty} p_n(T)U_\psi^* = \psi(\varphi(T)) = U_\varphi TU_\psi^*$$

where we have chosen $p_n$ such that $p_n \to \psi$. Thus the unitary operator $U_\varphi^* U_\varphi U_\psi$ commutes with the operator $T$, which is irreducible. Therefore

$$U_\varphi^* U_\varphi U_\psi = c(\varphi, \psi)I, \quad |c(\varphi, \psi)| = 1$$

and it follows that $\varphi \to U_\varphi$ is a projective representation.

This proof of the theorem was suggested by E. Azoff to the first author.

We have not been able to obtain a complete characterization of homogeneous $n$-tuples of operators. However in this paper, we obtain a characterization of the homogeneous $n$-tuples $T$ which are in the Cowen-Douglas class $P_1(\Omega)$. This class of operators was introduced in [3, p. 334], see also [4].

2. HOMOGENEOUS $n$-TUPLES IN COWEN-DOUGLAS CLASS $P_1(\Omega)$

Following Cowen-Douglas [3], we define $P_1(\Omega)$ to be the class of those pairwise commuting operators $T$ acting on $H$ such that

(i) $\dim \bigcap_{j=1}^n \ker(T_j - \omega_j) = 1$ for all $(\omega_1, \ldots, \omega_n)$ in $\Omega$; 

(ii) The operator $T_\omega : H \to H \oplus \ldots \oplus H$ defined by

$$T_\omega x = \bigoplus_{j=1}^n (T_j - \omega_j x)$$
has closed range; and

$$(iii) \bigvee_{\omega \in \Omega} \left\{ \bigcap_{j=1}^n \ker(T_j - \omega_j) \right\} = \mathcal{H}.$$

For $T$ in $P_1(\Omega)$, let $H(T_1, \ldots, T_n)$ denote $\bigcap_{j,k=1}^n \ker(T_j T_k)$ and define

$$N_j(\omega) = (T_j - \omega_j) H(T_1 - \omega_1, \ldots, T_n - \omega_n).$$

2.1. **Theorem** (Cowen and Douglas). The $n$-tuples $(T_1, \ldots, T_n)$ and $(\tilde{T}_1, \ldots, \tilde{T}_n)$ in $P_1(\Omega)$ are unitarily equivalent if and only if $\text{tr}(\tilde{N}_j(\omega) \tilde{N}_k(\omega)^*)$ is identically equal to $\text{tr}(N_j(\omega)N_k(\omega)^*)$.

It was shown in [3] that each $n$-tuple in $P_1(\Omega)$ determines a nonzero holomorphic map $\gamma : \Omega \rightarrow \mathcal{H}$ such that $\gamma(\omega) \in \bigcap_{j=1}^n \ker(T_j - \omega_j)$ for all $\omega$ in $\Omega$ and the curvature of $T$ is

$$\mathcal{K}_T(\omega) = \left[ \frac{\partial^2}{\partial \omega_i \partial \omega_j} \log \|\gamma(\omega)\|^2 \right].$$

As in [3, p. 336–337] it can be verified that

$$\mathcal{K}_T = (N_j(\omega)N_k(\omega)^*)^{-1}.$$

Thus the curvature is a complete unitary invariant of $T$.

2.2. We now recall some well known results about the Bergman kernel on $\Omega$. Most of what follows can be found in Helgason [5]. However, the following is from Inoue [7].

Since $G$ is simply connected we can uniquely define, for each $t \in \mathbb{R}$, the power $j(\varphi, z)^t$ with $j(e, z)^t = 1$ ($e$ is the identity element in $G$) for all $z$ in $\Omega$. As usual $j(\varphi, z)$, denotes the Jacobian of $\varphi$ at $z$. For $z, \omega$ in $G$, let $K(z, \omega)$ be the Bergman kernel for $\Omega$. We can define $K(z, \omega)^t$, so that $K(z, \omega)^t > 0$ for all $z$ in $\Omega$.

Note that

$$j(\varphi \psi, z)^t = j(\varphi, \psi z) j(\psi, z)^t \quad \text{for } \varphi, \psi \text{ in } G, z \text{ in } \Omega;$$

$$K(\varphi z, \psi \omega)^t = j(\varphi, z)^{-t} K(z, \omega) j(\psi, \omega)^{-t}$$

for $\varphi$ in $G$ and $z, \omega$ in $\Omega$; and for $\varphi$ with $\varphi(0) = z$,

$$\left[ \frac{\partial^2}{\partial \omega_i \partial \omega_j} \log K(z, z)^2 \right] = \text{D} \varphi(0) \text{D} \varphi(0)^a.$$
Let $\mu$ be the Lebesgue measure on $\Omega$. Then we have
\[ \int f(\varphi z) d\mu(z) - \int f(z) \frac{1}{|j(\varphi^{-1}, z)|^2} d\mu(z) \]
for all integrable $f$ on $\Omega$ and $\varphi$ in $G$. For $t$ in $\mathbb{R}$ define a measure $\mu_t$ on $\Omega$ by
\[ d\mu_t(z) = K(z, z)^{-t+1}. \]
It follows from the above that $\mu_t$ is invariant under the action of $G$.

Let $L^2(\Omega, \mu_t)$ be the $L^2$ space of square integrable functions on $\Omega$ with respect to the measure $\mu_t$. Denote the space of holomorphic functions on $\Omega$ by $\mathcal{H}(\Omega)$ and the space $L^2(\Omega, \mu_t) \cap \mathcal{H}(\Omega)$ by $H^2(\Omega, \mu_t)$. The following proposition was proved in [7, Lemma 2.13].

2.3. Proposition. For any $t \geq 1$, $H^2(\Omega, \mu_t)$ is nonzero and is a closed subspace of $L^2(\Omega, \mu_t)$. Furthermore, it possesses a kernel function, which is a constant multiple of $K(z, w)^t$.

The following theorem shows that for a homogeneous $n$-tuple the curvature function is determined once its value at zero is known. The proof however is elementary and can be viewed as a change of variable formula for the curvature.

2.4. Theorem. If $(T_1, \ldots, T_n)$ is a homogeneous $n$-tuple of operators in $\mathcal{P}(\Omega)$ admitting $cI\Omega$ as a spectral set, then
\[ \mathcal{K}_T(\omega) = D\varphi(\omega)\mathcal{K}_T(0)D\varphi(\omega)^*, \]
where $\varphi$ is an automorphism of $\Omega$ which carries $\omega$ to zero and $\mathcal{K}_T(0)$ must be of the form $cI$.

Proof. Let $\omega \rightarrow \gamma_\omega$ be a holomorphic map from $\Omega$ to $\mathcal{K}$ such that $\gamma_\omega$ is in $\bigcap_{j=1}^n \ker(T_j - \omega_j)$ for each $\omega = (\omega_1, \ldots, \omega_n)$ in $\Omega$. It is easy to verify that $\omega \rightarrow \gamma_{\varphi(\omega)}$ is a holomorphic map such that $\gamma_{\varphi(\omega)}$ is in $\bigcap_{j=1}^n \ker(\varphi'(t) - \varphi'(\omega))$ for each $\varphi$ in $G$.

Thus $\omega \rightarrow \gamma_{\varphi^{-1}(\omega)}$ is holomorphic and $\gamma_{\varphi^{-1}(\omega)}$ is in $\bigcap_{j=1}^n \ker(\varphi'(t) - \omega_j)$. Applying the chain rule we obtain
\[ \mathcal{K}_{\varphi(T)}(\omega) = D_j D_k \log \|\gamma^{-1}(\omega)\|^2 = \]
\[ = ((D\varphi^{-1})(\omega)\mathcal{K}_T(\varphi^{-1}(\omega))((D\varphi^{-1})(\omega))^*. \]
Evaluate both sides at zero and observe that the Cowen-Douglas theorem implies the equality of $\mathcal{K}_{\varphi(T)}(0)$ and $\mathcal{K}_T(0)$ for each $\varphi$ in $G$, whenever $T$ is homogeneous.
Thus,
\[ \mathcal{H}_T(\varphi^{-1}(0)) = ((D\varphi)(\varphi^{-1}(0)))^{\varphi} \mathcal{H}_T(0)( (D\varphi)(\varphi^{-1}(0)))^{\varphi} \]
and so \( \mathcal{H}_T(0) \) commutes with \( D\psi(0) \) for each \( \psi \) in \( G \) such that \( \psi(0) = 0 \). In each of the four classical domains of interest here straightforward calculations imply that \( \mathcal{H}_T(0) \) must be a constant multiple of the identity. Since \( G \) acts transitively on \( \Omega \) the proof is complete.

Let \( M_z = (M_{z_1}, \ldots, M_{z_n}) \) denote the multiplication operators given by \( (M_z f)(z) = z_j f(z) \). It was pointed out in [4] that \( T \) in \( P_1(\Omega) \) is unitarily equivalent to \( M_z = (M_{z_1}, \ldots, M_{z_n}) \) on a Hilbert space with a kernel function \( K_\gamma \). We recall from [4] that, if \( T \) is in \( P_1(\Omega) \) then there exists a holomorphic map \( \gamma: \Omega_0 \subset \Omega \to \mathcal{H} \) such that \( \gamma(\omega) \) is in \( \bigcap_{j=1}^n \ker(T_j - \omega_j) \). Define \( U: \mathcal{H} \to \text{Hol}(\Omega) \) by
\[ (Ux)(\omega) = \langle x, \gamma(\omega) \rangle, \quad x \in \mathcal{H}, \omega \in \Omega. \]
Let \( \mathcal{H}_\gamma = \text{range } U \) and define the bilinear form \( \langle , \rangle_\gamma \) on \( \mathcal{H}_\gamma \) by
\[ \langle Ux, Uy \rangle_\gamma = \langle x, y \rangle; \quad x, y \in \mathcal{H}. \]

The map \( U \) is linear and injective, \( \mathcal{H}_\gamma \) is a Hilbert space with inner product \( \langle , \rangle_\gamma \) and \( U \) is a Hilbert space isomorphism. Furthermore, the space \( \mathcal{H}_\gamma \) is invariant under multiplication by the coordinate functions \( z_j \) and the \( n \)-tuple \( M_z = (M_{z_1}, \ldots, M_{z_n}) \) of these multiplication operators belongs to \( L(\mathcal{H}_\gamma) \). Indeed \( U \) intertwiners \( T \) and \( M_z^* \). Evaluation at each point is a bounded linear functional from \( \mathcal{H}_\gamma \) to \( C \). Moreover, there exists a reproducing kernel for the space \( \mathcal{H}_\gamma \) given by \( K_\gamma(\lambda, \mu) = \langle \gamma(\lambda), \gamma(\mu) \rangle \) for \( \lambda, \mu \) in \( \Omega_0 \).

2.4. Theorem. For \( t \geq 1 \), let \( T_t \) denote the \( n \)-tuple \( M_t^* = (M_{t_1}^*, \ldots, M_{t_n}^*) \) on \( H^2(\Omega, \mu) \). Then \( T_t \) is in \( P_1(\Omega) \) and \( T_t \) is homogeneous. Moreover, there exists \( U_t \) satisfying \( U_t^* T_t U_t = \varphi(T) \) which is of the form
\[ (U_t\varphi f)(z) = j(\varphi^{-1}, \varphi^{-1}(z))^{-t} f(\varphi^{-1}(z)). \]

Proof. The fact that \( T_t \) is in \( P_1(\Omega) \) follows from [12, Proposition 4.1]. There it is shown that if \( \Omega \) is a pseudoconvex domain and \( H^2(\Omega, \nu) \) is the closed subspace of \( L^2(\Omega, e^{-\varphi} d\nu) \) consisting of analytic functions on \( \Omega \) then the Koszul complex determined by \( T_t \) is exact except at the end point, where
\[ \dim H_n(\Omega, \nu) = 1. \]

Since the domain \( \Omega \) we are looking at is a bounded symmetric domain, it is convex. In particular it is pseudoconvex. Also the measure \( \mu_t \) can be written in
the form \(\exp(-\log K(z, z)^{t-1})\). Thus, \(\varphi(z) = (t - 1)\log K(z, z)\) is continuous and plurisubharmonic since

\[
[\partial_z, \partial_{\bar{z}}] = \log K(z, z) \geq 0,
\]

cf. [5, p. 368]. This shows that \(T_t\) is in \(P_1(\Omega)\), cf. [4, Remark 2.4C]. Considering \(\varphi(T_t)\), we find that

\[
\gamma_{\varphi(T_t)}(\omega) = \gamma_{T_t}(\varphi^{-1}(\omega)).
\]

Thus, the map \(U: H^2(\Omega, \mu_t) \to \text{Hol}(\Omega)\) defined by

\[
UX(\omega) = \langle x, \gamma_{\varphi(T_t)} \rangle
\]

interwines \(\varphi(T_t)\) and \(M^*_t\) on \(H^2\). The kernel for \(H^2\), \(K(\lambda, \mu) = \langle \gamma(\lambda), \gamma(\mu) \rangle = (\varphi^{-1}(\lambda), \varphi^{-1}(\mu)) = K(\varphi^{-1}(\lambda), \varphi^{-1}(\mu))\).

However,

\[
K(\varphi^{-1}(\mu), \varphi^{-1}(\lambda)) = j(\varphi^{-1}, \mu)K(\lambda, \mu)f(\varphi^{-1}, \lambda).
\]

Lemma 4.8 of [4] implies that \(M^*_t\) on \(H^2\) is unitarily equivalent to \(T_t\). Furthermore by Lemma 3.9 of the same article [4] the map \(U_t: H^p \to H^p\) defined by

\[
U_tf(\omega) = j(\varphi^{-1}, \omega)f(\omega)
\]

interwines \(M^*_t\) on \(H^2(\mu_t)\) and \(M^*_t\) on \(H^2\). Thus \(U_\varphi = U_tU\) is an unitary map intertwining \(M^*_t\) on \(H^2(\mu_t)\) and \(\varphi(M^*_t)\). Observe that

\[
U_\varphi f(\omega) = U_tU f(\omega) = \langle U f, \gamma(\omega) \rangle = Uf(\gamma(\omega)) = j(\varphi^{-1}, \varphi^{-1}(\omega)) f(\varphi^{-1}(\omega)).
\]

2.5. Remark. When \(t\) is an integer greater than \(n\), the map \(\varphi \to U_{\varphi^{-1}}\) is an irreducible representation of \(G\) which is in the discrete series.

3. THE CASE OF THE UNIT BALL

In the following \(I = (i_1, \ldots, i_n)\) will always denote a multi-index of positive integers. Let \(\epsilon_k = (0, \ldots, 0, 1, 0, \ldots, 0)\) be the multi-index having \(i_j = 1\) or \(0\) according as \(j = k\) or otherwise. The multi-index \(I + k\) denotes \(\langle i_1, \ldots, i_k + k, \ldots, i_n \rangle\). Let \((\omega_l)\) be an orthogonal basis for a complex Hilbert space \(H^p\) and let \(\omega_{I,j}, j = 1, \ldots, n\), be a bounded sequence of complex numbers such that

\[
\omega_{I,k}\omega_{I+\epsilon_k,k} = \omega_{I,l}\omega_{I+\epsilon_l,k}.
\]
3.1. Definition. A system of $n$-variable weighted shifts is a family of $n$ operators $(T_1, \ldots, T_n)$ on $H$ such that

$$Te_i = \omega_{i, j} e_{i + e_j}.$$ 

As in the single operator case, a commuting system of $n$-variable weighted shifts is an $n$-tuple of multiplication operators on a suitable Hilbert space consisting of formal power series in $n$ variables defined as follows.

3.2. Definition. Let $\{\beta_I : I \geq 0\}$ be a set of strictly positive numbers with

$$H^2(\beta) = \{f(z) = \sum f_I z^I : \|f\|^2 = \sum \|f_I\|^2 \beta_I^2 < \infty\}.$$ 

Clearly, $H^2(\beta)$ is a Hilbert space with inner product

$$\langle f, g \rangle = \sum f_I g_I \beta_I^2.$$ 

The set $M_\beta^g$ is a commuting system of $n$-variable weighted shifts with $\omega_{i, j} = \beta_{i + e_j}(\beta_I)^{-1}$ and it is possible to go the other way round cf. [8].

Let $\beta(I)^{-2}$ denote the coefficient of $\omega^I \bar{\omega}^j$ in the multivariable binomial expansion of

$$(1 - \omega^I \bar{\omega}^j)^{-r}, \quad r \text{ real and } \omega \text{ in } B^g \subset C^n$$

It is then evident that the kernel function for $H^2(\mu_t)$ is $(1 - \langle z, \omega \rangle)^{-r}$. Following general considerations of Jewell and Lubin [8], we see that if $\omega = (\omega_1, \ldots, \omega_n)$ is in the ball, then $\omega_j$ is an eigenvalue for $M_\beta^g$ with joint eigenvector

$$K(z, \omega) = (1 - \langle z, \omega \rangle)^{-r} = (1 - \langle z, \omega \rangle)^{-(n+1)t}, \quad t = r/(n + 1).$$

Of course $(1 - \langle z, \omega \rangle)^{-(n+1)}$ is the Bergman kernel for the ball in $C^n$.

3.3. Theorem. $T_i$ is in $P_t(B^g)$ for $t \geq 1/(n + 1)$.

Proof. In view of [4], we have to only verify that for the weighted shift $\varphi(T_i) \rightarrow -\omega I$ satisfies

$$D \leq \sum_{j=1}^n |\omega_{i - r, j}|^2 \leq C$$

for all $\omega$ in the unit ball $C^n$. We consider the case of $\omega = 0$ and immediately see that

$$\beta(I) = \frac{|I_1 i_1! \ldots i_n!|}{t(t - 1) \ldots (t + |I| - 1)(i_1 + \ldots + i_n)} = \frac{i_1! \ldots i_n!}{t(t - 1) \ldots (t + |I| - 1)} = 1.$$
\[ \omega_{j,1} = \left( \frac{i_1! \cdots (i_j + 1)! \cdots i_n!}{t(t-1) \cdots (t+|I|)} \right) \bigg/ \left( \frac{i_1! \cdots i_n!}{t \cdots (t+|I|-1)} \right) = \frac{i_j + 1}{t + |I|}, \]

and

\[ \sum_{j=1}^{n} |\omega_{j,1}|^2 = \sum_{j=1}^{n} \frac{i_j}{t + |I| - 1} = \frac{|I|}{t + |I| - 1}, \]

which is both bounded below and above. Writing down the homogeneous expansion for \( K(z, \omega) \) around the point \((z_0, \omega_0)\), we can verify that \((M^*_z - \omega_0 I)\) also satisfies similar inequalities. Thus \( M^*_z \) is in \( P_1(B^n) \).

3.4. Remarks. a) In the case of the ball in \( \mathbb{C}^n \) for \( t \) in the set \( \{1/(n+1), \ldots, n/(n+1)\} \) we do get irreducible unitary representations of \( SU(n,1) \) in a very simple form. The fact that these representations are irreducible follows from a rather general result of Kunze [9]. However, these representations are no longer in the discrete series [13]. Of course the case of \( t = n/(n+1) \) corresponds to familiar Hardy space on the ball. Note that if \( t = k/(n+1), \quad 1 \leq k \leq n-1 \), it is not clear that \( M^*_z \) admits \( B^n \) as a spectral set, however \( \varphi(M^*_z) \) can still be defined to be \( (M^*_{\varphi^{-1}(z_1)}, \ldots, M^*_{\varphi^{-1}(z_n)}) \). To see that \( \varphi(M^*_z) \) defines an \( n \)-tuple of bounded linear operators, we merely note that \( \varphi(M^*_z) \) is unitarily equivalent to \( M^*_z \) on the Hilbert space \( \mathcal{H} \) with kernel function \( K(\varphi^{-1}(\lambda), \varphi^{-1}(\mu)) \), where the kernel function is some power of the Bergman kernel function; transformation properties of the kernel function (see, Section 2.2) imply that \( \varphi(M^*_z) \) is a bounded \( n \)-tuple of operators.

b) To treat the case of an arbitrary real \( t \), we have to use the notion of a Wallach set, which will be taken up in a subsequent paper.

REFERENCES


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