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Hartree–Fock variational bounds for ground state energy of chargeless fermions with finite magnetic moment in the presence of a hard core potential: A stable ferromagnetic state

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Abstract. We use different determinantal Hartree–Fock (HF) wave functions to calculate true variational upper bounds for the ground state energy of N spin-half fermions in volume V_0, with mass m, electric charge zero, and magnetic moment μ, interacting through magnetic dipole–dipole interaction. We find that at high densities when the average interparticle distance r_0 becomes small compared to the magnetic length $r_m \equiv 2m\mu^2/\hbar^2$, a ferromagnetic state with spheroidal occupation function $n_\uparrow(\vec{k})$, involving quadrupolar deformation, gives a lower upper bound compared to the variational energy for the uniform paramagnetic state or for the state with dipolar deformation. This system is unstable towards infinite density collapse, but we show explicitly that a suitable short-range repulsive (hard core) interaction of strength U_0 and range a can stop this collapse. The existence of a stable equilibrium high density ferromagnetic state with spheroidal occupation function is possible as long as the ratio of coupling constants $\Gamma_{cm} \equiv (U_0 a^3/\mu^2)$ is not very small compared to 1.

Keywords. Chargeless fermions; magnetic dipole–dipole interaction; Hartree–Fock bounds; ferromagnetic ground state; neutrino-like gas.

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1. Introduction

Because of its great relevance to the properties of ordinary matter, the problem of the nature of the ground state of many-particle electron systems, like the electron gas, has been studied extensively for more than seven decades now. However, the nature of the ground state of quantum spin-half chargeless fermion systems, like neutrinos, with finite magnetic dipole moment has not drawn too much attention. In an earlier paper [1], we had tried to address this problem involving spin-dependent long-range noncentral interaction between particles with no electric charge, without any short-range repulsive central hard core interaction. However, that calculation was not done with an N-particle determinantal HF wave function. We had used instead eigenvalues $n_\sigma(\vec{k})$ of a positive semi-definite single particle density matrix operator $\rho^{(1)} = \sum_{\vec{k}\sigma} n_\sigma(\vec{k})|\vec{k}\sigma\rangle\langle\vec{k}\sigma|$, with $0 \leq n_\sigma(\vec{k}) \leq 1$, as variational parameters. We discovered later that Lieb [2] had shown long back that unless the interaction is purely repulsive everywhere, which unlike in the case of an electron gas is not true in our problem, only a variational determinantal HF wave function would give an upper bound to the ground state energy of such a system, and not any arbitrary variational positive definite single particle density matrix. In other words, our earlier result may not correspond to any variational bound to the energy at all! In view of this, here we restrict ourselves to variational determinantal Hartree–Fock (HF) wave functions to investigate the nature of the ground state of such systems and calculate true variational upper bounds to their ground state energy. At high enough densities, a ferromagnetic state with a spheroidal shape of the boundary of occupied states in $\vec{k}\uparrow$-space ($n_\uparrow(\vec{k}) = 1$), to be referred as the JM deformed ferromagnetic HF state to distinguish it from the state used in [1], gives the upper bound to the ground state energy. This state, as expected for dipole–dipole interaction ($\sim r^{-3}$), is unstable towards a high density collapse. However, we will explicitly show that this instability of the system is stopped by an additional suitable short-range repulsive finite hardcore interaction. These results may not be of direct relevance to the neutrino cosmology at this stage, because of the tiny neutrino magnetic moment, but it may be a very interesting problem in itself in the study of quantum many-particle systems.

In §2 of this paper, we first briefly present the calculation of variational determinantal HF bounds for the ground state energy of N chargeless fermions of mass m, spin-1/2 and magnetic moment μ in volume $V_0 \equiv (4\pi/3)r_0^3N$, interacting through the long-range magnetic dipole–dipole interaction only. The quadrupolar deformation represented by the JM ferromagnetic state gives a lower upper bound compared to a HF ferromagnetic state with dipolar deformation. Then, we calculate the effect of including a short-range repulsive hard core interaction of strength U_0 and range a between the chargeless fermions, in addition to the magnetic dipole interaction. We find that if the hard core coupling constant U_0a^3 is not extremely weak compared to the magnetic coupling constant μ^2, there is no infinite density collapse of the JM ferromagnetic state. We thus show the possible existence of a stable equilibrium high density JM ferromagnetic ground state for such a system. Finally, we conclude with §3.

2. Single determinant HF variational calculations of ground state energy upper bounds

An N-particle determinantal HF wave function can be written in the form

$$\Psi_N(\vec{r}_1s_1,...,\vec{r}_Ns_N) = \langle\vec{r}_1s_1,...,\vec{r}_Ns_N|\Psi_N\rangle = \left(1/\sqrt{N!}\right)\det[f_{\vec{k}_i\sigma_i}(\vec{r}_js_j)] \tag{1}$$

as constructed from any N single particle orthonormal functions $f_{\vec{k}_i\sigma_i}(\vec{r}, s)$, where $\{\vec{r}, s\}$ refer to space-spin variables. The actual choice of the complete set of single particle orbitals f, in general, need not necessarily correspond to plane-wave states. The N-particle HF density matrix operator $\rho_{\rm HF}^{(N)} = |\Psi_N\rangle\langle\Psi_N|$ corresponds to a pure state. The matrix elements of the corresponding reduced single particle density matrix is given by the expression [2]

$$\langle\vec{r}s|\rho_{\rm HF}^{(1)}|\vec{r}\,'s'\rangle = \rho_{\rm HF}^{(1)}(\vec{r}s, \vec{r}\,'s') = \sum_{\vec{k}\sigma} n_{\vec{k}\sigma} f_{\vec{k}\sigma}(\vec{r}s) f^*_{\vec{k}\sigma}(\vec{r}\,'s'), \tag{2a}$$

where, in the particular case of plane-wave single particle orbitals and spin functions χ_σ, to be used here throughout,

$$f_{\vec{k}\sigma}(\vec{r}s) = \langle\vec{r}s|\vec{k}\sigma\rangle = (V_0)^{-1/2}\{\exp(i\vec{k}\cdot\vec{r})\}\chi_\sigma(s) \tag{2b}$$

and where $n_{\vec{k}\sigma} \equiv n_\sigma(\vec{k}) = 1$ for all the N orthonormal single particle states used in constructing the determinantal HF wave function, and $n_\sigma(\vec{k}) = 0$ for all the other states in the complete set of single particle states in the Hilbert space. Let us assume that we are dealing with a determinantal HF wave function Ψ_N of the type given by eq. (1) to calculate the variational energy $\langle\Psi_N|H|\Psi_N\rangle$, which is equivalent to using the single particle density matrix operator $\rho^{(1)} = \rho_{\rm HF}^{(1)}$ of eqs (2a) and (2b) with $n_\sigma(\vec{k}) = 1(0)$ for the occupied(unoccupied) states.

2.1 *Single determinant variational calculation with only magnetic dipole–dipole interaction*

First, let us consider the case with no hard core interaction. For the magnetic dipole–dipole interaction, the total Hamiltonian [1] is given by

$$H = \sum_{i=1}^{N}\frac{p_i^2}{2m} + \sum_{i<}^{N}\sum_{j}^{N} V(\vec{r}_i s_i, \vec{r}_j s_j), \tag{3a}$$

$$V(\vec{r}_1 s_1, \vec{r}_2 s_2) = \left(\frac{\mu^2}{r^3}\right)[\vec{s}_1\cdot\vec{s}_2 - 3\vec{s}_1\cdot\hat{r}\vec{s}_2\cdot\hat{r}], \tag{3b}$$

$$\vec{r} = \vec{r}_1 - \vec{r}_2, \quad r = |\vec{r}|, \quad \hat{r} = \vec{r}/r, \tag{3c}$$

$$\begin{aligned} V_{12}(\vec{q}) &\equiv \int \mathrm{d}^3 r \mathrm{e}^{-i\vec{q}\cdot\vec{r}} V(\vec{r}_1 s_1, \vec{r}_2 s_2) \\ &= \mu^2 \sum_{M=-2}^{+2} h_{-M} N_{12}^{(M)}(\vec{s}_1, \vec{s}_2) Y_{2,-M}(\hat{q})\{1 - \delta_{\vec{q},0}\}, \end{aligned} \tag{3d}$$

where h_{-M} are known numerical constants, $N_{12}^{(M)}(\vec{s}_1, \vec{s}_2)$ are spin operators corresponding to two spin-1/2 particles, and $Y_{2M}(\hat{q})$ are spherical harmonics of order 2. The interaction matrix element vanishes for the momentum transfer $\vec{q} = 0$ and is independent of the magnitude of the momentum transfer. The two-particle spin operator $N_{12}^{(M)}$ connects only those states for which the total z-components of the two spins differ by M. When one is taking the expectation value of V in any chosen determinantal state, only $M = 0$ term contributes. In fact, the total variational energy is then given by

$$E = \langle \Psi_N | H | \Psi_N \rangle = E_{\text{kin}} + E_{\text{exch}}, \tag{4}$$

$$E_{\text{kin}} = \sum_{\vec{k}\sigma} \left(\frac{\hbar^2 k^2}{2m} \right) n_\sigma(\vec{k}), \quad N \equiv \sum_{\vec{k}\sigma} n_\sigma(\vec{k}), \tag{5}$$

$$E_{\text{exch}} = -\left(\frac{\mu^2}{2} \right) h_0 \frac{1}{V_0} \sum_{\vec{k}} \sum_{\vec{q}} \sum_{\sigma_1} \sum_{\sigma_2} n_{\sigma_1}(\vec{k} + \vec{q}) \times n_{\sigma_2}(\vec{k}) Y_{20}(-\hat{q}) \bar{N}_{12}^{(0)}(\sigma_1, \sigma_2), \tag{6}$$

where

$$h_0 = \frac{4\pi}{3} \frac{\sqrt{16\pi}}{\sqrt{5}}; \quad \bar{N}_{12}^{(0)}(\sigma_1, \sigma_2) = \frac{1}{4} \delta_{\sigma_1,\sigma_2} - \frac{1}{4} (\delta_{\sigma_1,\downarrow} \delta_{\sigma_2,\uparrow} + \delta_{\sigma_1,\uparrow} \delta_{\sigma_2,\downarrow}), \tag{7}$$

$$Y_{20}(\hat{q}) = Y_{20}(-\hat{q}) = \left(\frac{5}{4\pi} \right)^{1/2} P_2(\cos\theta_{\hat{q}}) = \left(\frac{5}{4\pi} \right)^{1/2} (1/2)(3\cos\theta_{\hat{q}}^2 - 1). \tag{8}$$

As in the case of the familiar electron gas problem with uniform positive ionic background, there is no contribution due to the direct dipolar term because $V_{12}(\vec{q} = 0) = 0$. Note that in any variational paramagnetic state, with $n_\uparrow(\vec{k}) = n_\downarrow(\vec{k})$, the exchange contribution goes to zero because of summations over spins σ_1 and σ_2, due to the particular form of the spin matrix elements given in eq. (7). There is a contribution only from the kinetic energy part, and the best variational paramagnetic state is then nothing but the noninteracting uniform paramagnetic state with energy E_0. The exchange term of eq. (6) gives the maximum negative contribution when $\sigma_1 = \sigma_2$, and the direction of the momentum transfer is such that $\cos\theta_{\hat{q}}^2 < 1/3$, i.e., close to the spin quantization direction $\hat{z}$. Note that the summations over $\vec{k}$ and $\vec{q}$ give zero for the exchange contribution if the occupation function is spherical. To proceed further, let us assume that all spins are parallel (say, in the up-direction), with $n_\downarrow(\vec{k}) = 0$, but $n_\uparrow(\vec{k})$ depends on both the magnitude and the direction of $\vec{k}$. For example, one may consider the form

$$n_\uparrow(\vec{k}) = \Theta(k_{F\uparrow}^2 - k^2 + k^2(\delta_1 Y_{10}(\hat{k}) + \delta_2 Y_{20}(\hat{k}) + \cdots)); \quad \sum_{\vec{k}} n_\uparrow(\vec{k}) = N \tag{9}$$

in which $\delta_1, \delta_2, \ldots,$ lead to deformation from the spherical surface. Before proceeding further, it has to be noted that following the usual procedure [3], a simple calculation of the proper self-energy $\hbar\Sigma_\uparrow^*(\vec{k})$ can be easily done to the lowest order in the dipole–dipole interaction for a system of particles with only up-spins, starting with the unperturbed Green's function of the noninteracting gas corresponding to the occupation function $n_{0\sigma}(k) = \Theta(k_{F\uparrow} - k)\delta_{\sigma,\uparrow}$. Such a calculation shows that the resulting proper self-energy is proportional to $(-)Y_{20}(\hat{k})$ with a positive proportionality factor which is a function of the magnitude k. This implies that in the polar direction, the single particle energy $(\hbar^2/2m)k^2 + \hbar\Sigma_\uparrow^*(\vec{k})$ is lower compared to its value in the equatorial $k_x - k_y$ plane for the same value of the magnitude of the wave vector. Note, however, that the contribution to the total energy from this proper self-energy goes to zero, as already observed, since it requires the integration over $\vec{k}$ of the product of the self-energy and the unperturbed spherical function $n_{0\uparrow}(k)$. Thus, there is no alternative but to deform the occupation function from the spherical shape to get any nonvanishing contribution to E_{exch}. However, the simple self-energy calculation does suggest that it is more natural to consider the deformation of the surface bounding the occupied states to be of the quadrupolar type corresponding to a prolate spheroid. We consider this case first.

Let the occupied region of $\vec{k}$ be a prolate spheroid (a symmetrical egg) pointed towards the z-direction. The surface bounding the occupied region in the $\vec{k}$-space is then given by the equation

$$\frac{(k_x^2 + k_y^2)}{k_{\mathrm{F}x}^2} + \frac{k_z^2}{k_{\mathrm{F}z}^2} = 1, \quad k_{\mathrm{F}z} > k_{\mathrm{F}x} \tag{10}$$

so that

$$n_\uparrow(\vec{k}) = \Theta\left(1 - \frac{(k_x^2 + k_y^2)}{k_{\mathrm{F}x}^2} - \frac{k_z^2}{k_{\mathrm{F}z}^2}\right) = \Theta(k_{\mathrm{F}\uparrow}^2 - k^2(1 - \beta_2 P_2(\cos\theta_{\hat{k}}))) \tag{11}$$

in this state (to be called the JM ferromagnetic state), where

$$k_{\mathrm{F}\uparrow}^2 = \frac{3k_{\mathrm{F}x}^2 k_{\mathrm{F}z}^2}{(2k_{\mathrm{F}z}^2 + k_{\mathrm{F}x}^2)}; \quad \beta_2 \equiv \left(\frac{5}{4\pi}\right)^{1/2} \delta_2 = \frac{2(k_{\mathrm{F}z}^2 - k_{\mathrm{F}x}^2)}{(2k_{\mathrm{F}z}^2 + k_{\mathrm{F}x}^2)}, \tag{12a}$$

$$k_{\mathrm{F}x}^2 = \frac{k_{\mathrm{F}\uparrow}^2}{(1 + \beta_2/2)}; \quad k_{\mathrm{F}z}^2 = \frac{k_{\mathrm{F}\uparrow}^2}{(1 - \beta_2)} \tag{12b}$$

and where the volume of the k-space spheroid is given by

$$V_{\text{spheroid}} = \left(\frac{4\pi}{3}\right) k_{\text{F}x}^2 k_{\text{F}z} = \frac{(4\pi/3)k_{F\uparrow}^3}{[(1+\beta_2/2)(1-\beta_2)^{1/2}]}. \tag{12c}$$

Using the fact that for only up-spins present in the system,

$$\sum_{\sigma_1}\sum_{\sigma_2} \bar{N}_{12}^{(0)}(\sigma_1,\sigma_2) = 1/4. \tag{13}$$

E_{exch} for different types of deformation can be calculated from the expression

$$E_{\text{exch}} = -\left(\frac{\mu^2}{2}\right) h_0 \frac{1}{4V_0} \sum_{\vec{k}}\sum_{\vec{q}} n_\uparrow(\vec{k}+\vec{q}) n_\uparrow(\vec{k}) Y_{20}(-\hat{q}). \tag{14}$$

For the value of the deformation parameter such that $0 < \beta_2 < 1$, the form (11) for the JM state leads to the expression for the number density as

$$\begin{aligned}\frac{N}{V_0} &= \frac{1}{V_0}\sum_{\vec{k}} n_\uparrow(\vec{k}) = \frac{V_{\text{spheroid}}}{8\pi^3} = \frac{k_{\text{F}\uparrow}^3}{6\pi^2}\left[\frac{1}{(1+\beta_2/2)(1-\beta_2)^{1/2}}\right] \\ &\to \frac{k_{F\uparrow}^3}{6\pi^2}\left(1+\frac{3}{8}\beta_2^2+\cdots\right),\end{aligned} \tag{15}$$

where the noninteracting gas ferromagnetic state Fermi wave vector $k_{\text{F}\uparrow} = 2^{1/3}k_{\text{F}0}, k_{\text{F}0}$ being the Fermi wave vector for the non interacting particles in the uniform paramagnetic state, with $N/V_0 = k_{\text{F}0}^3/3\pi^2$. For the kinetic energy in this state one finds

$$\begin{aligned}E_{\text{kin}}^{(Q)}(\text{JM}) &= N\frac{\hbar^2 k_{\text{F}\uparrow}^2}{2m}(3/5)\frac{(1-\beta_2/2)}{(1+\beta_2/2)(1-\beta_2)} \\ &\to E_0\{2^{2/3}[1+(1/4)\beta_2^2+\cdots]\},\end{aligned} \tag{16}$$

where

$$E_0 = (3/5)\frac{\hbar^2 k_{\text{F}0}^2}{2m} = \left(\frac{2.21}{r_0^2}\right)\frac{\hbar^2}{2m} \tag{17}$$

is the energy of the best variational paramagnetic state with $n_\sigma(k) = \Theta(k_{\text{F}0}-k)$, which is just the ground state energy of noninteracting spin-half Fermi gas. For a general value of β_2 between 0 and 1, it is not easy to obtain the exchange energy contribution analytically. However, for small deformations one can expand the expression (11) for the occupation function in powers of β_2. As we have seen already, to the order linear in β_2 there are no corrections to the number density and to the kinetic energy. But, to the lowest order the exchange energy is linear in β_2, and a straightforward calculation to this order then leads to the expression of the total variational energy per particle

$$\frac{E^{(Q)}}{N}(\text{JM}) = \frac{E_{\text{kin}}^{(Q)}}{N} + \frac{E_{\text{exch}}^{Q}}{N} = \frac{E_0}{N}\left\{2^{2/3} - \left(\frac{3}{40}\right)\beta_2\frac{r_m}{(2.21)r_0}\right\}, \tag{18}$$

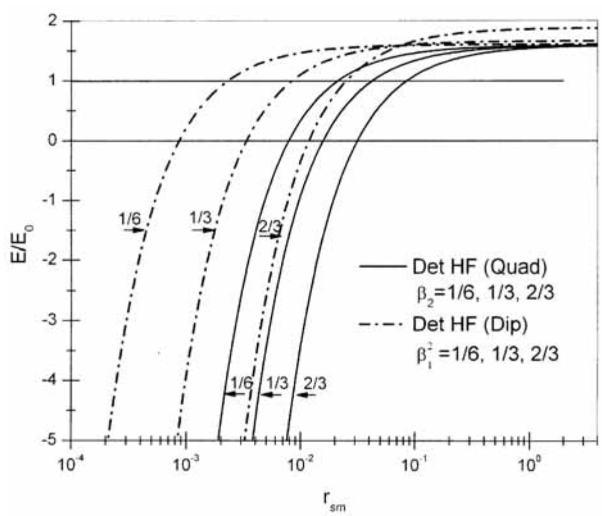

Figure 1. Comparison of the variational energy E in the ferromagnetic state calculated using determinantal HF wave functions with purely quadrupolar and purely dipolar deformations of the occupation function, as a function of the density parameter $r_{sm} = (2.21)r_0/r_m$, in the absence of any hard core interaction. The solid curves represent the ratio $E^{(Q)}/E_0$ for different quadrupolar deformation parameters β_2 in the JM state, whereas the dash-dot curves represent the ratio $E^{(\mathrm{dip.})}/E_0$ for the square of different dipolar parameters β_1^2.

where $r_m \equiv 2\mu^2 m/\hbar^2$. For different allowed positive values of the quadrupolar deformation parameter β_2, figure 1 shows a plot of $E^{(Q)}/E_0$, as a function of the dimensionless density parameter $r_{sm} = (2.21)r_0/r_m$. The total energy becomes less than E_0 of the uniform variational paramagnetic state for

$$r_{sm} < \frac{3}{40}\left[\frac{\beta_2}{(2^{2/3}-1)}\right], \quad \text{i.e., } r_0 < \frac{3}{40}\frac{\beta_2 r_m}{(2^{2/3}-1)(2.21)}. \tag{19}$$

When we increase the density further, the total energy becomes negative for

$$r_0(\text{critical}) \leq \frac{3}{40}\frac{\beta_2 r_m}{(2^{2/3})(2.21)} \tag{20}$$

and eventually the system will collapse to an infinite density state.

Although, the quadrupolar deformation was the most natural choice, it is instructive to consider also a purely dipolar deformation of the surface bounding the occupied spin-up only states in the $\vec{k}$-space. In this case, let us assume

$$n_\uparrow(\vec{k}) = \Theta(k_{\mathrm{F}\uparrow} - k\sqrt{1-\delta_1 Y_{10}(\hat{k})}) = \Theta(k_{\mathrm{F}\uparrow}^2 - k^2 + k^2\beta_1 P_1(\cos\theta_{\hat{k}})), \tag{21}$$

where

$$\beta_1 = \sqrt{3/4\pi}\delta_1, \quad 0 < \beta_1 < 1; \quad P_1(\cos\theta_{\vec{k}}) = \cos\theta_{\vec{k}}. \tag{22}$$

When the above form of $n_\uparrow(\vec{k})$ is used to calculate the number density, the kinetic energy and the exchange energy respectively, we find that the total energy correct to the order β_1^2 is then given by

$$\frac{E^{(\text{dip.})}}{N} = \frac{E_0}{N}\left[2^{2/3}\left(1+\left(\frac{5}{12}\right)\beta_1^2\right) - \frac{\beta_1^2}{20(2.21)}\frac{r_m}{r_0}\right]. \tag{23}$$

The above energy is lower than the uniform paramagnetic state energy E_0/N, for

$$r_0 < \frac{20\beta_1^2 r_m}{2^{2/3}(2.21)[1+(5/12)\beta_1^2]}. \tag{24}$$

As the density is increased further, the total energy with dipolar deformation also becomes negative, and eventually collapses to the infinite density state. However, note that the comparison of expressions (18) and (23) shows that at the same density $E^{(\text{dip.})}/N$ is higher than $E^{(Q)}/N$ of the JM state, even if one takes $|\beta_1|$ high enough so that $\beta_1^2 \approx \beta_2$. In fact in figure 1, we have also plotted $E^{(\text{dip.})}/E_0$ for an easy comparison. For small deformations, the result that the JM state involving quadrupolar deformation gives a lower energy is of course valid exactly, and one cannot really give too much importance to the fact that a comparison of these approximate expressions even for higher allowed values (close to 1) for the respective deformation parameters β_1^2, β_2 gives lower energy for the quadrupolar case. But, it seems very likely that the quadrupolar deformation of the surface bounding the occupied spin-up states in the JM state gives a better upper bound to the ground state energy.

2.2 *Variational HF ground state energy in the presence of a hard core potential*

Now, we consider the situation in which the chargeless fermion system with magnetic dipole interaction has also a short-range repulsive hard core interaction between the particles. This is to explore the condition under which the inclusion of this hard core interaction can stop the high density collapse of the JM ferromagnetic state. The most general form of the velocity-independent interaction between spin-half particles can be written in the form [3]

$$V_{12}(\vec{r}) = V_c(\vec{r}) + V_s(\vec{r})\vec{s}_1\cdot\vec{s}_2 + V_{\text{T}}(\vec{r})[\vec{s}_1\cdot\vec{s}_2 - 3\vec{s}_1\cdot\hat{r}\vec{s}_2\cdot\hat{r}]. \tag{25}$$

The last term (tensor interaction) has the form of the magnetic dipole–dipole interaction in our problem, with $V_{\text{T}} = \mu^2/r^3$. We had no other interaction until now. We will now add a very short-range (finite) repulsive interaction for calculating its expectation value in the following chosen determinantal states:

(a) The uniform paramagnetic state (PARA):

$$n_\sigma(\vec{k}) = n_0(k) = \Theta(k_{\text{F}0} - k). \tag{26}$$

(b) The fully polarized ferromagnetic state with quadrupolar deformation (JM state):

$$n_\sigma(\vec{k}) = n_\uparrow(\vec{k})\delta_{\sigma,\uparrow}; \quad n_\uparrow(\vec{k}) \equiv \Theta(k_{\mathrm{F}\uparrow}^2 - k^2 + k^2\beta_2 P_2(\cos\theta_{\vec{k}})). \tag{27}$$

We can choose the central short-range repulsive part, of the form as in the first or in the second term of eq. (25). In the PARA state, the second term does not contribute to the direct term, because of the spin summations. The exchange term has a factor of 1/2 due to spin summations. In the JM state, except for an additional factor of 1/4 due to the spin part, the contribution will be similar to the first term. Thus it is enough to consider the form of the only first term in eq. (25) for the repulsive short-range part, to get all the results. Therefore, let

$$V_c(\vec{r}) = U_0(\text{large \& positive}) \text{ for } r \leq a, V_c(\vec{r}) = 0, \quad r > a. \tag{28}$$

The range a is assumed to be small but finite. The Fourier transform in the $\vec{q}$-space of the above interaction is

$$V_c(\vec{q}) = \int \mathrm{d}^3 r \mathrm{e}^{-i\vec{q}\cdot\vec{r}} V_c(\vec{r}) = V_c(q=0)\left[3(\sin qa - qa\cos qa)/q^3a^3\right], \tag{29}$$

where

$$V_c(q=0) \equiv V_c(0) = \left(\frac{4\pi}{3}\right) U_0 a^3; \quad U_0 a^3 \equiv \left(\frac{\hbar^2}{2m}\right) 6a_0. \tag{30}$$

Note that there are two parameters in the potential, the strength U_0 and the range a. Sometimes, one replaces the coupling constant U_0a^3 by the actual low-energy scattering length a_0 [3], as indicated in eq. (30).

In general, the expectation value of the hard core interaction (29) in any HF state is

$$E_c = \frac{V_0}{2}\int\left(\frac{\mathrm{d}^3k_1}{8\pi^3}\right)\int\left(\frac{\mathrm{d}^3k_2}{8\pi^3}\right) \times \sum_{\sigma_1}\sum_{\sigma_2} n_{\sigma_1}(\vec{k}_1) n_{\sigma_2}(\vec{k}_2)[V_c(0) - \delta_{\sigma_1,\sigma_2}V_c(\vec{k}_1 - \vec{k}_2 \equiv \vec{q})]. \tag{31}$$

(a) In the PARA state, this gives

$$E_c(\text{PARA}) = \frac{V_0}{2}\times 2\int\left(\frac{\mathrm{d}^3k_1}{8\pi^3}\right)\int\left(\frac{\mathrm{d}^3k_2}{8\pi^3}\right)[2V_c(0) - V_c(q)]. \tag{32}$$

This leads to the usual result valid for any potential not depending on the direction of $\vec{q}$:

$$\frac{E_c(\text{PARA})}{N} = \frac{1}{2}\times\frac{1}{2\pi^3}\int \mathrm{d}^3q\left(1 - \frac{3}{2}x + \frac{1}{2}x^3\right)\Theta(1-x)[2V_c(0) - V_c(q)]; \quad x \equiv \frac{q}{2k_{\mathrm{F}0}}. \tag{33}$$

(b) In the JM state, we have the general result

$$E_c(\text{JM}) = \frac{V_0}{2(2\pi)^6}\int \mathrm{d}^3k \int \mathrm{d}^3q n_\uparrow(\vec{k})n_\uparrow(\vec{k}+\vec{q})[V_c(0) - V_c(q)]. \tag{34}$$

To the linear order in the quadrupolar deformation parameter β_2 of the JM state, there is no need to recalculate the kinetic energy term and the exchange term. They remain the same as in the expression (18). One has to recalculate only the repulsive core contribution without any expansion in the parameter a/r_0. We find

$$\frac{E_c(\text{PARA})}{E_0} = \frac{2mU_0a^2}{\hbar^2}\left(\frac{19.2}{54\pi}\right)\left(\frac{a}{r_0}\right)[1 + T(\alpha_0)]; \quad \alpha_0 \equiv 2k_{\text{F}0}a = 2\left(\frac{1.92a}{r_0}\right);$$
$$E_0 \equiv \left(\frac{\hbar^2}{2m}\right)\left(\frac{2.21}{r_0^2}\right); \quad 1.92 \cong \left(\frac{9\pi}{4}\right)^{1/3}, \tag{35}$$

where the function $T(s)$ is defined as

$$T(s) \equiv 1 - \frac{72}{s^3}[(Si(s) - \sin s) - \frac{3}{2s}(2 - 2\cos s - s\sin s)$$
$$+\frac{1}{2s^3}(8s\sin s + 8\cos s - 8 - 4s^2\cos s - s^3\sin s)] \tag{36}$$

and where the sine integral function

$$Si(s) = \int_0^s \mathrm{d}x\left(\frac{\sin x}{x}\right) = s - \frac{s^3}{3!3} + \frac{s^5}{5!5} - \frac{s^7}{7!7} + \cdots. \tag{37}$$

Similarly, one finds

$$\frac{E_c(\text{JM})}{E_0} = \frac{2mU_0a^2}{\hbar^2}\left(\frac{19.2}{27\pi}\right)\left(\frac{a}{r_0}\right)T(\alpha_\uparrow);$$
$$\alpha_\uparrow \equiv 2k_{\text{F}\uparrow}a = 2^{4/3}\left(\frac{1.92a}{r_0}\right). \tag{38}$$

Thus in the presence of the hard core interaction, the expression (18) giving the ratio of the total energy E and E_0 in the JM state is now replaced by

$$\frac{E}{E_0}(\text{JM}) = \left\{2^{2/3} + \frac{(2m/\hbar^2)U_0a^3}{r_0}\left(\frac{19.2}{27\pi}\right)T(\alpha_\uparrow)\right.$$
$$\left. - \left(\frac{3}{40}\right)\beta_2\frac{(2m/\hbar^2)\mu^2}{(2.21)r_0}\right\}, \tag{39}$$

where as defined before,

$$\frac{(2.21)r_0}{(2m\mu^2\hbar^{-2})} \equiv r_{sm}.$$

From the definition of the function $T(s)$ given by eqs (36) and (37), it is easy to see that it is a smooth function, with $T(s) \to 1$, as $s \to \infty$ and $T(s) \to 3s^2/100$,

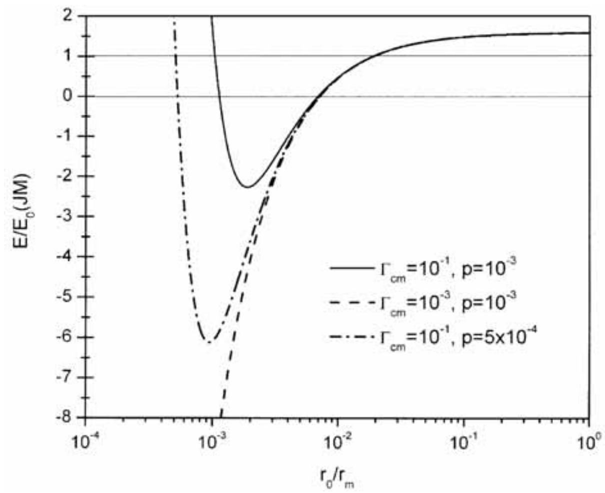

Figure 2. Plot of the total variational HF energy E in the ferromagnetic JM state, in units of the energy E_0 of the corresponding paramagnetic non-interacting gas, in the presence of the magnetic dipole interaction as well as a repulsive short-range hard core interaction of range a and strength U_0, as a function of r_0/r_m. Plots are for two values of the ratio of the two coupling constants $\Gamma_{cm} \equiv U_0 a^3/\mu^2$, in which the label $p \equiv a/r_m$ is the range a in units of the magnetic length $r_m \equiv 2m\mu^2/\hbar^2$.

for $s \ll 1$. With the definition given in eq. (38) for $\alpha_\uparrow \sim 1/r_0$, the expression (39) immediately shows us that for the ratio $U_0 a^3/\mu^2$ of the coupling constants not extremely small compared to 1, the total energy becomes positive as $r_0 \to 0$, and there is no longer the high density collapse of §2.1, as explicitly shown in figure 2. The JM ferromagnetic state is thus a possible stable equilibrium ground state of the system.

3. Concluding remarks

Our interest in the nature of the ground state of chargeless fermions with a finite magnetic moment initially arose because of the suggestion by Yajnik [4] that the state of the universal relic background neutrinos might be a ferromagnetic state with domain walls, made in the context of big-bang cosmology. However, we find that in the nonrelativistic case, at $T = 0$ K, the density required for the ferromagnetic transition, with a spheroidal occupation function, is too high for satisfying the condition $r_0 < r_m = 2\mu^2 m/\hbar^2$. For chargeless fermions with an atomic mass of 10^4 to 10^5 times the electron mass and magnetic moment μ of the order of the Bohr magneton, $r_m \approx 10^{-9}$–10^{-8} cm. For neutrinos with a mass in the range of 0.01 eV and a very tiny magnetic moment [5], r_m is extremely small. Note that in the absence of the hard core interaction, our HF calculation of the true upper bound for the ground state energy shows that the true ground state energy must go to $(-)\infty$ at infinite density. The high density collapse is real. No exotic state of any kind, spatially homogeneous or not [6] can lead to a ground state energy greater than $(-)\infty$, as density goes to infinity. However, in the presence of a repulsive short-range hard core potential this collapse is arrested, and one can get, e.g., the stable JM deformed ferromagnetic state as the ground state of the system.

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