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Mean square number fluctuation for a fermion source and its dependence on neutrino mass for the universal cosmic neutrino background

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Abstract. Using the general formulation for obtaining chemical potential μ of an ideal Fermi gas of particles at temperature T, with particle rest mass m_0 and average density $\langle N\rangle/V$, the dependence of the mean square number fluctuation $\langle \Delta N^2\rangle/V$ on the particle mass m_0 has been calculated explicitly. The numerical calculations are exact in all cases whether rest mass energy m_0c^2 is very large (non-relativistic case), very small (ultra-relativistic case) or of the same order as the thermal energy $k_{\rm B}T$. Application of our results to the detection of the universal very low energy cosmic neutrino background (CNB), from any of the three species of neutrinos, shows that it is possible to estimate the neutrino mass of these species if from approximate experimental measurements of their momentum distribution one can extract, someday, not only the density $\langle N_\nu\rangle/V$ but also the mean square fluctuation $\langle \Delta N_\nu^2\rangle/V$. If at the present epoch, the universe is expanding much faster than thermalization rate for CNB, it is shown that our analysis leads to a scaled neutrino mass m_ν instead of the actual mass $m_{0\nu}$.

Keywords. Cosmic neutrino background; relic neutrinos; mean square fluctuation; neutrino mass.

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1. Introduction

While measuring the flux of any particle arising from a thermal source at a given temperature T, one is always concerned with the background noise due to the root mean square number fluctuation of the particles as observed by the detector. For particles with the classical Maxwell–Boltzmann (MB) distribution, the mean square fluctuation per unit volume, $\langle \Delta N^2\rangle/V$, is equal to the average density $\langle N\rangle/V$, where the number dispersion $\langle \Delta N^2\rangle = \langle N^2\rangle - \langle N\rangle^2$. In this case, it is independent of the rest mass m_0 of the particles being detected, for a fixed density $n = \langle N\rangle/V$. However, it is well-known that for an ideal Bose gas, the mean square fluctuation $\langle \Delta N^2\rangle/V$ is larger than $\langle N\rangle/V$, whereas for an ideal Fermi gas it is smaller than the classical value [1,2]. The deviation of the fluctuation from the classical value or the total dispersion $\langle \Delta N^2 \rangle / V$ depends, in general, on the mass m_0 of the particle in the gas and the temperature T. In this paper, our aim is to examine this dependence explicitly for particles which can be treated as an ideal Fermi gas with a given mass m_0, both in the non-relativistic limit as well as in the relativistic regime. The idea is to study its possible role in estimating the mass of different types of neutrinos in the universal cosmic neutrino background (CNB), also called relic neutrinos [3].

In the standard Big-Bang model of cosmology [3], which is supported by the observation of the cosmic microwave background radiation (CMBR) at the present temperature of 2.725 K and by various other predictions, the universe started from a very high density, and high temperature state. It cooled down while it expanded and during this early process particles of different species and masses got decoupled one after another. At temperatures of about a few MeV ($\sim 2 \times 10^{10}$ K), one had only neutrinos, electrons, positrons and photons in thermal equilibrium with each other. As the temperature cooled down further, the neutrinos of all the three types decoupled one by one from the rest of the matter within the first few seconds or so after the Big-Bang. At the time of this decoupling, the temperature (very high) was same for the particles of all species and they were highly relativistic. Below the temperature of about 0.5 MeV ($\sim 5 \times 10^9$ K), the electron–positron annihilation process created more photons and heated the photon gas in the early universe. One may assume that the universe expanded in thermal equilibrium with constant entropy which was proportional to $N_\mathrm{T} T^3$ in the highly relativistic regime. Here, N_T is the total of the effective numbers for each of the species, which take into account their spin and polarization degrees of freedom and effect of their specific quantum statistics. Taking into account the spin degeneracies of electrons and positrons, two polarizations of the photons and the additional factor 7/8 in the energy of the highly relativistic fermions [4] as compared to the black-body radiation (photons) at the same temperature and volume, the effective total number changed from $(2 \times 2) \times (7/8) + 2 = 11/2$ to 2 after the annihilation of the electron–positron pairs into photons. This leads to the heating of the photon gas and to the well-known estimate of the CNB temperature T_ν to be equal to $(4/11)^{1/3} T_\mathrm{CMBR}$, which for the present time gives $T_\nu = (4/11)^{1/3} \times 2.725\ \mathrm{K} \simeq 1.95$ K. The ratio of the number density of neutrinos to that of photons [5,6] is given by $\langle N_\nu / V \rangle / \langle N/V \rangle_\mathrm{CMBR} = (3/4) \times (T_\nu / T_\mathrm{CMBR})^3 = 3/11$, for each of the three species of neutrinos. The factor (3/4) in the above ratio is due to the Fermi statistics of each of the highly relativistic neutrinos at the time of decoupling, compared to Bose statistics of massless photons [4]. We have assumed that neutrinos have finite mass so that helicity factor 2 cancels the factor 2 from the polarization of photons. The above estimate gives the value of CNB density for each type of neutrino to be about $10^2\ \mathrm{cm}^{-3} = 10^8\ \mathrm{m}^{-3}$. The assumption that all the neutrinos decouple before any heating of the photon gas due to electron–positron annihilations took place is not strictly valid, and neutrinos also take part in the heating. There are some more corrections to the above estimates, but all these corrections [6,7] increase the above estimate for T_ν and $\langle N_\nu \rangle / V$ by no more than about one to two per cent.

At the present time, the temperature T_ν of CNB is only about 1.95 K ($\sim 1.7 \times 10^{-4}$ eV), and depending upon the actual mass $m_{0\nu}$ of different types of neutrinos, the gas may either be in the non-relativistic ($m_{0\nu} c^2 \gg k_\mathrm{B} T_\nu$) or in the relativistic regime ($m_{0\nu} c^2 \sim k_\mathrm{B} T_\nu$). Of course at such low densities, the Fermi gas is expected to be away from the degenerate case. In any case, our general analysis of dispersion and average density of neutrinos of each of the species must explicitly take into account the variation of the chemical potential parameter μ with particle mass and the temperature T, for a given density $\langle N\rangle/V$.

Before proceeding further, we must point out that while applying our analysis to the case of the cosmic neutrino background (CNB), one has to be very careful. In an expanding universe, when the relaxation rate for reaching thermal equilibrium for the weakly interacting CNB becomes less than the rate of expansion of the universe, the neutrino gas will no longer remain in thermal equilibrium. As explained, for example, by Pal and Kar [8], for simplicity one can assume that this decoupling of CNB takes place when these rates are equal, which may be specified by the epoch where the cosmological scale parameter was $\tilde{a}_D$. Till then, one can assume that the CNB momentum distribution $\langle N_D(\vec{k})\rangle$ was described by the usual thermal Fermi–Dirac energy distribution function $\langle N_D(\vec{k})\rangle = f_D(\epsilon(k, m_{0\nu}), \mu_D, T_D) = [\exp[(\epsilon(k, m_{0\nu}) - \mu_D)/k_{\mathrm{B}}T_D] + 1]^{-1}$ at temperature T_D and chemical potential μ_D. Here, $\epsilon(k, m_{0\nu})$ is the usual single particle energy of the neutrino of rest mass $m_{0\nu}$ and momentum $\hbar k$, which is given by $(\hbar^2k^2c^2 + m_{0\nu}^2c^4)^{1/2} - m_{0\nu}c^2$, if it is measured with respect to $m_{0\nu}c^2$, in the most general relativistic case. After that epoch, if nothing happens to the decoupled CNB except for the gravitational red-shift in the expanding universe, i.e. if no thermalization process continues, the new distribution function at the present epoch, specified by the cosmological scale parameter $\tilde{a}_0 = \tilde{a}_D/r$, is given simply by [8], $\langle N_0(\vec{k})\rangle = f_D(\epsilon(k/r, m_{0\nu}), \mu_D, T_D)$. The energy distribution is no longer a thermal distribution. However, because of the form of the Fermi–Dirac (FD) distribution function f_D and the form of the single particle energy ϵ as a function of k and $m_{0\nu}$ (whether it is the general relativistic case or the non-relativistic case or the ultra-relativistic case), the new distribution function can be rewritten as $\langle N_0(\vec{k})\rangle = f_D(\epsilon(k, m_\nu \equiv rm_{0\nu}), \mu \equiv r\mu_D, T_\nu \equiv rT_D)$. This looks exactly like a thermal FD distribution function except that now one has the scaled rest mass $m_\nu \equiv rm_{0\nu}$, scaled temperature $T_\nu = rT_D$ and the scaled chemical potential parameter $\mu = r\mu_D$. We will call this distribution function as the scaled FD distribution function. Note that the new distribution function still depends only on the magnitude $\hbar k$ of the momentum, i.e. $\langle N_0(\vec{k})\rangle = \langle N_0(k)\rangle$, and is always positive with values less than or equal to 1. This, of course, can be obtained from the *scaled* grand canonical density matrix or the *scaled* grand canonical partition function, as one does in the usual case, with the same scaling of rest mass, temperature and the chemical potential parameter. However, the expression for the single particle energy has still the unscaled rest mass $m_{0\nu}$ everywhere except inside the scaled FD function, the scaled grand canonical partition function, etc. In that sense, the scaled μ has no longer the usual physical significance of the chemical potential. But as long as we do not calculate the average value of any function which contains the rest mass $m_{0\nu}$, e.g., energy density, etc., and we calculate only the integrated density $\langle N_\nu\rangle/V$ and dispersion $\langle \Delta N_\nu^2\rangle/V$ using the scaled FD distribution or the scaled grand canonical partition function, our analysis will remain exactly the same as in the pure thermal FD case. Since we take T_ν to be $(4/11)^{1/3}T_{\mathrm{CMBR}} = rT_D$ in our analysis, as long as we consider the resulting neutrino mass parameter m_ν to be the scaled mass $rm_{0\nu}$, our analysis will remain unchanged for both types of distribution. More explicitly, in terms of the scaled grand canonical partition function $Z_{\rm GC}(m_\nu, \mu, T_\nu)$, the average value $\langle N_\nu\rangle$ of the non-interacting neutrinos is still given by the first partial derivative of $k_{\rm B}T_\nu \ln Z_{\rm GC}$ with respect to the scaled parameter μ and the dispersion $\langle \Delta N_\nu^2\rangle$ is still obtained from the second partial derivative of $k_{\rm B}^2 T_\nu^2 \ln Z_{\rm GC}$ with respect to μ. Here, one still has the usual form [2], $Z_{\rm GC} = \sum_N \sum_j \exp[(\mu N - E_j^N(m_\nu))/k_{\rm B}T_\nu]$, where $E_j^N(m_\nu)$ are the possible energies of N free particles of mass m_ν. It is also clear from the mathematical formulation described in the next section that even for the scaled distribution function $\langle N_\nu(k)\rangle$, the relations $\langle N_\nu\rangle = \sum_{\vec{k}} \sum_\sigma \langle N_\nu(k)\rangle$ and $\langle \Delta N_\nu^2\rangle = \sum_{\vec{k}} \sum_\sigma \langle N_\nu(k)\rangle[1 - \langle N_\nu(k)\rangle]$ are still valid. Here, the summation over σ denotes summation over spins and summation over $\vec{k}$ may be converted to integration in the usual way. These relations allow extraction of $\langle N_\nu\rangle/V$ and $\langle \Delta N_\nu^2\rangle/V$ from experimental measurement of the distribution function $\langle N_\nu(k)\rangle$. For a given $\langle N_\nu\rangle/V$, the scaled chemical potential parameter μ is determined completely, once T_ν and m_ν are specified. We will, therefore, continue to call μ as the chemical potential parameter even if the distribution is the scaled FD distribution.

In reality, even after the so-called decoupling described above, the CNB will continue to try to move towards a thermal equilibrium even if the interaction with the external world and the interaction amongst themselves are very weak. How far away is the actual distribution function of CNB from thermal equilibrium or how close the CNB distribution is to the scaled FD distribution at the present epoch can also be tested experimentally. Even for this very important testing it becomes more important to develop a very good low-energy neutrino detector with adequate momentum and energy resolution in the 10^{-4} eV range. Only then can one perform measurements of CNB distribution, similar to the detection of CMBR photons, as functions of energy and momentum $\hbar k$. It is not enough to detect just a few isolated events involving cosmic neutrinos. It should be emphasized here that even for determining the validity of the scaled FD distribution function at the present epoch, our mathematical analysis described in this paper will be required. In what follows, we calculate here only $\langle N_\nu\rangle/V$ and $\langle \Delta N_\nu^2\rangle/V$ and not the average value of any function involving the rest mass of the particles. Whether the distribution is the usual FD or the scaled FD distribution, our numerical analysis of the problem will remain the same. One thing we must keep in mind is that if the distribution at the present epoch turns out to be closer to the scaled FD distribution, the resulting m_ν is not $m_{0\nu}$ but $rm_{0\nu}$.

Because of very weak interaction of relic neutrinos with matter and its extremely low energy, as of now, no one has been able to even detect CNB. There are small indirect effects due to the presence of sufficiently large amount of CNB everywhere [9], but unlike the case of high-energy ($\sim$ MeV) neutrinos, e.g., solar neutrinos, any direct detection of 10^{-4} eV neutrinos is extremely difficult [9]. One possibility which is being talked about is to observe the decay products (particle–antiparticle pairs) of the massive Z-bosons produced by the collision of low-energy relic neutrinos with ultra-high energy anti-neutrinos. But such high energy neutrinos or anti-neutrinos are not available easily, except in cosmic rays with very low flux. It is indeed one of the greatest challenges of the present century to directly detect CNB and its momentum and energy distributions, for a proper understanding of the nature of the early universe.

Not underestimating the great difficulty in developing a very low energy neutrino detector with good momentum and energy resolution for observing CNB, we feel that we can still talk about not only extracting the total integrated CNB number density but also about extracting its number dispersion. By proper interpolation, these two parameters can be extracted fairly well even if the experimental measurement of the distribution $\langle N_\nu(k)\rangle$ is not very detailed. One has to use the relations, $\langle N_\nu\rangle = \sum_{\vec{k}}\sum_\sigma \langle N_\nu(k)\rangle$ and $\langle \Delta N_\nu^2\rangle = \sum_{\vec{k}}\sum_\sigma \langle N_\nu(k)\rangle[1-\langle N_\nu(k)\rangle]$, as explained earlier. Let us assume that some day this will happen. This will allow us to get an idea of the mass for each of the species of cosmic neutrinos.

In §2 of this paper, we present the mathematical formulation of the problem of calculating the chemical potential μ as a function of m_0 and T for a given density $\langle N\rangle/V$, for an usual thermal FD distribution. This is the crucial step in our numerical analysis of dispersion, and this is done both for a non-relativistic Fermi gas as well as a relativistic Fermi gas. We also obtain the resulting mean square fluctuation $\langle \Delta N^2\rangle/V$ in each case. In §3, we apply our analysis to the case relevant to relic neutrinos at temperature T_ν. We present explicitly numerical results for the chemical potential parameter μ and $\langle \Delta N_\nu^2\rangle/V$ as a function of the average density $\langle N_\nu\rangle/V$, for different values of neutrino mass parameter m_ν which enter in the distribution function. We show that $[\langle N_\nu\rangle/V - \langle \Delta N_\nu^2\rangle/V]$ depends appreciably on the mass parameter m_ν, and a reliable extraction of this quantity from experimental measurement of distribution function $N_\nu(\vec{k})$, can lead to a realistic estimate of mass parameter m_ν. Let us emphasize again that if the actual distribution of CNB is closer to the scaled FD distribution, the resulting m_ν will be scaled as $m_\nu = r m_{0\nu}$. Otherwise, for a purely thermal FD distribution, $m_\nu = m_{0\nu}$. We also discuss our results in the concluding section.

2. Mathematical formulation of the problem

In terms of the grand canonical partition function Z_{GC}, the thermodynamic potential $\Omega(T,V,\mu)$ for a statistical system is defined by

$$\Omega(T,V,\mu) = -\frac{1}{\beta}\ln Z_{\mathrm{GC}}, \quad \beta \equiv \frac{1}{k_{\mathrm{B}}T}, \tag{1}$$

where μ is the chemical potential and k_{B} is the Boltzmann constant. The average number of particles and the mean square fluctuation in the number are determined by [2]

$$\langle N\rangle = -\left(\frac{\partial\Omega}{\partial\mu}\right)_{T,V}, \tag{2}$$

$$\langle \Delta N^2\rangle = \langle N^2\rangle - \langle N\rangle^2 = -\frac{1}{\beta}\left(\frac{\partial^2\Omega}{\partial\mu^2}\right)_{T,V} = \frac{1}{\beta}\left(\frac{\partial\langle N\rangle}{\partial\mu}\right)_{T,V}. \tag{3}$$

For an ideal Fermi gas with single particle energy ϵ_i for the state i, one has [2]

$$\Omega = \sum_i \Omega_i; \quad \Omega_i = -\frac{1}{\beta}\ln[1+\exp[-\beta(\epsilon_i-\mu)], \tag{4}$$

$$\langle N\rangle = \sum_i \langle N_i\rangle; \quad \langle N_i\rangle = [\exp[\beta(\epsilon_i-\mu)+1]^{-1}, \tag{5}$$

$$\langle \Delta N^2\rangle = \sum_i \langle \Delta N_i^2\rangle;$$

$$\langle \Delta N_i^2\rangle = \frac{\exp[\beta(\epsilon_i-\mu)]}{(\exp[\beta(\epsilon_i-\mu)+1])^2} = \langle N_i\rangle[1-\langle N_i\rangle]. \tag{6}$$

2.1 *Non-relativistic energy spectrum*

For a non-relativistic fermion gas of particles of mass m_0 confined in a box of very large volume V,

$$\epsilon_{\vec{k}} = \frac{\hbar^2 k^2}{2m_0} \tag{7}$$

with discrete values of $\vec{k}$. Going over to the continuous energy spectrum in the limit of large volume V one then finds

$$\frac{\langle N\rangle}{V} = \frac{g}{4\pi^2}\left(\frac{2m_0}{\hbar^2}\right)^{3/2}\int_0^\infty \frac{\mathrm{d}\epsilon\ \epsilon^{1/2}}{[\exp[\beta(\epsilon-\mu)]+1]}, \tag{8}$$

where g is the spin degeneracy factor (2 for spin-$\frac{1}{2}$ particle). In terms of thermal wavelength λ_{T} defined by

$$\lambda_{\mathrm{T}} = \left(\frac{2\pi\hbar^2}{m_0 k_{\mathrm{B}} T}\right)^{1/2}, \tag{9}$$

the above expression is rewritten in a more common form:

$$\frac{\langle N\rangle}{V} = \frac{g}{\lambda_{\mathrm{T}}^3} f_{3/2}^{+}(\exp y), \tag{10}$$

$$f_{3/2}^{+}(\exp y) = \frac{2}{\sqrt{\pi}}\int_0^\infty \frac{\mathrm{d}x\ x^{1/2}}{[\exp[x-y]+1]}, \tag{11}$$

where $x = \beta\epsilon$ and

$$y = \beta\mu = \frac{\mu}{k_{\mathrm{B}}T}. \tag{12}$$

There is no general analytical method to obtain chemical potential μ for a given value of $\langle N\rangle/V$, as a function of m_0 and T, i.e. for inverting eq. (8). Only in the two limiting cases of (i) the high temperature limit (which reduces to the case of the classical MB result) and (ii) the low temperature limit (which reduces to the case of the degenerate Fermi gas result), one can obtain analytic expressions for μ. Since in general we are faced with the intermediate region, we must use numerical methods [10] to determine $\mu(T, m_0)$ for a given $\langle N\rangle/V$.

In the non-relativistic case being considered, the mean square fluctuation is given by

$$\frac{\langle \Delta N^2\rangle}{V} = \frac{g}{4\pi^2}\left(\frac{2m_0}{\hbar^2}\right)^{3/2}\int_0^\infty \frac{\mathrm{d}\epsilon\ \epsilon^{1/2}\exp[\beta(\epsilon-\mu)]}{[\exp[\beta(\epsilon-\mu)+1]^2} \tag{13}$$

$$= \frac{g}{\lambda_{\mathrm{T}}^3}\frac{2}{\sqrt{\pi}}\int_0^\infty \frac{\mathrm{d}x\ x^{1/2}\exp[x-y]}{[\exp[x-y]+1]^2} \tag{14}$$

with $x = \beta\epsilon$ and, as before

$$y = \beta\mu. \tag{15}$$

For a given value of $\langle N\rangle/V$, once μ is determined from eqs (10) to (12), one can find the dispersion $\langle \Delta N^2\rangle/V$ from eq. (13). It can be shown that the non-relativistic expressions will give good results as long as $m_0c^2 \gg k_{\mathrm{B}}T$.

2.2 *Relativistic energy spectrum*

In the general case of relativistic dynamics, one has

$$\epsilon_{\vec{k}} = (\hbar^2k^2c^2 + m_0^2c^4)^{1/2} - m_0c^2, \tag{16}$$

where we use the convention in which the single particle energies and the chemical potential μ will be measured with respect to m_0c^2. This convention is very convenient here because it makes it much simpler to compare our numerical results and expressions with the non-relativistic case, when the non-relativistic limit is applicable. For this case, in the continuous limit, we get

$$\frac{\langle N\rangle}{V} = \frac{4\pi g}{h^3c^3}\int_0^\infty \frac{\mathrm{d}\epsilon(\epsilon+m_0c^2)[(\epsilon+m_0c^2)^2 - m_0^2c^4]^{1/2}}{[\exp[\beta(\epsilon-\mu)]+1]} \tag{17}$$

$$= \frac{4\pi g}{(\beta hc)^3}\int_0^\infty \frac{\mathrm{d}x(x+a)(x^2+2ax)^{1/2}}{[\exp[x-y]+1]}, \tag{18}$$

where again $x = \beta\epsilon$, and

$$a = \beta m_0c^2, \quad y = \beta\mu. \tag{19}$$

One must always remember here that we are measuring ϵ and μ with respect to m_0c^2. Under our convention, if we take the non-relativistic limit, $a = \beta m_0c^2 \gg 1$ in eq. (18), we recover exactly eq. (8) or equivalently eqs (10)–(12) derived for the non-relativistic case. Again, for a given $\langle N\rangle/V$, μ has to be determined numerically in the relativistic case also, by using eqs (18) and (19).

In the general case of relativistic dynamics, the mean square fluctuation is determined by

$$\frac{\langle \Delta N^2 \rangle}{V} = \frac{4\pi g}{(\beta hc)^3} \int_0^\infty \frac{\mathrm{d}x (x+a)(x^2+2ax)^{1/2} \exp[x-y]}{[\exp[x-y]+1]^2} \tag{20}$$

which again reduces to the non-relativistic expression, eq. (14), in the limit $a = \beta m_0 c^2 \gg 1$.

2.3 *Ultra-relativistic limit*

The ultra-relativistic regime corresponds to the limit $a = \beta m_0 c^2 \ll 1$, in the general relativistic expressions (18)–(20). In this case, the single particle spectrum reduces to

$$\epsilon_{\vec{k}} = \hbar k c \tag{21}$$

and one finds

$$\frac{\langle N \rangle}{V} = \frac{4\pi g}{(\beta hc)^3} \int_0^\infty \frac{\mathrm{d}x\ x^2}{[\exp[x-y]+1]}, \tag{22}$$

where y is again $\beta\mu$ and $x = \beta\epsilon$. Note that in this case the mass parameter disappears and the chemical potential is independent of m_0. The mean square fluctuation is now given by

$$\frac{\langle \Delta N^2 \rangle}{V} = \frac{4\pi g}{(\beta hc)^3} \int_0^\infty \frac{\mathrm{d}x\ x^2 \exp[x-y]}{[\exp[x-y]+1]^2}, \tag{23}$$

where $y = \beta\mu$. In this limit, the fluctuation is independent of m_0.

In what follows, in the next section we will first check our non-relativistic results for calculation of μ in the range of parameters relevant to electrons and holes in semiconductors and metals for which numerical tables are already available readily [10]. We will then apply our general expressions to the case of parameters relevant to neutrinos in CNB.

3. Numerical calculations and results for cosmic neutrino background

For a non-relativistic ideal Fermi gas, numerical tables of Blakemore [10] are already available for determining the chemical potential μ for a given temperature T and density $n = \langle N \rangle / V$. However, parameters used in those tables are in the range of densities, temperature and mass relevant to electrons or holes in semiconductors and metals. Nevertheless, this allows us to check our numerical code for calculating μ in the non-relativistic case, using expressions (10)–(12). Our numerical results coincide with the numerical results of Blakemore, for parameters tabulated by him. Here, we are, of course, concerned with a different range of parameters relevant to CNB, where we have to deal with low densities in the range $\langle N \rangle / V \sim 10^2$ $\mathrm{cm}^{-3} = 10^8\ \mathrm{m}^{-3}$, low temperatures in the range of 1.95 K ($k_\mathrm{B} T_\nu \sim 1.7 \times 10^{-4}$ eV) and unknown low mass parameter $m_\nu c^2$ in the range of 10 eV to 10^{-6} eV. The non-relativistic results are expected to be valid for $m_\nu c^2 \gg 10^{-4}$ eV. Note that if

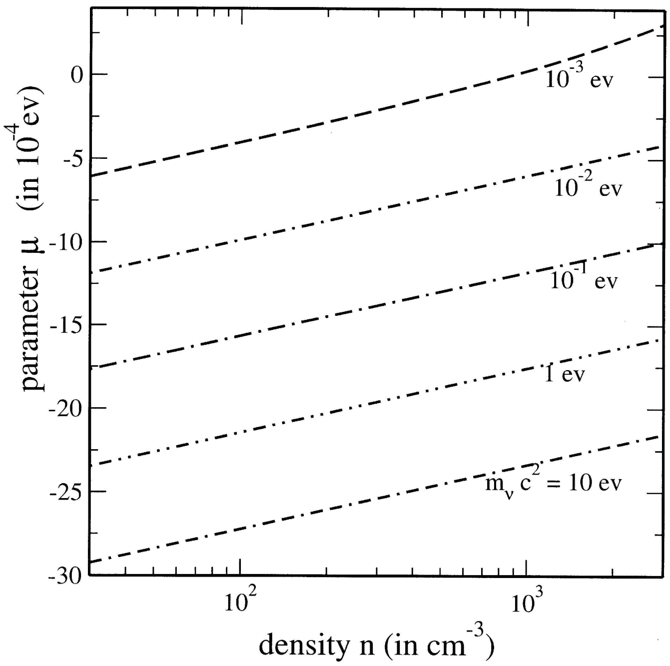

Figure 1. A plot of calculated values of the chemical potential μ showing its variation with the number density $n = (\langle N/V \rangle)$ of an ideal non-relativistic gas at temperature $T = 1.95$ K (i.e., $k_{\rm B}T = 1.7 \times 10^{-4}$ eV), for different assumed neutrino masses $m_\nu c^2$.

the actual CNB distribution turns out to be closer to the scaled FD distribution, $m_\nu = rm_{0\nu}$, where $m_{0\nu}$ is the actual rest mass of the neutrino.

In figure 1, we show our result of the calculation of the chemical potential parameter μ as a function of density $\langle N \rangle / V$ in the range 5×10 to 10^3 cm^{-3}, for temperature $T_\nu = 1.95$ K, for different neutrino mass parameter $m_\nu c^2$. Note that for such low densities, the neutrino gas is far from a degenerate Fermi gas and the chemical potential parameter μ is negative in most of the range considered. There is a substantial variation in parameter μ as a function of $m_\nu c^2$. These results allow us to calculate the mean square fluctuation $\langle \Delta N^2 \rangle / V$, using eqs (13) and (14), in the non-relativistic case. However, we do not present these results here explicitly, but we have used these numbers to check our general results using the relativistic expression (20), to be presented below. Since in the non-relativistic limit, $m_\nu c^2 \gg k_{\rm B} T_\nu$, the results are identical, there is no need for a separate plot here.

For the general case, we must use our relativistic expressions (18)–(20) to calculate the chemical potential parameter μ, for a given $n = \langle N \rangle / V$, and the corresponding mean square fluctuation $\langle \Delta N^2 \rangle / V$. These results are valid for any value of $m_\nu c^2$, i.e. for any ratio $a = m_\nu c^2 / k_{\rm B} T_\nu$. In figure 2, we plot the variation of the chemical potential μ as a function of density $\langle N \rangle / V$, for different values of $m_\nu c^2$ at $T = 1.95$ K (i.e. $k_{\rm B} T \simeq 1.7 \times 10^{-4}$) eV. For $a = \beta m_\nu c^2 \gg 1$, these results are

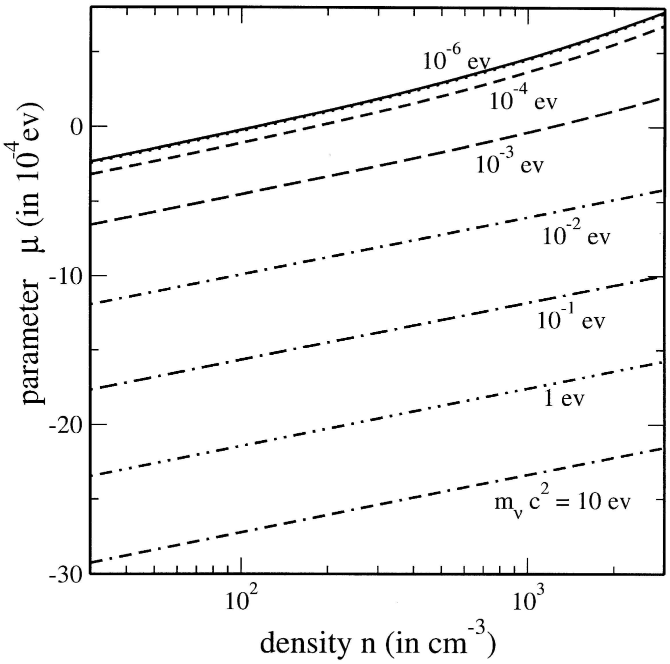

Figure 2. A plot of chemical potential μ showing its variation with number density $n = (\langle N/V \rangle)$ of an ideal relativistic gas at temperature $k_{\rm B}T = 1.7 \times 10^{-4}$ eV, for different masses $m_\nu c^2$.

identical to the plot in figure 1, as expected. Using these general values of μ, the deviation $\langle N_\nu \rangle / V - \langle \Delta N_\nu^2 \rangle / V$ in the mean square fluctuation from the classical value is plotted in figure 3 as a function of the density $\langle N_\nu \rangle / V$, for different values of $m_\nu c^2$ in the range 10 eV–10^{-6} eV. As stated before, we have checked that these general results are identical to the non-relativistic calculations of $\langle N_\nu \rangle / V - \langle \Delta N_\nu^2 \rangle / V$ in the limit $a = \beta m_0 c^2 \gg 1$. If we examine the plot in figure 3 carefully, we find a strong dependence of the dispersion on $m_\nu c^2$, except in the ultra-relativistic case when $a = \beta m_0 c^2 \ll 1$, i.e. when the dependence of μ and the mean square fluctuation on $m_\nu c^2$ disappears (see, expressions (22) and (23)). To make it more explicit, we plot in figure 4 the deviation of the mean square fluctuation from its classical value as a function of $m_\nu c^2$, for fixed $k_{\rm B}T = 1.7 \times 10^{-4}$ eV and fixed density $\langle N_\nu \rangle / V = 10^2$ cm^{-3}.

Our exact numerical calculations, which take into account the variation of the chemical potential parameter μ as a function of mass and temperature for the extremely low energy relic neutrinos, for a given density $\langle N \rangle / V$, show that the mean square number fluctuation is a sensitive function of the neutrino mass. In conclusion, we have shown that if from experimental measurement of CNB momentum distribution one can at least extract the mean square fluctuation $\langle \Delta N_\nu^2 \rangle / V$ and the integrated density $\langle N_\nu \rangle / V$ of each type of neutrinos it can give a good estimate of the corresponding neutrino mass directly. If detection involves measurement of total density and mean square fluctuation of all the three types of neutrinos at the

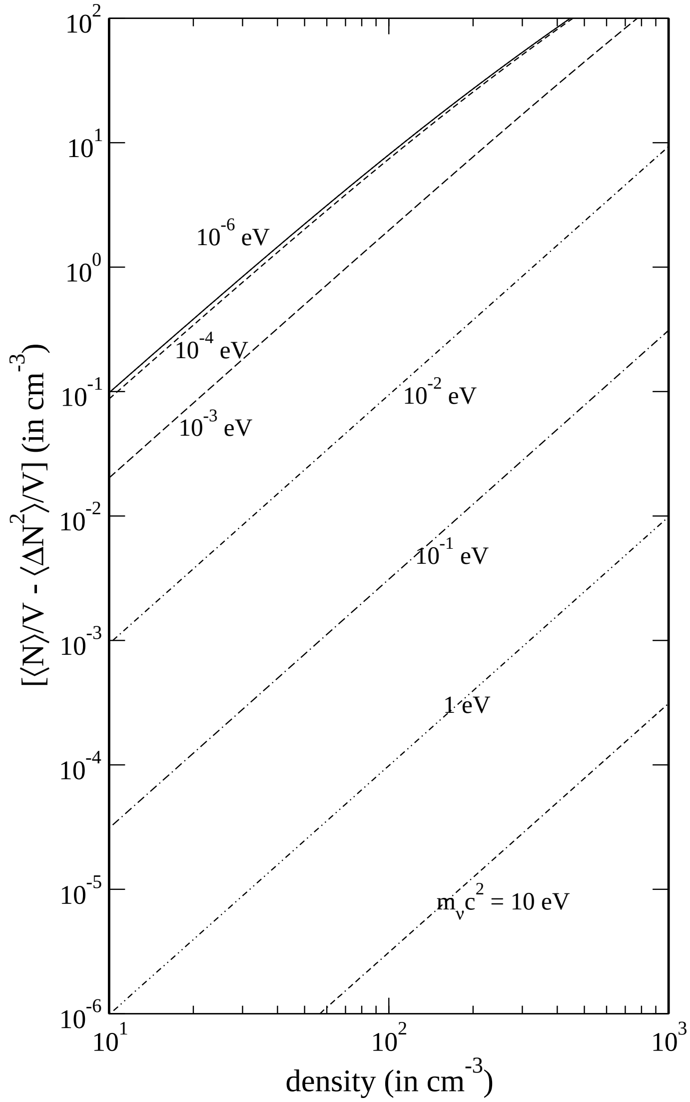

Figure 3. A plot of the deviation $\langle N\rangle/V - \langle\Delta N^2\rangle/V$ of the mean square fluctuation of density $n = \langle N/V\rangle$ for an ideal relativistic Fermi gas at temperature $k_{\mathrm{B}}T = 1.7\times 10^{-4}$ eV, for different neutrino mass parameter $m_\nu c^2$.

same time, our method will, of course, fail to estimate the mass of the different types of neutrinos separately. However, it is hoped that some day in the future it would become possible to extract both $\langle N_\nu\rangle/V$ and $\langle\Delta N_\nu^2\rangle/V$ from experimental measurements of momentum distribution of each type of neutrinos. It requires, as we said before, the development of sensitive low-energy neutrino detectors for each type of neutrinos with good momentum and energy resolution in the 10^{-4} eV range to make these direct measurements for each type of neutrinos. In any scat-

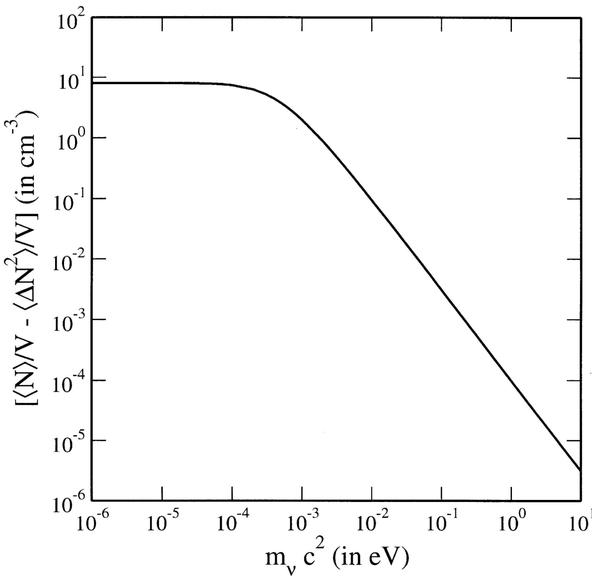

Figure 4. Variation of the deviation $\langle N \rangle/V - \langle \Delta N^2 \rangle/V$ of the mean square fluctuation as a function of the neutrino mass parameter $m_\nu c^2$, for $k_{\mathrm{B}}T = 1.7 \times 10^{-4}$ eV and $\langle N_\nu \rangle/V = 10^2$ cm^{-2}.

terring experiment of particles with CNB, involving momentum transfer $\hbar\vec{k}$, which couples through the number density of CNB, one can obtain $\langle N(k) \rangle^2$ directly from the cross-section. However, any other experiment which can give $\langle N(k) \rangle$ is good enough to find $\langle N_\nu \rangle/V$ and $\langle \Delta N_\nu^2 \rangle/V$. Although, the present emphasis on precise measurements of high-energy ($\sim$ MeV) neutrinos from the Sun and other sources is important, it is a much greater challenge for all of us to make a direct detection of CNB distribution, which has a flux of about 10^{11}–10^{12} particles per cm^2 per s, everywhere.

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