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Dual charged solution in curved space-time

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Abstract. A dual charged solution carrying both electric and magnetic charge is formulated in $SU(2)\times U(1)$ gauge theory without making use of the topological characteristics of Higgs fields. When Dirac quantisation condition is imposed, two consequences follow: (i) Weinberg angle is restricted to the value $\sin^2\theta=1/2$ and (ii) the solution cannot have fractional electric change, but must have integer items the basic electric charge of the theory. The infinity inherent in the theory is removed at the classical level by the use of gravitational effects by obtaining the same solution in the curved space-time. The resultant metric is of Reissner-Nordström form.

Keywords. Dyon; monopoles; gauge theories; curved space-time.

1. Introduction

The study of solutions to classical Yang-Mills field equations has received much attention in recent times due to the realisation of their important role in understanding various aspects of hadron physics. An initial momentum to this was provided by 't Hooft's observation that solutions with characteristics of a magnetic monopole exist for YM field systems with spontaneous symmetry breaking ('t Hooft 1974). The magnetic charge is contained at the zeros of the Higgs field (Arafune *et al* 1975). 't Hooft's work was extended to solutions which are both electrically and magnetically charged (Julia *et al* 1975, Prasad *et al* 1975). The existence of dual charged particles was earlier considered by Schwinger (1969) as a possible answer to many intriguing questions. One hopes that quarks with fractional electric charge are entities of this variety.

The introduction of elementary scalar field to spontaneously break the symmetry of a non-abelian gauge theory describing strong interactions is likely to affect properties such as asymptotic freedom, possessed by unbroken Yang Mills field. Therefore, much effort has been put recently in considering dynamical symmetry breaking of such theories. From this point of view, the work of Hsu (1976), exhibiting monopole solutions in Weinberg's unified theory without any crucial use of Higg's scalar becomes significant.

In this paper we look foı a similar but dual charged solutions in Weinberg's model (1967). We observe that the electric charge of the solution depends on the Weinberg angle. It is an integral multiple of electron charge if $\sin^2\theta = \frac{1}{2}$ but is otherwise arbitrary. One of the drawbacks of these solutions is that unlike 't Hooft monopoles these are not finite energy solutions of the theory. The total energy in the field configuration is infinite, leading to infinite self mass. One would then hope that quantum corrections and renormalizations would make it finite. Alternatively we may look for solutions, which are of finite mass at the classical level, by incorporating the effect of gravity. This can be effectively implemented by considering the dual charged solution in curved space time. The resultant Einstein equations are solved together with Maxwell like equations for the gauge fields. A metric identical to the Reissner-Nordström metric is obtained for the solution characterizing a dyon, in conformity with Birkhoff's theorem. (Birkhoff 1923, See also Weinberg 1972).

2. Dual-charged solution

Consider the Weinberg's unified theory (1967) based on the gauge group $SU(2)\times U(1)$. The equations of motion for the classical fields $\mathbf{A}_\mu$, B_μ, ϕ and $\phi\dagger$ are

$$\partial_\mu \mathbf{F}^{\mu\nu} - f\mathbf{F}^{\mu\nu} \times \mathbf{A}_\mu + if\phi\dagger^\nu \vec{\tau}/2\,\theta - if\phi\dagger\,\vec{\tau}/2\,\phi^\nu = 0 \tag{1}$$

$$\partial_\mu B^{\mu\nu} - \tfrac{1}{2}\, if'\,\phi\dagger\phi^\nu + \tfrac{1}{2}\, if'\,\phi\dagger^\nu\phi = 0 \tag{2}$$

$$\partial_\mu \phi^\nu - M_1^2\phi + 2h\,\phi\dagger\phi^2 - [i\,F A_\mu\,\vec{\tau}/2 + \tfrac{1}{2}\, if'\, B_\mu]\phi^\mu = 0 \tag{3}$$

where

$$\phi = \begin{bmatrix} \phi\dagger \\ (\phi_1^0 + \sqrt{2}\lambda + i\phi_2^0)/\sqrt{2} \end{bmatrix}$$

$$\mathbf{F}_{\mu\nu} = \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu + f\mathbf{A}_\mu \times \mathbf{A}_\nu$$

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$$

$$\phi^\mu = \partial^\mu\phi - if\mathbf{A}^\mu\,(\vec{\tau}/2)\,\phi - \tfrac{1}{2}\, if'\, B^\mu\phi$$

and $\vec{\tau}$ are the usual Pauli matrics.

Leptons have been neglected for simplicity. The photon field A_μ^{em} and the neutral massive vector field Z_μ are given by

$$A_\mu^{\mathrm{em}} = A_\mu^3 \sin\theta + B_\mu \cos\theta \tag{4}$$

$$Z_\mu = A_\mu^3 \cos\theta - B_\mu \sin\theta.$$

The Weinberg angle θ and the electronic charge e_0 are given by

$$\tan\theta = f/f' \text{ and } e_0 = -f\sin\theta. \tag{5}$$

The Higgs scalar fields obviously possess the following trivial solutions:

$$\phi^\pm = 0, \quad \phi_2^0 = 0, \quad \phi_1^0 = -\sqrt{2}\,\lambda = -\frac{2\sqrt{2}M_w}{f}$$

where M_w is the mass of $W_\mu^\pm = (A^1_\mu \pm iA^2_\mu)/\sqrt{2}$. Consequently, they decouple from the field eqs (1) and (2) which then become,

$$\partial_\mu \mathbf{F}^{\mu\nu} - f\mathbf{F}^{\mu\nu} \times A_\nu = 0. \tag{6}$$

$$\partial_\mu B^{\mu\nu} = 0. \tag{7}$$

Now the local gauge invariance of the theory allows us to introduce a unit isovector $\nu^a(x)$ (Hsu 1976). Therefore, one can look for the static spherically symmetric solutions of the form

$$B^i = \nu^i B(r),\ \nu^i = r^i/r.,\ B_0 = 0 \tag{8}$$

$$A_i^a = \epsilon_{iab}\, \nu^b A(r),\ A_0^a = \nu^a A_0(r) \tag{9}$$

$$i, a, b = 1, 2, 3.$$

On substituting (8) and (9) in the field equations we find that eq. (7) does not restrict the radial functions $B(r)$. However, eq. (6) reduces to

$$r^2 \frac{d^2A}{dr^2} + 2r \frac{dA}{dr} - A\,(1+fAr)\,(2+fAr) + fr\, A_0^2\,(1+fr\,A) = 0 \tag{10}$$

$$r^2 \frac{d^2A_0}{dr^2} + 2r \frac{dA_0}{dr} - 2A_0\,(1+fr\,A)^2 = 0. \tag{11}$$

A special solution of eq. (10) is

$$A(r) = -(1/fr). \tag{12}$$

The structure of the equations of motion is such that this solution decouples the radial functions $A(r)$ and $A_0(r)$. Hence one can also look for non-trivial solution of $A_0(r)$. The other solution $A_0 = -2/fr$ considered by Hsu (1976) does not admit non-trivial value of $A_0(r)$. Equation (11) is obviously satisfied by

$$A\,(r) = J/r \tag{13}$$

with J as an arbitrary constant. To understand the meaning of this classical solution a gauge invariant generalised electromagnetic field tensor is defined as follows:

$$\bar{F}_{\mu\nu} = \nu^a F^a{}_{\mu\nu} \sin\theta + B_{\mu\nu} \cos\theta - \left(\frac{\sin\theta}{f}\right) \epsilon^{abc}\, \nu^a\, D_\mu \nu^b D_\nu \nu^c \tag{14}$$

where

$$\nu^a(x)\, \nu^a(x) = 1 \text{ and}$$

$$D_\mu \nu^a = \partial_\mu \nu^a + f\, \epsilon^{abc}\, A_\mu^b \nu^c.$$

It is easily verified that $\bar{F}_{\mu\nu} = \partial_\mu A_\nu^{\text{em}} - \partial_\nu A_\mu^{\text{em}}$ when v^a points in the z-direction at each space time point. In the present case we find that the electric field is

$$E_j = \bar{F}_{jo} = \sin\theta \frac{d}{dr}(J/r)\, v_j$$

$$\mathbf{E} = -(J\sin\theta/r^2)\,\mathbf{v}.$$

This originates from an electric charge given by

$$Q_D = \frac{1}{4\pi}\int \mathbf{E}\cdot d\mathbf{s} = -J\sin\theta. \tag{15}$$

Since the basic electric charge in Weinberg's model is $e_0 = -f\sin\theta$, this represents an electric charge of J/f units of basic electric charge. Similarly the magnetic field is given by

$$H_j = \tfrac{1}{2}\,\epsilon_{jkl}\,\bar{F}^{kl} = -(\sin\theta/fr^2)\, v_j$$

or $$\mathbf{H} = -(\sin\theta/fr^2)\,\mathbf{v}.$$

This form implies a monopole of strength $-g_0 = -\sin\theta/f$. Further Dirac quantization condition for electric and magnetic charges imply that

$$e_i\, g_j - e_j\, g_i = n/2 \tag{16}$$

where e_i and g_i are respectively the electric and magnetic charges on the ith particle. If i refers to a particle that carries just the basic charge e_0 and no magnetic charge and j refers to a monopole of strength $g_0 = \sin\theta/f$, Dirac's condition that

$$e_0\, g_0 = n/2$$

implies $\sin^2\theta = n/2$. Only $n = 0, 1, 2$ is thus compatible. While $n = 0$ and $n = 2$ will mean absence of either electromagnetic or weak coupling, only consistent value, we may have for $\sin^2\theta$ is $1/2$ corresponding to the product $e_0\, g_0$ having a value $1/2$. When we apply eq. (16) with our dyon solution as particle i and a pure monopole as particle j, we arrive at

$$Q_D\, g_0 = n'/2$$

or $(J/f)\, e_0\, g_0 = n'/2$. Since $e_0\, g_0 = 1/2$, we find $J/f = n'$ which is an integer. The electric charge carried by the dyon is an integral multiple of the basic electronic charge. In particular, this precludes the possible association of fractionally charged quarks with these solutions.

3. Curved space-time

We now proceed to consider the nature of spacetime containing the dual charged solution. In view of the spherical symmetry of the following form:

$$g_{\mu\nu} = \text{diag}\,(-e^{2\eta},\, e^{2\Lambda},\, r^2,\, r^2 \sin^2\theta) \tag{17}$$

where Λ and η are unknown functions of r. The field eqs (6) and (7) by virtue of the antisymmetric nature of $F^a{}_{\mu\nu}$ become

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\nu}(\sqrt{-g}\,F_a^{\mu\nu}) - f\,\epsilon_{abc}\,F_b^{\mu\nu}\,g_{\alpha\nu}\,A_c^\alpha = 0 \tag{18}$$

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\nu}(\sqrt{-g}\,B^{\mu\nu}) = 0 \tag{19}$$

where x_μ refers to the curvilinear coordinates and g is the det $g_{\mu\nu}$. We again look for solutions of the forms (8) and (9). Equation (19) does not restrict $B(r)$. Choosing spherical coordinates eq. (18) reduces to the following two coupled equations.

$$e^{-2\Lambda}[r^2A'' + 2rA' + (r^2A' + rA)(\eta' + \Lambda')]$$

$$-A\,(1+frA)\,(2+frA) - e^{-2\eta}\,f\,A_0^2\,r\,(1+fAr) = 0. \tag{20}$$

$$e^{-2\Lambda}\,[r^2A_0'' + 2r\,A_0' + r^2\,A_0'\,(\eta' + \Lambda')] - 2A_0\,(1+frA)^2 = 0. \tag{21}$$

These equations reduce to (10) and (11) in flat spacetime as they should. The fact that A_0 is not zero leads to a η-dependent term in eq. (20). Moreover we have an additional eq. (21) containing η and Λ. This makes it different from the situation in the case of 't Hooft's monopole in curved spacetime considered by Bais and Russel (1975).

Note the above equations again decouple in A_0 and A for $A(r) = -1/fr$ and become linear. The term containing exponential in η also disappears though A_0 is not zero. Moreover, the coefficient of $(\eta'+\Lambda')$ term is such that eq. (20) is satisfied without in any way restricting the functions η and Λ. But the substitution of A_0 from (13) leads to the following constraint condition

$$\eta' + \Lambda' = 0. \tag{22}$$

Thus, for the dual charged solution the field equations do not decouple from the Einstein's equations of the gravitational field in contrast to the monopole case (Bais and Russel 1975).

We now consider the Einstein's equations,

$$R_{\mu\nu} - \tfrac{1}{2}\,g_{\mu\nu}\,R = T_{\mu\nu} \tag{23}$$

where $R_{\mu\nu}$ is the Ricci tensor and R its trace. $T_{\mu\nu}$ is the energy momentum tensor which in our case is given by

$$T_{\mu\nu}=\tfrac{1}{4}[g^{\alpha\beta}F^{a}{}_{\mu\alpha}F^{a}{}_{\nu\beta}-\tfrac{1}{4}g_{\mu\nu}g^{\sigma\tau}g^{\alpha\beta}F^{a}{}_{\alpha\sigma}F^{a}{}_{\beta\tau}$$
$$+g^{\alpha\beta}B_{\mu\alpha}B_{\nu\beta}-\tfrac{1}{4}g_{\mu\nu}g^{\alpha\beta}B_{\alpha\sigma}B_{\beta\tau}] \tag{24}$$

Inserting (8) and (9) with (12) and (13) we obtain

$$B_{\mu\nu}=0$$

$$F^{a}{}_{oj}=\frac{J}{r^4}r_a r_j$$

$$F^{a}{}_{mj}=\frac{1}{fr^4}2(\epsilon_{jaa'}r_m-\epsilon_{maa'}r_j)r^{a'}+2\epsilon_{jma}r^2-\epsilon_{jmc}r^c r_a. \tag{25}$$

Thus the highly symmetric solution of B_μ does not contribute to $T_{\mu\nu}$. Expressing (25) in spherical coordinates we find the following result for the components of stress energy tensor:

$$T_{tt}=\frac{1}{8\pi f^2r^4}[e^{2\eta}+J^2f^2e^{-2\Lambda}]$$

$$T_{rr}=\frac{1}{8\pi f^2r^4}[e^{2\Lambda}+J^2f^2e^{-2\eta}]$$

$$T_{\theta\theta}=\frac{1}{8\pi f^2r^4}[1+J^2f^2e^{-2(\eta+\Lambda)}]$$

$$T_{\phi\phi}=\sin^2\theta\, T_{\theta\theta}.$$

All other components vanish. For monopole solution, i.e. $J=0$ the above relations reduce to those obtained by Bais and Russel (1975). Similarly one can also compute the components of Ricci tensor using (17). Substituting these the Einstein's equations get transformed into the following form:

$$e^{-2\Lambda}\left(\frac{1}{r^2}-\frac{2\Lambda'}{r}\right)-\frac{1}{r^2}=\frac{k}{f^2r^4}(1+J^2f^2e^{-2(\eta+\Lambda)}) \tag{26}$$

$$e^{-2\Lambda}\left(\frac{1}{r^2}-\frac{2\eta'}{r}\right)-\frac{1}{r^2}=-\frac{k}{f^2r^4}(1+J^2f^2e^{-2(\eta+\Lambda)}) \tag{27}$$

$$r^2e^{-2\Lambda}\left(\eta''+\eta'^2-\Lambda'\eta'+\frac{\eta'-\Lambda'}{r}\right)=\frac{k}{f^2r^2}(1+J^2f^2e^{-2(\eta+\Lambda)}). \tag{28}$$

These equations are to be solved with the constraint eq. (22) i.e.

$$\eta' + \Lambda' = 0$$

or

$$\eta(r) = -\Lambda(r) + C,$$

C is the constant of integration, which can be fixed by demanding that asymptotically the space becomes flat. This implies $C = 0$. so that

$$\eta(r) + \Lambda(r) = 0. \tag{29}$$

Substituting this in (26) — (28)

$$e^{-2\Lambda}\left(\frac{1}{r^2} - \frac{2\Lambda'}{r}\right) - \frac{1}{r^2} = -\frac{k}{f^2 r^4}(1+J^2 f^2) \tag{30}$$

$$e^{-2\Lambda}\left(\frac{1}{r^2} - \frac{2\eta'}{r}\right) - \frac{1}{r^2} = -\frac{k}{f^2 r^4}(1+J^2 f^2) \tag{31}$$

$$r^2 e^{-2\Lambda}\left(\eta'' + \eta'^2 - \Lambda'\eta' + \frac{\eta' - \Lambda'}{r}\right) = \frac{k}{f^2 r^2}(1+J^2 f^2). \tag{32}$$

Equation (30) can be rewritten as

$$\frac{d}{dr}(re^{-2\Lambda}) = 1 - \frac{k}{f^2 r^2}(1+J^2 f^2) \tag{33}$$

On integration this gives

$$e^{-2\Lambda} = 1 - \frac{M_D k}{r} + \frac{k(1+J^2 f^2)}{f^2 r^2} \tag{34}$$

where M_D is the constant of integration and is identified with the mass of the dyon. This also satisfies eqs (31) and (32) in conjunction with the condition $\Lambda(r) = -\eta(r)$ imposed by the field equations. Expressing J and f in terms of electric and magnetic charges the metric can be written as

$$ds^2 = -\left(1 - \frac{M_D k}{r} + \frac{k(g_m^2 + Q_D^2)}{r^2 \sin^2\theta}\right) dt^2$$

$$+ \left(1 - \frac{M_D k}{r} + \frac{k(g_m^2 + Q_D^2)}{r^2 \sin^2\theta}\right)^{-1} dr^2 + r^2 d\Omega^2. \tag{35}$$

This is the standard Reissner-Nordström metric. According to Birkhoff's theorem a spherically symmetric distribution of mass and electromagnetic sources would lead to a unique metric. This result is in conformity with this theorem. Note this metric is valid only for $r \geqslant kM_D$ since kM_D is the event horizon. Thus, we have effectively given an extension to our dyon, so that the energy content or mass of the dyon is no longer infinite. Indeed, the mass of the dyon M_D, being the integration constant in eq. (33), is arbitrary and an external parameter.

4. Comments

We have obtained here an exact dual charged solution in Weinberg's theory. While the motivation for this was for finding solutions with fractional electric charge, the charge of the dyon turns out to be an integer multiple of electronic charge. The Weinberg's angle is restricted to a value $\sin^2\theta = \frac{1}{2}$. We then obtain an explicit solution of Einstein's equations for the dual charged solution of the non-Abelian gauge theory. The solution of the system becomes possible due to the fact that the linearisation and decoupling of the equations occur just as easily in the curved space-time. Moreover, the field equations imposed a very simple restriction (eq. 22) on the solution of the Einstein's equation. This, in fact, is responsible for our metric to be identical with the Reissner-Nordström metric for a particle endowed with both electric and magnetic charge. Notice that from eqs (26) – (28) the condition (22) results in equal weightage for the two charges in the metric.

The magnetic charge of this solution is not of topological origin. On the contrary, such a solution, because of the singularity at the origin for a point (electric or magnetic or both) charges has infinite self mass at the classical level. By combining the electromagnetic properties with the gravitational effects we have been able to give a finite extension to the sources and thereby remove the infinities at the classical level.

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