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Radiative decays of charged leptons in arbitrary order

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Abstract. In radiative decays of charged leptons induced by non-degenerate neutrino masses and consequent lepton flavour-mixing, the dominant suppression factor at the 1-loop level is $(\Sigma\, m_a^2)/m_W^2$, m_a being the mass of the ath neutrino. We show that this suppression factor is present *in all orders* in a class of models including the standard model.

Keywords. Charged leptons; flavour mixing; radiative decays; neutrino masses; decoupling theorem.

1. Introduction

There has been a number of calculations (Petcov 1977; Bilenky *et al* 1977; Cheng and Li 1977a; Marciano and Sanda 1977), over the last few years of "flavour"-changing radiative decays, such as $\mu \to e\gamma$ or $\tau \to \mu\gamma$, $e\gamma$. Especially noteworthy is the work of Cheng and Li (1977a,b), extended recently by them to cover the general situation in which the neutrinos have both Dirac (D) and Majorana (M) mass terms in the Lagrangian (Cheng and Li 1980). Their general conclusion is that these rates are minuscule for both D and M neutrinos except for a special choice of some parameters in the case when both D and M mass terms are present.

The mechanism responsible for the suppression are entirely different for the two cases of D and M neutrinos. For D neutrinos, the amplitude for, say, the decay $\mu \to e\gamma$ is given in the lowest non-vanishing order (1-loop) by Cheng and Li (1977a)

$$A^{(1)}_{\mu e} = \frac{G_F}{32\pi^2\,(2)^{1/2}}\, e \cos\theta \sin\theta \left[\frac{m_1^2 - m_2^2}{m_W^2}\right] \bar{u}_e\, m_\mu\, \sigma_{\beta\alpha}\, q_\alpha\, (1+\gamma_5)\, u_\mu\, \varepsilon_\beta, \quad (1)$$

in the case of two mixing families of leptons (*i.e.* two distinct neutrinos of masses m_1 and m_2), where θ is the Cabibbo-like mixing angle. The amplitude is therefore bounded by

$$|A^{(1)}_{\mu e}| < G_F\, e \left[\frac{\delta m^2}{m_W^2}\right] |M^{(1)}_{\mu e}|, \quad (2)$$

where $$M^{(1)}_{\mu e} = \frac{1}{32\pi^2\,(2)^{1/2}}\, \bar{u}_e\, m_\mu\, \sigma_{\beta\alpha}\, q_\alpha\, (1+\gamma_5)\, u_\mu\, \epsilon_\beta\, (q).$$

In the general case of N mixing leptonic families (the N neutrinos having masses $m_1, \ldots, m_N$), the formula for the 1-loop radiative decay of the charged lepton l_a to $l_b + \gamma$ becomes

$$A_{ab}^{(1)} = G_F\, e \sum_{c=1}^{N} C_{ac}\, C_{bc}^{*} \left[\frac{m_c^2}{m_W^2}\right] M_{ab}^{(1)}, \tag{3}$$

so that we still have

$$|\, A_{ab}^{(1)} \,| < G_F\, e \left[\frac{\Sigma\, m_c^2}{m_W^2}\right] |\, M_{ab}^{(1)} \,| \equiv G_F\, e\, \delta\, |\, M_{ab}^{(1)} \,|, \tag{4}$$

because of the unitarity of the Cabibbo-like mixing matrix C whose elements are C_{ab}. It is the factor δ which is the cause of hopelessly small radiative decay rates, for realistic neutrino masses: for neutrino masses of the order of 100 eV,

$$\delta \simeq 10^{-18}, \tag{5}$$

a number which is much smaller than the fine structure constant; if δ were to be of the order 1, the branching ratio for $\mu \to e\gamma$ would be of the order of

$$R_\mu \equiv \Gamma\,(\mu \to e\gamma)/\Gamma\,(\mu \to e\nu\bar{\nu}) \sim 10^{-5}. \tag{6}$$

This circumstance raises the following theoretically and experimentally interesting question: is it possible that the contribution of some n-loop graph is free from the suppression factor δ *i.e.* that δ is replaced by a $\delta^{(n)} \gg \delta$ so that $\alpha^{n-1}\, \delta^{(n)} > \delta$? Purely for illustration, if there is an n-loop contribution to $A_{\mu e}$ which has the form $G_F\, \alpha^{n-1/2}\, \delta^{1/2}$ $(\delta^{(n)} = \delta^{1/2})$, then $A_{\mu e}^{(n)}$ rather than $A_{\mu e}^{(1)}$ will dominate the full amplitude $A_{\mu e}$ if $n < 6$. It is clearly necessary to settle this question before we can accept the lowest order calculation as giving the true magnitude of radiative decay rates.

2. Two assertions on the suppression factor in the decay rate

In this note we show that in a class of models including the standard model, the possibility envisaged above does not in fact occur: *e.g.*, in the standard model and apart from numerical factors, $|\, A_{ab}^{(n)} \,|$ is bounded by $G_F\, \alpha^{n-1/2}\, \delta$. More precisely, it is shown, *in every order*, that (i) in models with one (neutral) Higgs particle ϕ° and Z bosons coupling always to diagonal lepton currents, the amplitude $A_{ab}^{(n)}$ vanishes in the limit of degenerate neutrino masses; and (ii) when neutrino mass differences are " turned on ", $A_{ab}^{(n)}$ is proportional to $\sum_c C_{ac}\, C_{bc}^{*}\, m_c^2/m_W^2$ if the Z-bosons couple only to left handed neutral currents, as in the standard SU(2) $\times$ U(1) theory. It may be recalled that relatively profuse radiative decays can occur in models with more than one Higgs particle (Bjorken and Weinberg 1977). It will be clear at the appropriate point in which way other features of the standard model can be changed if we wish to have "large " radiative decay rates. It may also be noted here that these considerations are not relevant to Majorana neutrinos, since the suppression mechanism in that case is of a very different kind (Cheng and Li 1980).

There is a general belief that our statement (i) above can be trivially proved by working on the basis for the set of neutrino fields that diagonalises flavour *i.e.*, those associated with the charged lepton fields of definite mass to form doublets. We shall indicate later why such reasoning is inapplicable in the present context and why a more detailed proof such as the one we give below is necessary.

3. Proof of the assertions

Our demonstration of statement (i) goes as follows. The class of models considered has a pair of charged W bosons ($W^{\pm}$) coupling to the charged current

$$J_a^+ = \sum_{a,\, b=1} \bar{l}_a^- \gamma_\alpha \tfrac{1}{2}(1+\gamma_5)\, C_{ab}\, \nu_b, \tag{7}$$

and its conjugate *via* an interaction Lagrangian

$$\mathscr{L}_{cc} = g J_a^+ W_a^- + \text{h.c.} \tag{8}$$

where l_a and ν_a are the fields for the charged leptons and neutrinos which have the correct asymptotic limits with definite masses and C is the Cabibbo-like flavour-mixing matrix. There can be any number of neutral Z-bosons as long as they all couple to diagonal currents and it is assumed that there is only one (neutral) Higgs particle ϕ^0 whose interaction with (mass eigenstate) leptons is necessarily diagonal. In every (connected) diagram contributing to $l_a \to l_b + \gamma$, there is a continuous lepton line with initial label a and final label b ($\neq a$). By assumption, Z and ϕ° vertices along this line conserve the label. So there must occur $W^{\pm}$ vertices, in pairs to conserve electric charge. For any vertex on this line at which l_c^- emits a W^- the nearest subsequent charge changing vertex must absorb a W^- to convert an incoming neutrino back into a charged lepton l_d^-. Of all such pairs of "nearest neighbour" charged vertices, at least one pair must be such that $c \neq d$, for overall change of flavour. Let P and P' be one such pair. The line PP' is neutrino line, with the same label, say t, all the way from P to P'—even though there can be γ, Z and ϕ° vertices between P and P', they cannot change the label of ν. There is a set of N different diagrams for the N distinct neutrinos that connect P to P', *i.e.* the set of all diagrams which contribute in a given order can be divided into subsets containing N diagrams each, which differ from one another only in the label of the neutrino line from P to P'. When all neutrinos have the same mass, the contribution of each of these diagrams is the same (before and after renormalization) except for the vertex factors gC_{ct} at P and gC_{dt}^* at P'. The contribution of any such set of N diagrams is therefore proportional to $\Sigma g^2 C_{ct} C_{dt}^* = g^2 \delta_{cd}$ ($=0$ for $c \neq d$), because of the unitarity of C. This proves our assertion (i).

What happens when the neutrinos are no longer mass-degenerate is clearly seen by recalling the origin of the factor $\sum_c m_c^2$ at the l-loop level (Cheng and Li 1977a). The contribution of a typical diagram with a ν_t internal line has an integrand proportional to

$$I_t(k) = C_{at} C^*_{bt} \gamma_\alpha (l + \gamma_5)(\gamma \cdot k + m_t) \gamma_\beta (l + \gamma_5)(k^2 - m_t^2)^{-1}, \quad (9)$$

where k will be integrated over. Because $m_t \ll m_W$, this may be expanded in powers of m_t and because of helicity conservation at each vertex terms with odd powers of m_t vanish. Thus

$$I_t(k) = C_{at} C^*_{bt} \gamma_\alpha (l + \gamma_5) \gamma \cdot k \gamma_\beta (l + \gamma_5) k^{-2} (l + m_t^2 k^{-2} + \ldots). \quad (10)$$

On summing over t, the first term in the expansion vanishes and the leading non-vanishing contribution is proportional to $\sum_t C_{at} C^*_{bt} m_t^2$.

This argument carries over immediately, with minor changes, to diagrams of arbitrary order. As in the degenerate mass case, diagrams fall into sets of N, identical except for the neutrino line from P to P'. Let us first consider a set such that there are no vertices at all between P and P'. The contribution of the diagram with a ν_t line from such a set has, in the integrand, a factor identical to (9) where k is now one of the many loop momenta which will subsequently be integrated over. The reasoning which gave the factor $\Sigma C_{ct} C^*_{dt} m_t^2$ earlier goes through as before.

What happens when these are Z and ϕ^0 vertices between P and P'? Consider first the insertion of one Z-vertex on the ν_t line from P to P'. In the standard model the neutrino current coupling to Z is V-A (and also, of course, flavour preserving). The contribution of a typical diagram has an integrand which has a factor

$$C_{ct} C^*_{dt} \gamma_\alpha (l + \gamma_5)(\gamma \cdot k + m_t) \gamma_\delta (l + \gamma_5)(\gamma \cdot k' + m_t) \gamma_\delta$$
$$(l + \gamma_5)(k^2 - m_t^2)^{-1} (k'^2 - m_t^2)^{-1}, \quad (11)$$

with both k and k' to be integrated over. Because the Z-ν_t interaction is helicity-preserving, only even powers of m_t occur in an expansion of (11) and, as before, the m_t independent term is identically zero on summing over t. Exactly the same situation occurs for an arbitrary number of Z-vertices between P and P'.

On the other hand, Higgs couplings to fermions in the standard model are, in addition to being diagonal, also proportional to fermion masses. Consequently ϕ^0 vertices on the ν_t line contribute extra powers of m_t/m_W in the amplitude and so may be ignored.

It is intuitively obvious, for purely dimensional reasons, that the (neutrino mass)^2 factors occur in the general n-loop amplitude in the combination $C_{ac} C^*_{bc} m_c^2/m_W^2$, exactly as in the l-loop amplitude. This feeling can be made precisely by a simple appeal to the decoupling theorem (Symanzik 1973; Appelquist and Carrazone 1975). If we take the leptons and the photon to be the " light " particles in the sense of the theorem and " integrate " amplitudes over the " heavy " $W^\pm$ fields (integration over the Z and ϕ^0 fields is optional) in, for example, a functional integral representation (Weinberg 1980), the resulting effective renormalisable low-energy Lagrangian will not allow for radiative leptonic decays. Nor will they come from the first term in the residual effective non-renormalisable Lagrangian of order m_W^{-2} (*i. e.*, the "4-fermion" Lagrangian) because of the GIM cancellation. They can only arise from effective Lagrangians of the form $m_W^{-2r} L_{\text{eff}}^{(2r)}$ $(r > 2)$ where $L_{\text{eff}}^{(2r)}$ is of dimension $4 + 2r$. The l-loop calculation already illustrates this and the way in which the full l-loop amplitude is approximated to get the answer (1) can be looked upon as an illustrative proof of the decoupling theorem to order g^3 (or $G_F\alpha^{1/2}$) in the particular context of the SU(2) $\times$ U(1) theory with massive neutrinos. Even though the duplication of that argument directly in arbitrarily high orders is impractical, the recent general proofs (Kazama and Yao 1981) of the decoupling expansion make it unnecessary. The only complication is the possible occurrence of factors of ln $(m_W)^2$ for sufficiently high orders. But such factors are more than compensated for by the extra powers of α that they entail.

This completes our demonstration that if neutrinos are purely Dirac particles, radiative decays such as $\mu \to e\gamma$ and $\tau \to \mu\gamma$ or $e\gamma$ will always remain beyond direct experimental detection by many orders of magnitude.

4. Concluding remarks

If all neutrinos were of strictly degenerate mass and if we assume that any linear combination of ν_a can then be identified with the physical neutrino fields (those whose asymptotic limits create and annihilate physical neutrino states), then our statement (i) would not need the rather elaborate demonstration that we have given: in the basis which diagonalises flavour, no part of the Lagrangian would mix flavour. Such a line of reasoning* cannot be applied to the situation we are dealing within which the masses of the neutrinos are not the same. Since the asymptotic fields in terms of which the Lagrangian is to be written (and, therefore, propagators in diagrams) must have a definite mass, we are not free to transform to a basis which diagonalises flavour. The detailed demonstration we have given of our statement (i) cannot, therefore, be dispensed with.

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*There is also the point that even if the neutrinos were degenerate in mass, the particular linear combination of the single-neutrino states which are the true physical states (all of the same mass) may be fixed by considerations which go beyond the gauge theory of their weak interactions, *e.g.*, by a grand unification model and its first state of symmetry breaking. There is no reason for the so determined 'physical' neutrino fields to coincide with these which combine with the different charged leptons to make the charged currents.

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