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A decomposition theorem for SU(n) and its application to CP-violation through quark mass diagonalisation

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Abstract. It is proved that the group $G = \mathrm{SU}(n)$ has a decomposition $G = FCF$ where F is a maximal abelian subgroup and C is an $(n-1)^2$ parameter subset of matrices. The result is applied to the problem of absorbing the maximum possible number of phases in the mass-diagonalising matrix of the charged weak current into the quark fields; i.e., of determining the exact number of CP-violating phases for arbitrary number of generations. The inadequacies of the usual way of solving this problem are discussed. The $n = 3$ case is worked out in detail as an example of the constructive procedure furnished by the proof of the decomposition theorem.

Keywords. Semisimple Lie algebras; Cartan decomposition; CP-violating phases; Kobayashi-Maskawa matrix; decomposition theorem; quark mass diagonalisation.

1. Introduction

To incorporate CP-noninvariance within the standard gauge model of weak interactions, the currently favoured procedure is to write the charged weak current as

$$J_\mu = \bar{U}\gamma_\mu \tfrac{1}{2}(1-\gamma_5)\, cD, \tag{1}$$

where $U = (u_1, u_2, \ldots, u_n)$ and $D = (d_1, d_2, \ldots, d_n)$ are, respectively, sets of 'up' quarks of charge $+2/3$ and 'down' quarks of charge $-1/3$. These quark fields are defined as the eigenvectors of the quark mass operators. The unitary $n \times n$ matrix c will incorporate CP-violation if its elements include complex numbers whose phases cannot be eliminated by redefining the phase of individual quark fields. The matrix c is to be determined as follows: the Yukawa coupling of the gauge group doublets (u_i^0, d_i^0) with the Higgs field(s) leads to the mass matrices $M(U)$ and $M(D)$ in the bases U^0, D^0 respectively. These matrices are diagonalised by unitary matrices v_U, v_D respectively, with

$$v_U U^0 = U, \quad v_D D^0 = D.$$

The charged current is then

$$J_\mu = \bar{U}^0 \gamma_\mu \tfrac{1}{2}(1-\gamma_5)\, D^0 = \bar{U}\gamma_\mu \tfrac{1}{2}(1-\gamma_5)\, vD, \tag{2}$$

where v is, in general, an n^2-parameter unitary matrix. We are then to factorise v in such a way as to separate an overall phase to be absorbed into the W_μ to which J_μ

couples and to similarly absorb as many additional phases as possible into the individual quark fields. It is the last problem (raised, especially, by the phrase 'as many as possible') that is the concern of this paper. What is required is a decomposition of a general unitary matrix in the form

$$v = fcf' \exp(i\chi), \tag{3}$$

where f and f' are 'maximal' diagonal unitary matrices, χ is real and c is the matrix required in (1). Clearly, we may choose $\exp(i\chi) = \det v$. Since the maximal abelian subgroup of SU(n) has $(n-1)$ parameters, (3) will imply that c is a matrix with *at least* $n^2 - 1 - 2(n-1) = (n-1)^2$ parameters. On the other hand if c were real, it will be an orthogonal matrix and so can have *at most* $\frac{1}{2} n(n-1)$ parameters ('Euler angles') so that the general c will have *at least* $\frac{1}{2}(n-1)(n-2)$ 'phase angles'. These angles are responsible for CP-violation.

For $n = 3$, a parametrisation of c in terms of these Euler angles and one phase angle was first written down by Kobayashi and Maskawa (1973). Following them, it has generally been taken, on insufficient grounds, that there are *precisely* $\frac{1}{2}(n-1)(n-2)$ phase angles in the general case. It is obvious from the above that this number is only a lower bound on the number of phase angles – simple counting is not sufficient to establish the exact number of CP-violating phases. To justify the counting, what is required is a proof of the decomposition theorem (3) (a more explicit critique of the usual incomplete argument as given, e.g., in two recent expository articles (Harari 1976; Ellis 1978) will be found in the concluding section). We supply such a proof here. Our proof is also constructive; it lets us write down systematically the matrix c as a function of Euler and phase angles. The case $n = 2$ is of course trivial. The $n = 3$ case is sufficiently complicated to illustrate fully the general procedure; extension to $n > 3$, if and when more generations of fermions are discovered, only costs more labour.

Because of the nature of the problem, this paper is mathematical in content and form. Its direct relevance to the description of an important physical phenomenon, that of CP-violation in weak-electromagnetic gauge theories should, however, be clear from the remarks above.

2. The general decomposition theorem

The basic mathematical tool we use is a decomposition of a connected semi-simple Lie group with a finite centre into factors which are its one-parameter subgroups. This decomposition itself follows from the cartan decomposition of semi simple Lie algebras (see, e.g., Hermann 1966). The result we wish to arrive at is equation (3) or, more precisely (after first factoring out $\det v = \exp(i\chi)$), the following:

Theorem: The group SU(n) $= G$ has a decomposition $G = FCF$, where F is a ($(n-1)$ dimensional) maximal abelian subgroup and C is a ($(n-1)^2$-parameter) subset of SU(n).

Our proof of this theorem proceeds by first working out a suitable decomposition into one-parameter subgroups followed by a reordering of factors. For completeness, and as an aid to easy understanding, we give below in a subsection a brief summary of relevant standard general results without proofs (two books we have found useful are Helgason 1962 and Hermann 1966). The subsequent subsections of this section prove our theorem with all details given.

Notation: A capital letter (e.g. G) will denote a group and the same letter in bold face ($\mathbf{G}$) the corresponding Lie algebra. Lower case letters stand for group (or, when in bold face, Lie algebra) elements (g and $\mathbf{g}$ respectively).

2.1. *Summary of relevant general theory*

Cartan's fundamental theorem on decompositions of semi-simple Lie algebras is the starting point:

Theorem 1: A semisimple Lie algebra $\mathbf{G}$ has a direct sum decomposition into a subalgebra $\mathbf{K}$ and a vector subspace $\mathbf{P}$ satisfying (i) $[\mathbf{K}, \mathbf{K}] \subset \mathbf{K}$ (i.e., K is a subalgebra); (ii) $[\mathbf{K}, \mathbf{P}] \subset \mathbf{P}$ (i.e., $ad_{\mathbf{K}}$ leaves $\mathbf{P}$ invariant)*; and (iii) $[\mathbf{P}, \mathbf{P}] \subset \mathbf{K}$.

A $\mathbf{K}$ satisfying these conditions is a *symmetric subalgebra*. The corresponding subgroup K of G is a *symmetric subgroup* and the coset space G/K a *symmetric space*.

A maximal abelian subalgebra of $\mathbf{P}$ is called a *Cartan subalgebra* and denoted, typically, by $\mathbf{A}$.

The analogue of theorem 1 for Lie groups is

Theorem 2: Let G be a connected Lie group with finite centre whose Lie algebra is $\mathbf{G}$, K the connected subgroup whose Lie algebra is $\mathbf{K}$, and P the exponentiation of $\mathbf{P}$. Then G has the decomposition $G = KP$.

$\mathbf{K}$ and $\mathbf{P}$ (and hence K and P) may be defined by the action of a linear automorphism Φ on $\mathbf{G}$: $\Phi(\mathbf{k}) = \mathbf{k}$ for $\mathbf{k} \in \mathbf{K}$, $\Phi(\mathbf{p}) = -\mathbf{p}$ for $\mathbf{p} \in \mathbf{P}$, $\Phi^2 =$ identity. Under the exponential map, these conditions become $\phi(k) = k$, $k \in K$ and $\phi(p) = \phi(\exp \mathbf{p}) = \exp[\Phi(\mathbf{p})] = \exp(-\mathbf{p}) = p^{-1}$, $p \in P$, on the automorphism ϕ on G. A ϕ (or Φ) satisfying these properties, called a symmetric automorphism, always exists under the conditions stated. Finding a symmetric automorphism is a convenient practical way of carrying out a Cartan decomposition, a way we shall follow. The requirement that G must have a finite centre gives one more reason for working with SU(n) rather than U(n).

One further result we need is

Theorem 3: If $\mathbf{A}$ is a Cartan subalgebra and $\mathbf{A}'$ is any abelian subalgebra of $\mathbf{P}$, then there exists a $k \in K$ such that $Ad_k(A') \subset A$.

Given a decomposition $G = KP$, we may decompose K and P further. In the case of K, since it is a subgroup, one simply carries the KP decomposition a step further. As for P, it follows from Theorem 3 that every element of $\mathbf{P}$, considered as a one-dimensional (abelian) subalgebra of P, can be written as $ad_{\mathbf{K}}(\mathbf{a})$ for $\mathbf{a}$ in a fixed cartan subalgebra $\mathbf{A}$ and some $\mathbf{k} \in \mathbf{K}$. Applying the exponential map, we then have $P = Ad_K(A) = KAK$ (i.e., $p = k'ak'^{-1}$ for some $k' \in K$, $a \in A$), so that

$$G = KAK \equiv K_1A_1K_1,$$

is the first step in the required decomposition. Now let K_2 be a symmetric subgroup of K_1 (and A_2 correspondingly). The next stage of the decomposition is

$$G = K_2A_2K_2A_1K_2A_2K_2,$$

*The definitions and some properties of the adjoint maps we need here are given in the Appendix.

and so on. Eventually we have a decomposition into one-parameter subgroups, the particular decomposition depending on particular choices of symmetric subgroups at each level. The familiar Euler angle decomposition of SO(3, R) is a simple application of this procedure, as shown in Hermann (1966).

2.2. *The case* $G = SU(n)$

For $G = \mathrm{SU}(n)$, consider the map $\phi(g) = g_0 g g_0^{-1}$, with $g_0^2 =$ identity. The choice*

$$g_{0n} = \begin{pmatrix} 1 & \\ & -1_{n-1} \end{pmatrix}, \tag{4}$$

is the most convenient [even though g_0 is not an element of SU (n) for even n, ϕ is always an (outer-) automorphism]. It is easily checked that the set $K = \{k \mid \phi(k) = k\}$ is a subgroup, in fact the group S [U (1) $\times$ U (n—1)]; $k \in K$ has the general form

$$k = \begin{pmatrix} \exp(i\alpha) & \\ & \exp[-i\alpha/(n-1)]\, v_{n-1} \end{pmatrix}, \tag{5}$$

where α is real and $v_{n-1} \in \mathrm{SU}(n-1)$.

We now determine $\mathbf{P} = \{\mathbf{p} \mid \Phi(\mathbf{p}) = -\mathbf{p}\}$ and verify at the same time that $\mathbf{G} = \mathbf{K} \oplus \mathbf{P}$ is a cartan decomposition. Firstly, write $g_0 = \exp \mathbf{g}_0$ with

$$\mathbf{g}_{0n} = \begin{pmatrix} 0 & \\ & i\pi 1_{n-1} \end{pmatrix}, \tag{6}$$

and $g = \exp \mathbf{g}$. Then $\phi(g) = g_0 \exp(g\, g_0^{-1}) = \exp(g_0\, g\, g_0^{-1}) = \exp[\phi(g)]$ (see the appendix). For $\phi(g)$, we use the formula

$$\Phi(\mathbf{g}) = g_0\, \mathbf{g}\, g_0^{-1} = \exp \mathbf{g}_0\, \mathbf{g} \exp(-\mathbf{g}_0),$$

$$= \mathbf{g} + [\mathbf{g}_0, \mathbf{g}] + \frac{1}{2!} [\mathbf{g}_0, [\mathbf{g}_0, \mathbf{g}]] + \ldots$$

Now, from (5), any $\mathbf{k} \in \mathbf{K}$ which commutes with $\mathbf{g}_0$ has the form

$$\mathbf{k}_n = \begin{pmatrix} i\alpha & \\ & -[i\alpha/(n-1)]\, \mathbf{v}_{n-1} \end{pmatrix}, \tag{7}$$

*Here and in the following, we indicate the dimensionality of matrices by subscripts whenever it is necessary to be explicit. All blank entries in matrices stand for zero.

with $\mathbf{v}_{n-1} \in \mathrm{SU}\,(n-1)$ and α real. Clearly, **P**, the orthogonal complement of **K**, consists of all skew-hermitian matrices of the form

$$\mathbf{P}_n = \begin{pmatrix} & B_{n-1} \\ -B^+_{n-1} & \end{pmatrix}, \tag{8}$$

where B_{n-1} is a $(n-1)$ dimensional row vector. It follows that $[\mathbf{g}_0, \mathbf{g}] = i\pi\mathbf{p}$ and using the formula for $\Phi\,(\mathbf{g})$ above, we get

$$\Phi\,(\mathbf{g}) = \mathbf{g} + i\pi\mathbf{p} + \frac{(i\pi)^2}{2!}\mathbf{p} + \ldots$$

$$= \mathbf{g} - \mathbf{p} + \mathbf{p}\exp\,(i\pi) = \mathbf{g} - 2\mathbf{p}.$$

When $\mathbf{g} = \mathbf{k}$, $\mathbf{p} = 0$ and we have $\Phi\,(\mathbf{k}) = \mathbf{k}$; and when $\mathbf{g} = \mathbf{p}$, we have $\Phi\,(\mathbf{p}) = \mathbf{p} - 2\mathbf{p} = -\mathbf{p}$, thus verifying that what we have exhibited is indeed a cartan decomposition. Finally, exponentiating the right side of (8), P is itself the set of all matrices of the form

$$P_n = \begin{pmatrix} \cos\beta & B_{n-1}\sin\beta \\ -\beta\sin\beta\, B^+_{n-1} & 1_{n-1} + b_{n-1}(\cos\beta - 1)/\beta^2 \end{pmatrix},$$

where β is the non-negative real number $(B_{n-1}B^+_{n-1})^{1/2}$ and b_{n-1} is the matrix $B^+_{n-1}B_{n-1}$.

Having found the decomposition $G = KP \equiv K_1 P_1$, we have to split K_1 and P_1 further. K_1 can be further split by repeating our earlier procedure on the $\mathrm{SU}(n-1)$ submatrix. Choose

$$g_{0,n-1} = \begin{pmatrix} 1 & \\ & -1_{n-2} \end{pmatrix},$$

so that $k_2 \in K_2$ is of the form

$$k_2 = \begin{pmatrix} \exp\,(i\alpha) & & \\ & \exp\,[-i\alpha/(n-1)+i\beta] & \\ & & \exp\,[-i\alpha/(n-1)-i\beta/(n-2)]\,v_{n-2} \end{pmatrix},$$

where $v_{n-2} \in \mathrm{SU}\,(n-2)$, for the decomposition $K_1 = K_2 P_2$. Correspondingly, $p_2 \in P_2$ is obtained by exponentiating

$$\mathbf{p}_2 = \begin{pmatrix} 0 & & \\ & 0 & B_{n-2} \\ & -B^+_{n-2} & 0 \end{pmatrix}.$$

Further choices of g_0 are now obvious. The decomposition of P_i in the sequence is achieved by writing $p_i \in P_i$ in the form $k'_i\, a_i\, k'^{-1}_i$ for $a_i \in A_i$, the maximal abelian subgroup of P_i, and $k'_i \in K_i$.

The maximal abelian subgroup of P_1 (in fact, of any P_i) is easily seen to be one-dimensional. $\mathbf{A}_1$ is of the form

$$\mathbf{a}_1 = \begin{bmatrix} 0 & b & 0 & \cdots \\ -b & 0 & & \\ 0 & & & \\ \vdots & & & 0 \end{bmatrix}, \quad b \text{ real.}$$

On exponentiation, we have, then

$$a_1 = \begin{pmatrix} r_2(b) & \\ & 1_{n-2} \end{pmatrix},$$

where r_2 is the SO(2, R) matrix

$$r_2(b) = \begin{pmatrix} \cos b & \sin b \\ -\sin b & \cos b \end{pmatrix}.$$

The decomposition into one-parameter subgroups then goes in the following sequence.

$$\begin{aligned} g &= k_1 p_1, \\ &= k_2 p_2 k'_1 a_1 k'^{-1}_1, \\ &= k_2 k'_2 a_2 k'^{-1}_2 k'_1 a_1 k'^{-1}_1 \\ &= \tilde{k}_2 a_2 k'^{-1}_2 k'_1 a_1 k'^{-1}_1 && (\tilde{k}_2 = k_2 k'_2), \\ &= \tilde{k}_2 a_2 k'^{-1}_2 k'_1 a_1 k''_2\, a'_2 k''^{-1}_2 k'^{-1}_2 \\ &= \tilde{k}_2 a_2 k'^{-1}_2 k'_1 a_1 k''_2\, a'_2 \tilde{k}'^{-1}_2 && (\tilde{k}'_2 = k'_2 k''_2). \\ &= \text{etc.} \end{aligned}$$

It is clear that for SU(n), the subscripts on the $\tilde{k}$ at either extreme will be (n—1) (i.e., they will be products of n—1 factors) and that each k_{n-1} is a diagonal one-parameter matrix. We have thus the result we wanted; the extreme factors are elements of the maximal abelian subgroup of SU(n). The factors in the middle recombine to form the

'Cabibbo-Kobayashi-Maskawa' matrix. Concrete examples treated in the next section make matters even clearer.

3. Explicit construction for n = 2, 3

3.1 $n=2$

The trivial SU(2) example is already instructive: Write $g \in SU(2)$ as*

$$g = \begin{pmatrix} c(\theta)\exp(i\delta_1) & s(\theta)\exp(i\delta_2) \\ -s(\theta)\exp(-i\delta_2) & c(\theta)\exp(-i\delta_1) \end{pmatrix} \equiv v_2(\theta;\delta_1,\delta_2).$$

For our choice $g_0 = \text{diag}(1, -1)$, we have

$$k_1 = \text{diag}(\exp(i\delta_1), \exp(-i\delta_1));\ p_1 = v_2(\theta; 0, \delta'),\ (\delta' = \tfrac{1}{2}(\delta_2 - \delta_1)).$$

A_1 in this case is SO(2) and p_1 has the decomposition

$$p_1 = k_1' r_2(\theta) k_1'^{-1},\ k_1' = \text{diag}(\exp(i\delta'), \exp(-i\delta')).$$

Hence, the general U(2) matrix decomposes as

$$\exp(i\chi)\, v_2(\theta;\delta_1,\delta_2) = \text{diag}[\exp(i\chi + i\delta), \exp(i\chi - i\delta)]$$
$$\times\, r_2(\theta)\, \text{diag}[\exp(-i\delta'), \exp(i\delta')\ [\delta = \tfrac{1}{2}(\delta_1 + \delta_2)].$$

Thus, the three phase parameters χ, δ and δ' occur only in the extreme factors and can be 'absorbed' by redefining the individual quarks. Alternatively, the overall phase can be got rid of by redefining the $W^{\pm}$ field to which the current constructed by sandwiching the U(2) matrix couples and δ_1 and δ_2 by redefining the quarks. The essential point is that even though we have five non-hermitian fields, only three phases can be got rid of by redefining them, the reason being that the maximal abelian subgroup of SU(2) is one-dimensional—a point which is obscured in the usual discussions. A U(2) matrix has of course only three phase parameters and so there is no CP-violation.

3.2 $n = 3$

For $v_3 = g \in SU(3)$, g_0 is diag $(1, -1, -1)$, $k_1 \in K_1$ is of the form

$$k_1 = \begin{pmatrix} \exp(i\phi_1) & \\ & \exp(-i\phi_1/2)\, v_2 \end{pmatrix},\ v_2 \in SU(2),$$

*To avoid or at least to shorten, whenever feasible, the explicit writing down of big matrices, we follow from now on some additional notational abbreviations: $v_2(\theta;\delta_1,\delta_2)$ stands for the general SU(2) matrix parametrised as above ($v_2(\theta; 0, 0) = r_2(\theta)$, the rotation matrix), diag $(x, y, \ldots)$ for the diagonal matrix with diagonal entries $x, y, \ldots,$ and $c(\theta)$ and $s(\theta)$ for cos θ and sin θ.

and $p_1 \in P_1$ is generated by

$$\mathbf{p}_1 = \begin{pmatrix} 0 & b_1 & b_2 \\ -b_1^* & & \\ -b_2^* & & \end{pmatrix},$$

which is a 4-parameter matrix. A_1 is one-dimensional, in fact SO(2), and we choose the parametrisation

$$a_1 = \begin{pmatrix} r_2(\theta_1) & \\ & 1 \end{pmatrix}.$$

p_1 can then be written $p_1 = k_1' a_1 k_1'^{-1}$, $k_1' \in \mathrm{SU}(2) \subset K_1$, i.e.,

$$k_1' = \begin{pmatrix} 1 & \\ & v_2(\theta_3;\, a_1, a_2) \end{pmatrix}.$$

Therefore, $$v_3 = k_1 k_1' a_1 k_1'^{-1} = \begin{pmatrix} \exp(i\phi_1) & \\ & \exp(i\phi_1/2)\, v_2(\theta_2;\, \beta_1, \beta_2) \end{pmatrix}$$

$$\times \begin{pmatrix} r_2(\theta_1) & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ & v_2(-\theta_3;\, -a_1, a_2) \end{pmatrix}.$$

Now, each of the SU(2) matrices can be further factorised as in § 3.1 to give

$$v_3 = \mathrm{diag}\,[\exp(i\phi_1), \exp(-i\phi_1/2)]\ \mathrm{diag}\,[\exp(i\beta), \exp(-i\beta)]$$

$$\times \begin{pmatrix} 1 & \\ & r_2(\theta_2) \end{pmatrix} \mathrm{diag}\,[1, \exp(-i\beta'), \exp(i\beta')]$$

$$\times \begin{pmatrix} r_2(\theta_1) & \\ & 1 \end{pmatrix} \mathrm{diag}\,[1, \exp(i a), \exp(-i a)]$$

$$\times \begin{pmatrix} 1 & \\ & r_2(-\theta_3) \end{pmatrix} \mathrm{diag}\,[1, \exp(-i a'), \exp(i a')],$$

where $a = \frac{1}{2}(a_1 + a_2)$, $\beta = \frac{1}{2}(\beta_1 + \beta_2)$, $a' = \frac{1}{2}(a_1 - a_2)$, $\beta' = \frac{1}{2}(\beta_1 - \beta_2)$.

The eight parameters θ_1, θ_2, θ_3, α, β, α', β' and ϕ_1 characterise v_3 fully. We now reorder some of the factors in order to exhibit another one parameter matrix on the right extreme. This can be done in many ways: the one we choose is

$$v_3 = \text{diag}\,[\exp(i\phi_1),\ \exp(i\beta - i\phi_1/2 - i\beta'),\ \exp(-i\beta - i\phi_1/2 - i\beta')$$

$$\times \begin{pmatrix} 1 & \\ & r_2(\theta_2) \end{pmatrix} \begin{pmatrix} r_2(\theta_1) & \\ & 1 \end{pmatrix} \text{diag}\,[1, 1, \exp(i\delta)]$$

$$\times \begin{pmatrix} 1 & \\ & r_2(-\theta_3) \end{pmatrix} \text{diag}\,[1, \exp(i\alpha - i\alpha'), \exp(i\alpha + i\alpha')],$$

where $\delta = 2(\beta' - \alpha)$.

The product of the middle four matrices is the Kobayashi-Maskawa matrix which, in full glory, is

$$c_3 = \begin{bmatrix} c(\theta_1) & s(\theta_1)\,c(\theta_3) & -s(\theta_1)\,s(\theta_3) \\ -s(\theta_1)\,c(\theta_2) & c(\theta_1)\,c(\theta_2)\,c(\theta_3) + s(\theta_2)\,s(\theta_3)\exp(i\delta) & -c(\theta_1)\,c(\theta_2)\,s(\theta_3) + s(\theta_2)\,c(\theta_3)\exp(i\delta) \\ s(\theta_1)\,s(\theta_2) & -c(\theta_1)\,s(\theta_2)\,c(\theta_3) + c(\theta_2)\,s(\theta_3)\exp(i\delta) & c(\theta_1)\,s(\theta_2)\,s(\theta_3) + c(\theta_2)\,c(\theta_3)\exp(i\delta) \end{bmatrix}.$$

From the point of view of the general decomposition of § 2, this is not quite what we were after: neither the matrix c_3 nor the diagonal matrices to its left and right is a unimodular matrix, even though the product v_3 is (there are only eight parameters; δ is not independent). Once we choose to exhibit c_3 in the Kobayashi-Maskawa form, the phase matrices will, in general, not be unimodular. There are alternative ways of decomposition which will have a different δ-dependence and be unimodular. In any case, what is required is a decomposition of U(n) and not SU(n). But since the general theorems are applicable to SU(n) and not to U(n), we were forced to factorise the determinant out. It can be restored at the end without any difficulty as we have seen.

4. Conclusions

As we stated in the introduction, there are a number of discussions in the literature on the question dealt with in this paper, at least as far as the counting of absorbable phases (which in turn gives the number of CP-violating phases) is concerned. They can be summarised in one sentence: in general, the $2n$ quark fields on either side of the unitary matrix arising from the mass diagonalisation can all have their phases redefined *except for one overall phase*, giving $2n-1$ absorbable phases. This argument is fallacious for a number of reasons. Firstly, as we saw in the introduction, it only gives the maximum number of absorbable phases and the minimum number of

CP-phases. More importantly, the phrase italicised above is misleading because if the issue was settled directly by the number of (non-hermitian) fields available, there would be $2n+1$ absorbable phases: one overall (goes into $W^{\pm}$) and $2n$ for the quarks, which is obviously absurd. The point is that the overall phase is not 'except' but 'in addition', and the correct counting is not $(2n)$—1 but $1+2(n-1)$, 1 overall and $(n-1)$ for the rank (the dimension of the maximal abelian subgroup) of SU(n).

In any case, the constructive procedure we have described for $n=3$ can be carried through for larger n in exactly the same way.

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Appendix

If $g_0 \in G$, the inner automorphism $Ad_{g_0}(g): g \to g_0 g g_0^{-1}$ is an analytic isomorphism of G onto itself. We write $\mathbf{Ad}_{g_0}$ for the differential of this map near the identity, which is an automorphism on the tangent space (i.e. $\mathbf{G}$, the Lie algebra of G) such that $\exp(\mathbf{Ad}_{g_0}(\mathbf{g})) = g_0 \exp \mathbf{g}\, g_0^{-1}$. The set of automorphisms $\mathbf{Ad}_G$ form a group GL (dim $\mathbf{G}$) and the map $g_0 \to \mathbf{Ad}_{g_0}$ is a group homomorphism of G into GL (dim $\mathbf{G}$). The differential of this map near the identity, a homomorphism of G into GL (dim $\mathbf{G}$), is written as $\mathbf{ad}_{\mathbf{g}_0}$ and is given by $\mathbf{ad}_{\mathbf{g}_0}(g): \mathbf{g} \to [\mathbf{g}_0, \mathbf{g}]$ (Helgason 1962).

Consider the automorphism given by $\phi_{g_0}(g) = g_0 g g_0^{-1}$ where $g_0^2 = I$ and $g \in G$ (which is of the kind used in the text). For matrix groups (which is what concerns us in this paper) we will show that this defines $\Phi_{g_0}(\mathbf{g}) = g_0 \mathbf{g} g_0^{-1}$ with $\mathbf{g} \in \mathbf{G}$.

Writing $g(t) = \exp(t\mathbf{g})$ we have

$$\Phi_{g_0}(g(t)) = g_0 \exp(t\mathbf{g})\, g_0^{-1}$$
$$= g_0 \left[1 + t\mathbf{g} + \frac{t^2}{2!}\mathbf{g}^2 + \ldots\right] g_0^{-1}.$$

(For the case of matrix groups the product $g_0\, \mathbf{g}$ is defined). The differential map of this close to $t=0$ will define $\phi_{g_0}(\mathbf{g})$ on the Lie algebra. This is easily seen to be given by

$$\Phi_{g_0}(\mathbf{g}) = g_0\, \mathbf{g} g_0^{-1},$$

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