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On injective tensor products whose duals are isometric to $L^1(\mu)$

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ON INJECTIVE TENSOR PRODUCTS WHOSE DUALS ARE ISOMETRIC TO $L^1(\mu)$

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Abstract. Following the well-known classification scheme of function spaces whose duals are isometric to $L^1(\mu)$, due to Lindenstrauss, Wulbert and Olsen ([5],[7]), in this paper we study the geometric properties of Banach spaces under the assumption that the injective tensor product of them is in one of the classes described by Lindenstrauss, Wulbert and Olsen.

1. Introduction

Let E be a complex Banach space such that E^* is isometric to $L^1(\mu)$ for some positive measure μ. Such spaces are called L^1-preduals or Lindenstrauss spaces. Study of their structure and classification attracted a lot of attention during the 70'. Lindenstrauss and Wulbert ([5]) gave a classification scheme for characterizing several known classes of function spaces among the preduals of L^1. These results were extended to complex Banach spaces by Olsen ([7]). See the monograph [4] for more details. It was shown in [8] that if E, F are L^1-preduals, then so is the injective tensor product space $E \otimes_\epsilon F$.

For a Banach space E by $E_1, S(E), \partial_e E_1$ we denote the closed unit ball, the unit sphere and the set of extreme points of the unit ball respectively. In this paper we are interested in considering spaces E, F such that $E \otimes_\epsilon F$ is in one of the classes in the classification scheme and decide whether E and F are also in the same class. We note that since being an L^1-predual is preserved by ranges of projections of norm one (see L Chapter 6 and [7] Corollary 5) and since E, F are isometric to the range of a projection of norm one in $E \otimes_\epsilon F$, we have that E and F are L^1-preduals. We will be using the identification of $\partial_e(E \otimes_\epsilon F)_1^*$ as vectors of the form $e^* \otimes f^*$ where

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$e^* \in \partial_e E_1^*$ and $f^* \in \partial_e F_1^*$ (see [10],[6]). Many of the classification results from [5] and [7] are based on the properties of the extreme points of the unit ball or dual unit ball. By using the description of the extreme points of the dual unit ball of injective tensor product spaces, it is not difficult to show that if $E \otimes_\epsilon F$ is a G-space, C_Σ-space or a C_σ space then so are the component spaces. For a compact set X if $E \otimes_\epsilon F$ is isometric to $C(X)$, it follows from the arguments given during the proof of Theorem 2 in [9] that E, F are isometric to $C(M)$ and $C(N)$ respectively for some compact sets M, N.

Let X be a locally compact space then $C_0(X)$, the space of continuous functions vanishing at infinity, is an L^1-predual space. This is the only function space in the classification scheme whose characterization is not wholly based on extremal structure of the dual unit ball or the unit ball. It has been classified as (see [7] section 6) a complex simplex space, i.e., there exists a maximal face $F \subset S(C_0(X)_1^*)$ of the dual unit ball , such that the convex hull $CO(F \cup \{0\})$ is weak*-closed and $\partial_e C_0(X)_1^* \cup \{0\}$ is a weak*-closed set. It can also be seen that when this happens for any maximal face F of the unit sphere, $CO(F \cup \{0\})$ is weak*-closed.

Let E, F be Banach spaces so that $E \otimes_\epsilon F$ is isometric to $C_0(X)$ for a locally compact space X. As already noted this implies that E, F are L^1-preduals. We do not know if our assumption implies that E and F are also C_0-spaces.

2. Main result

We give a positive solution to the problem posed above, when the dual of one component space is discrete.

Theorem 1. *Let E, F be Banach spaces such that $E^* = \ell^1$. Suppose $E \otimes_\epsilon F$ is isometric to $C_0(X)$ for a locally compact and non-compact set X. Then E is isometric to $C_0(L)$ for a dispersed locally compact metric space L and F is isometric to $C_0(N)$ for a locally compact space N.*

Proof. We will first show that E is isometric to $C_0(L)$. Since $E \otimes_\epsilon F$ is a $C_0(X)$ space, in view of the results from [7] we have that $\partial_e (E \otimes_\epsilon F)_1^* \cup \{0\}$ is a weak*-closed set and for any maximal face $G \subset S((E \otimes_\epsilon F)^*)$, $CO(G \cup \{0\})$ is weak*-closed.

Let $\{e^*_\alpha\} \subset \partial_e E^*_1$ be a net such that $e^*_\alpha \to e^*$ in the weak*-topology. Fix $f^* \in \partial_e F^*_1$. It is easy to see that $e^*_\alpha \otimes f^* \to e^* \otimes f^*$ in the weak*-topology. Thus $e^* \otimes f^* \in \partial_e (E \otimes_\epsilon F)^*_1$ or $e^* \otimes f^* = 0$. Hence we have that $\partial_e E^*_1 \cup \{0\}$ is weak*-closed.

We note that any L^1-predual space has the metric approximation property (see [4] Chapter 6). Since E^* has the Radon-Nikodym property, it follows from [2] Chapter VIII that $(E \otimes_\epsilon F)^* = E^* \otimes_\pi F^* = \ell^1 \otimes_\pi L^1(\mu)$, for some positive measure μ on a measurable space $(\Omega, \mathcal{A})$, with $F^* = L^1(\mu)$. We assume without loss of generality that $(\Omega, \mathcal{A}, \mu)$ is a complete measure space.

Next let $G \subset S(E^*)$ be a maximal face. We use the identification $(E \otimes_\epsilon F)^* = \ell^1 \otimes_\pi L^1(\mu) = L^1(\nu \times \mu)$ where ν is the counting measure on $\mathbf{N}$. To show that $CO(G \cup \{0\})$ is weak*-closed, we use the description of maximal faces of the surface of the unit ball of $L^1(\lambda)$ given on page 247 of [7]. Thus $G = \{x\phi : x \geq 0, \ \|x\| = 1\}$ where $\phi \in \ell^\infty$ with $|\phi| = 1$. Define $\phi' : \mathbf{N} \times \Omega \to \mathbf{C}$ by $\phi'((n, \omega)) = \phi(n)$. Then $\phi' \in L^\infty(\nu \times \mu)$ and $|\phi'| = 1$ so that $G' = \{f\phi' : f \geq 0, \ \|f\| = 1\}$ is a maximal face. Hence by the hypothesis again, $CO(G' \cup \{0\})$ is weak*-closed. Let $\lambda_\alpha g_\alpha \subset CO(G \cup \{0\})$ be a net such that $\lambda_\alpha g_\alpha \to g \neq 0$,where $g_\alpha \in G$ and $\lambda_\alpha \in [0, 1]$. Let $g_\alpha = x_\alpha \phi$. Fix a $f_0 \in L^1(\mu)$ with $f \geq 0$ and $\|f_0\| = 1$. We note that $x_\alpha \phi \otimes f_0 = (x_\alpha \otimes f_o)\phi'$. As the nets involved are norm bounded, it is easy to see that $\lambda_\alpha x_\alpha \phi \otimes f_0 \to g \otimes f_0$ in the weak*-topology (convergence need to be checked only at the elements of the dense set $E \otimes F$). Clearly $x_\alpha \phi \otimes f_0 = (x_\alpha \otimes f_o)\phi' \in G'$. Therefore $g \otimes f_0 = \lambda \phi' h$ for some $h \geq 0$ and $\int h d(\nu \times \mu) = 1$ and $\lambda \in (0, 1]$. Let $x_0(n) = \int h(n, \omega) d\mu(\omega)$. Then by Fubini' theorem we get that $g = \lambda x_0 \phi$. Hence $CO(G \cup \{0\})$ is weak*-closed.

Therefore by the remarks in section 6 of [7] we get that E is isometric to $C_0(L)$ for a locally compact metric space L (since E is separable). Since $E^* = \ell^1$, $C_0(L)^*$ does not have any non-atomic measures. Thus L is a dispersed metric space.

We now have, $E \otimes_\epsilon F = C_0(L) \otimes_\epsilon F = C_0(L, F)$. Since L is dispersed it has an isolated point, say l_0. Next note that $f \to \chi_{l_0} f$ is a projection in $C_0(L, F)$ such that $C_0(L, F) = F \oplus_\infty F'$ (ℓ^∞-direct sum) for some closed subspace F'. Now since $C(L, F)$ is a $C_0(.)$-space, it follows from Example 1.1.4(a) of [3] which describes ℓ^∞-summands in $C_0(.)$-spaces, that F is isometric to a $C_0(N)$ for a locally compact space N. $\square$

Remark 2. *It is clear from the above proof that same arguments work when $E^* = \ell^1(\Gamma)$ for an uncountable discrete set Γ. In general if one takes $E = C_0(K)$ for a locally compact set K, then it is well known that for any Banach space F, $E \otimes_\epsilon F = C_0(K, F)$, the space of continuous F-valued functions vanishing at infinity. However we do not know how maximal faces of the unit sphere of $C_0(K, F)^*$ look like? When K is infinite and F is infinite dimensional, it follows from* [1] *that if the identification $(C_0(K) \otimes_\epsilon F)^* = C_0(K)^* \otimes_\pi F^*$ holds then either K is dispersed or F^* has the Radon-Nikodym property. If F is an L^1-predual, this implies that $F^* = \ell^1(\Gamma)$ for a discrete set Γ. These are the two cases considered above.*

References

[1] M. Cambern and P. Greim, *The dual of a space of vector measures*, Math. Z. 180 (1982) 373–378.

[2] J. Diestel and J. J. Uhl, *Vector measures. With a foreword by B. J. Pettis*, Mathematical Surveys, No. 15. American Mathematical Society, Providence, R.I., 1977. xiii+322 pp.

[3] P. Harmand, D. Werner and W. Werner, *M-ideals in Banach spaces and Banach algebras*, Lecture Notes in Mathematics, 1547. Springer-Verlag, Berlin, 1993. viii+387 pp

[4] H. E. Lacey, *The isometric theory of classical Banach spaces*, Die Grundlehren der mathematischen Wissenschaften, Band 208. Springer-Verlag, New York-Heidelberg, 1974. x+270 pp.

[5] J. Lindenstrauss and D. E. Wulbert, *On the classification of the Banach spaces whose duals are L_1 spaces*, J. Functional Analysis 4 (1969) 332–349.

[6] A. Lima and G. H. Olsen, *Extreme points in duals of complex operator spaces*, Proc. Amer. Math. Soc. 94 (1985) 437–440.

[7] G. H. Olsen, *On the classification of complex Lindenstrauss spaces*, Math. Scand. 35 (1974) 237–258.

[8] T. S. S. R. K. Rao, A. K. Roy and K. Sundaresan, *Intersection properties of balls in tensor products of some Banach spaces*, Math. Scand. 65 (1989) 103–118.

[9] T. S. S. R. K. Rao, *A note on a paper of Ercan and Önal*, preprint, 2008.

[10] W. Ruess , *Duality and geometry of spaces of compact operators*, Functional analysis: surveys and recent results, III (Paderborn, 1983) 59–78, North-Holland Math. Stud., 90, North-Holland, Amsterdam, 1984.

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