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On simultaneously remotal sets in spaces of vector-valued functions

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ON SIMULTANEOUSLY REMOTAL SETS IN SPACES OF VECTOR-VALUED FUNCTIONS

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ABSTRACT. In this paper we formulate the notions of simultaneously remotal and that of simultaneously densely remotal sets. We exhibit large classes of Banach spaces which have subspaces, whose unit ball is a simultaneously remotal set. We also study them in spaces of vectorvalued function spaces.

1. INTRODUCTION

Let X be a real Banach space and let $G \subset X$ be a bounded set. We recall that G is said to be remotal in X, if for every $x \in X$, there exists a $g_0 \in G$ (possibly depending on x) such that $sup_{g\in G}||x - g|| = ||x - g_0||$. g_0 is called a farthest point from x. $F_x(G) = \{h \in G : ||h - x|| = sup_{g\in G}||g - x||\}$ is called the set of farthest points. The study of remotal sets has attracted a lot of attention recently. See [7], [6], [2] and [4]. It was shown by Martin and Rao that every infinite dimensional Banach space has a convex, closed and bounded non-remotal set. G is said to be densely remotal if $\{x : F_G(x) \neq \emptyset\}$ is dense in X. It was shown by Lau [3] that any weakly compact set is densely remotal.

Analogous to the notion of simultaneous best approximation studied in the literature, in this paper we are interested in formulating and studying the notion of simultaneously remotal and densely simultaneously remotal sets.

Definition 1. A closed and bounded set $G \subset X$ is said to be simultaneously remotal, if for every positive integer m, and for all $x_1, x_2, ..., x_m \in X$, there

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exists a $g_0 \in G$ such that

 $max_{1 \le i \le m} \ sup_{q \in G} \ (\|x_i - g\|) = max_{1 \le i \le m} (\|x_i - g_0\|).$

Clearly when m = 1, this coincides with the usual definition of a remotal set.

See [5] and [1] for some other forms of defining the notion of simultaneously remotal. We show that for any Banach space Y, its unit ball Y_1 is simultaneously remotal in Y.

Remark 2. Note that $\max_{1 \leq i \leq m} \sup_{g \in G} (||x_i - g||) = \sup_{g \in G} \max_{1 \leq i \leq m} ((||x_i - g||)$. Now it is easy to see that a bounded set G is simultaneously remotal if and only if the diagonal set $\Delta(G) = \{(g, g, ...g) : g \in G\}$ is remotal in $\otimes^m X$ (product of m-copies of X, equipped with the maximum norm), for all m. With the above formulation, if G is a compact set, since $\Delta(G)$ is compact, it is remotal in $\otimes^m X$, for all m.

This observation gives a way of defining the notion of simultaneously densely remotal.

Definition 3. A closed and bounded set $G \subset X$ is said to be simultaneously densely remotal, if $\Delta(G)$ is densely remotal in $\otimes^m X$ for all m.

Remark 4. Thus if G is a weakly compact set then since $\Delta(G)$ is also weakly compact in $\otimes^m X$, by a result of Lau, [3] G, is simultaneously densely remotal. We show in the next section that the unit ball of a factor reflexive, proximinal subspace of a c₀-direct sum of reflexive Banach spaces is, simultaneously densely remotal.

Remark 5. The authors of [1] also tried to define the notion of simultaneously densely remotal set. However their definition of $F_G(x)$ does not involve x and is thus flawed. This effects several of the results in [1].

In Section 2 we give several examples of sets that are simultaneously remotal and simultaneously densely remotal. We show that for a subspace $Y \subset X$, whose unit ball is a remotal set and which satisfies $sup_{y \in Y_1}(||x - y||) = 1 + ||x||$ for all $x \in X$, Y_1 is a simultaneously densely remotal. In [2], the authors have exhibited large classes of spaces that satisfy the norm condition assumed here, see Proposition 2.4 and Example 2.3 in [2]. We show that the unit ball of a proximinal and factor reflexive subspace of a c_0 -direct sum of reflexive Banach spaces is, simultaneously densely remotal.

In Section 3 we give a general formulation for sets in function spaces to be simultaneously remotal. In the last section we show that for a σ finite measure space $(\Omega, \mathcal{A}, \mu)$ and for a separable simultaneously remotal set $G \subset X$, the space of essentially bounded strongly measurable functions, $L^{\infty}(\mu, G)$ is simultaneously remotal in $L^{\infty}(\mu, X)$.

2. Examples

Let $Y \subset X$ be a closed subspace. Let Y_1 denote the closed unit ball. It is easy to see that for $0 \neq y_0 \in Y$,

$$sup_{y \in Y_1}(||y_0 - y||) = 1 + ||y_0|| = ||y_0 - (-\frac{y_0}{||y_0||})||$$

Thus Y_1 is always remotal in Y. In the following Proposition we note that Y_1 is also simultaneously remotal in Y.

Proposition 6. For any Banach space Y, Y_1 is simultaneously remotal in Y.

Proof. Let $0 \neq (y_1, ..., y_m) \in \otimes^m Y$. In the following estimates, instead of taking supremum over $\Delta(Y_1)$, we take supremum over Y_1 . Clearly,

$$\sup_{y \in Y_1} \max_{1 \le i \le m} (\|y_i - y\|) = \max_{1 \le i \le m} (1 + \|y_i\|).$$

Suppose $max_{1 \le i \le m}(1 + ||y_i||) = 1 + ||y_{i_o}||$. Then clearly $y_{i_0} \ne 0$. Consider the vector $y^{\wedge} = (\frac{-y_{i_0}}{||y_{i_0}||}, ..., \frac{-y_{i_0}}{||y_{i_0}||}) \in \Delta(Y_1)$. We claim that $max_{1 \le i \le m}(1 + ||y_i||) = 1 + ||y_{i_0}|| = max_{1 \le i \le m}(||y_i + \frac{y_{i_0}}{||y_{i_0}||}||)$. Clearly,

$$max_{1 \le i \le m}(1 + ||y_i||) = 1 + ||y_{i_0}|| \le max_{1 \le i \le m}(||y_i + \frac{y_{i_0}}{||y_{i_0}||}|).$$

On the other hand suppose this latter maximum is attained at $||y_j + \frac{y_{i_0}}{||y_{i_0}||}||$ for some *j*. Now $||y_j + \frac{y_{i_0}}{||y_{i_0}||}|| \le \frac{|||y_{i_0}||y_{j+y_{i_0}}||}{||y_{i_0}||} \le 1 + ||y_j|| \le 1 + ||y_{i_0}||$. Thus y^{\wedge} is a farthest point.

This completes the proof.

Remark 7. Suppose $X = Y \oplus Z$ and there exists a monotone map φ : $\mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ such that if x = y + z, then $||x|| = \varphi(||y||, ||z||)$. Let $E \subset Y$ and $F \subset Z$ be simultaneously remotal sets, then it follows from arguments similar to those given during the proof of Lemma 3.1 in [2] that E + F is simultaneously remotal in X. In particular, Y_1 and Z_1 are simultaneously remotal in X. It is also easy to see that if φ is the maximum function, then if E, F are simultaneously densely remotal then so is E + F.

Our next result is similar to Theorem 3.7 in [2]. We only briefly indicate its proof. For any family $\{X_j\}_{j\in J}$ of reflexive Banach spaces, let $X = \bigotimes_{c_0} X_j$ denote the c_0 -direct sum, equipped with the supremum norm. We recall that a subspace $Y \subset X$ is proximinal if there exists a $y \in Y$ such that d(x, Y) = ||x - y|| for every $x \in X$ and is said to be factor reflexive if the quotient space X/Y is reflexive.

Proposition 8. Let $\{X_j\}_{i \in J}$ be any family of reflexive spaces. Let $X = \bigotimes_{c_0} X_j$. For any factor reflexive proximinal subspace $Y \subseteq X$, Y_1 is simultaneously densely remotal in X.

Proof. As Y is proximinal and factor reflexive, it follows from the proof of Theorem 3.7 in [2] that $Y = (Y \cap X') \otimes_{\infty} X''$, where $X = X' \otimes_{\infty} X''$, is such that X' is reflexive. As $(Y \cap X')_1$ being a weakly compact set, is simultaneously densely remotal, it follows from our remarks above that Y_1 is simultaneously densely remotal in X.

A class of subspaces Y for which remotality of the unit ball was studied in [2] are those spaces Y for which Y_1 satisfies the formula

$$sup_{y \in Y_1}(||x - y||) = 1 + ||x||$$

for all $x \in X$. See Example 2.3 of [2] for several examples of this kind. In particular for any Banach space X, X_1 satisfies the formula, in the canonical embedding, in the bidual, X^{**} . The following proposition is easy to prove from arguments similar to those given during the proof of Proposition 6.

Proposition 9. Let $Y \subset X$ be a closed subspace such that $\sup_{y \in Y_1} (||x - y||) = 1 + ||x||$ for all $x \in X$. Then for any $(x_1, ..., x_m) \in \otimes^m X$,

$$sup_{\|y\|\leq 1} \max_{1\leq i\leq m}(\|x_i - y\|) = \max_{1\leq i\leq m}(1 + \|x_i\|)$$
$$= 1 + \max_{1\leq i\leq m}(\|x_i\|) = sup_{\|y_i\|\leq 1} \max_{1\leq i\leq m}(\|x_i - y_i\|)$$

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Thus both the sets $\Delta(Y)$ and $\otimes^m Y_1$ satisfy the above formula in the space $\otimes^m X$.

Theorem 10. Let $Y \subset X$ be a closed subspace such that Y_1 is a remotal set and $\sup_{y \in Y_1}(||x - y||) = 1 + ||x||$ for all $x \in X$. Then Y_1 is simultaneously remotal.

Proof. Let $x_1, ..., x_m \in X$. Since Y_1 is remotal, let $y_i^0 \in Y_1$ be such that $\sup_{y \in Y_1}(||x_i - y||) = 1 + ||x_i|| = ||x_i - y_i^0||$. Note that

 $max_{1 \le i \le m}(1 + \|x_i\|) = 1 + max_{1 \le i \le m}(\|x_i\|) = max_{1 \le i \le m} \sup_{\{\|y_i\| \le 1, 1 \le i \le m\}}(\|x_i - y_i\|)$

$$= \sup_{\{||y_i|| \le 1, 1 \le i \le m\}} \max_{1 \le i \le m} (||x_i - y_i||) = \max_{1 \le i \le m} (||x_i - y_i^0||).$$

Thus $\otimes^m Y_1$ is a remotal set in $\otimes^m X$. We next show that $\Delta(Y_1)$ is a remotal set using the above proposition.

We recall that $(\otimes^m X)^* = \otimes^m X^*$, product of m copies of X^* with the norm, $||(x_1^*, ..., x_m^*)|| = \sum_1^m ||x_i^*||$. It is well known that the extreme points of the unit ball of this space are of the form $(0, ..., x_i^*, 0, ..0)$ for some extreme point of X_1^* . We also recall that any non-zero vector attains its norm at an extreme point of the dual unit ball. Let Λ be an extreme point such that

 $max_{1 \le i \le m}(\|x_i - y_i^0\|) = \|(x_1 - y_1, ..., x_m - y_m\|) = \Lambda((x_1 - y_1, ..., x_m - y_m)).$

Now suppose $\Lambda = (0, ..., x_j^*, 0, ..., 0)$ where the extreme point $x_j^* \in X_1^*$ is at the *j*th place. Thus

$$1 + \max_{1 \le i \le m} (\|x_i\|) = \max_{1 \le i \le m} (\|x_i - y_i^0\|) = x_j^* (x_j - y_j^0) = x_j^* (x_j) + x_j^* (-y_j^0),$$
$$\le \|x_j\| + 1,$$

since $y_j^0 \in Y_1$. Therefore $x_j^*(x_j) = ||x_j|| = \max_{1 \le i \le m} (||x_i||)$ and $x_j^*(-y_j^0) = 1$.

We now show that $(y_i^0, y_i^0, ..., y_i^0) \in \Delta(Y_1)$ is the farthest point.

 $\max_{1 \le i \le m} (\|x_i - y_j^0\|) \ge \|x_j - y_j^0\| \ge x_j^* (x_j - y_j^0) = 1 + \max_{1 \le i \le m} (\|x_i\|).$

This completes the proof.

3. General setup

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Let Ω be a set and let $B(\Omega, X)$ be the set of bounded X-valued functions with the supremum norm. Let $G \subset X$ be a closed and bounded set. Let $\mathcal{F} \subset B(\Omega, X)$ be a collection of functions taking values in G and containing all G-valued constant functions and is also a closed set.

Then it is easy to see that for any $f_1, ..., f_n \in B(\Omega, X)$,

 $sup_{w\in\Omega} \ sup_{h\in G} \ max_{1\leq i\leq n}(\|f_i(w) - h\|) = sup_{F\in\mathcal{F}} \ max_{1\leq i\leq n}(\|f_i - F\|).$

Now suppose G is simultaneously remotal in X, then define a choice function $H : \Omega \to G$ by, for $w \in \Omega$, $\sup_{g \in G} \max_{1 \leq i \leq n} (\|f_i(w) - g\|) = \max_{1 \leq i \leq n} \|f_i(w) - H(w)\|$. Clearly H is a bounded G-valued function.

It follows from the above estimate,

$$\sup_{F \in \mathcal{F}} \max_{1 \le i \le n} (\|f_i - F\|) = \max_{1 \le i \le n} (\|f_i - H\|).$$

Thus if $H \in \mathcal{F}$, we get that \mathcal{F} is simultaneously remotal at $f_1, ..., f_n$.

Remark 11. Taking Ω as the set of positive integers, it is now easy to see that, for a remotal set $G \subset X$, the ℓ^{∞} -direct sum $\bigoplus_{\infty} G$, with the supremum norm, is simultaneously remotal in $\bigoplus_{\infty} X$.

4. Main Results

In what follows we use this idea to get simultaneously remotal sets in spaces of vector-valued functions. The considerations below are similar to the abstract set-up described above.

Let $G \subset X$ be a closed and bounded sets. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space, where μ is a σ -finite positive measure. Since we have already covered the discrete case in Remark 11, we may assume that μ is a nonatomic measure. Let $L^{\infty}(\Omega, X)$ denote the space of equivalence classes of X-valued strongly measurable, essentially bounded functions equipped with the essential supremum norm. Consider the closed and bounded set $L^{\infty}(\Omega, G) \subset L^{\infty}(\Omega, X)$.

The authors of [1] make an attempt to study this problem for $L^{\infty}(I, X)$, where I is the unit interval with Lebesgue measure and claim that $L^{\infty}(I, G)$ is simultaneously densely remotal in $L^{\infty}(I, X)$ ' (Theorem 3). But as their ON SIMULTANEOUSLY REMOTAL SETS IN SPACES OF VECTOR-VALUED FUNCTIONS

definition of simultaneously densely remotal is flawed, the arguments do not make sense.

For $f_1, ..., f_n \in L^{\infty}(\Omega, X)$, to decide the simultaneous remotality of $L^{\infty}(\Omega, G)$ at this set, note that since supremum over a set can be achieved by taking limits of sequences from the set, there are only countably many functions involved in the above estimate. Thus after discarding a common null set, the following theorem is easy to deduce (for the essential supremum norm) from arguments similar to the point-wise estimates given above. Also note that since only sequences of measurable functions are involved, point-wise supremum (maximum) functions are measurable and give raise to essentially bounded functions. The following theorem is now easy to prove.

Theorem 12. Let $f_1, ..., f_n \in L^{\infty}(\Omega, X)$. Suppose there exists a $H \in L^{\infty}(\Omega, G)$ such $\sup_{F \in L^{\infty}(\Omega, G)} \max_{1 \leq i \leq n} (\|f_i(t) - F(t)\|) = \max_{1 \leq i \leq n} (\|f_i(t) - H(t)\|)$ a.e. Then $L^{\infty}(I, G)$ is simultaneously remotal at $f_1, ..., f_n$.

Theorem 13. Let X be a Banach space and let G be a separable, simultaneously remotal set. Then $L^{\infty}(\Omega, G)$ is simultaneously remotal in $L^{\infty}(\Omega, X)$.

Proof. Let $f_1, ..., f_n \in L^{\infty}(\Omega, X)$. We need to show that the choice function H in the discussion in Section 3 can be chosen to be measurable. Then we get that $H \in L^{\infty}(\Omega, G)$ and thus by Theorem 12, $L^{\infty}(\Omega, G)$ is simultaneously remotal at $f_1, ..., f_n$.

As before w. l. o. g we assume that discarding a null set, functions involved in the discussion are point-wise defined. We also assume that the σ -field is countably generated.

Define a set-valued function, $\mathcal{H}: \Omega \to 2^G$ by

$$\mathcal{H}(t) = \{g^{\wedge} \in G : \max_{1 \le i \le n} (\|f_i(t) - g^{\wedge}\|) = \sup_{g \in G} \max_{1 \le i \le n} (\|f_i(t) - g\|)\}.$$

for $t \in \Omega$. Since G is simultaneously remotal, $\mathcal{H}(t) \neq \emptyset$ for all t. In order to apply von-Neumann measurable selection theorem, we need to show that the graph of \mathcal{H} , $\{(t, g^{\wedge}) : g^{\wedge} \in \mathcal{H}(t)\}$ is a measurable set in the product space. Since G is separable, let $\{g_n\}_{n\geq 1} \subset G$ be a dense sequence. Thus $sup_{g\in G} \max_{1\leq i\leq n} (||f_i(t) - g|| = sup_{n\geq 1} \max_{1\leq i\leq n} (||f_i(t) - g_n||)$ for any t. Therefore the graph is a measurable set. Thus by Von-Neumann' theorem RAO

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([8] Corollary 5.5.8) there is a measurable function $H: I \to G$ such that $H(t) \in \mathcal{H}(t)$ for all t. This completes the proof.

Corollary 14. Let $G \subset X$ be a compact set. Then $L^{\infty}(\Omega, G)$ is simultaneously remotal in $L^{\infty}(\Omega, X)$.

Remark 15. It may be noted that since remotal sets are in particular simultaneously remotal, the arguments above show that $L^{\infty}(\Omega, G)$ is remotal in $L^{\infty}(\Omega, X)$, for a separable remotal set G. We also note that in Theorem 7 of [1], the authors assume that G is a simultaneously remotal set with span(G) finite dimensional subspace of X. However as G is closed and bounded, this implies that G is compact. Thus the above corollary gives correct formulation and proof of Theorem 7 that $L^{\infty}(\Omega, G)$ is simultaneously remotal in $L^{\infty}(\Omega, X)$.

We next consider these questions in the space $C(\Omega, X)$, the space of Xvalued continuous functions defined on totally disconnected Hausdorff spaces Ω , equipped with the supremum norm. Our result shows that for any weakly compact set $G, C(\Omega, G)$ is simultaneously densely remotal in $C(\Omega, X)$.

Theorem 16. Let Ω be a totally disconnected compact Hausdorff space. Let $G \subset X$ be a simultaneously densely remotal set. Then $C(\Omega, G)$ is simultaneously densely remotal in $C(\Omega, X)$.

Proof. For a fixed m, we need to show that $\Delta(C(\Omega, G))$ is densely remotal in $\otimes^m C(\Omega, X)$. It is easy to see that the canonical map $(f_1, ..., f_m) \rightarrow$ $(f_1(w), ..., f_m(w))$ for $w \in \Omega$, is an isometry between $\otimes^m C(\Omega, X)$ and $C(\Omega, \otimes^m X)$. Thus it is enough to show that $C(\Omega, \Delta(G))$ is densely remotal in $C(\Omega, \otimes^m X)$. Since by hypothesis, $\Delta(G)$ is densely remotal in $\otimes^m X$, it follows from Thereom 3.11 in [2] that $C(\Omega, \Delta(G))$ is densely remotal in $C(\Omega, \otimes^m X)$.

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