Spaces of Operators as Continuous Function Spaces

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1. Introduction

Let $X$ and $Y$ be Banach spaces. Let $\mathcal{L}(X, Y)$ and $\mathcal{K}(X, Y)$ denote spaces of bounded and compact linear operators respectively. Let $X_1$ denote the closed unit ball of $X$ and $\partial X_1$ the set of extreme points of $X_1$. For any $x^{**} \in X^{**}$, $y^* \in Y^*$ let $x^{**} \otimes y^*$ denote the functional defined on $\mathcal{L}(X, Y)(\mathcal{K}(X, Y))$ by $(x^{**} \otimes y^*)(T) = x^{**}(T^*(y^*))$. Let $\Omega = X^{**}_1 \times Y^*_1$. If one equips the component spaces with the $w^*$-topology, $\Omega$ is a compact space in the product topology. A well known and useful isometric embedding of $\mathcal{K}(X, Y)$ into $C(\Omega)$ is given by the mapping $\phi: \mathcal{K}(X, Y) \to C(\Omega)$ defined by $\phi(T)(x^{**}, y^*) = (x^{**} \otimes y^*)(T)$ (see [4]). It is easy to see that $\phi$ is not onto $\phi(T)(0, y^*) = 0 = \phi(T)(x^{**}, 0)$ for any $T$. This raises the question, is it possible to choose smaller compact subsets of the respective dual balls so that this correspondence is onto? It is easy to see that if $X^* = C(\Omega_1)$ and $Y = C(\Omega_2)$ (isometrically) for some compact sets $\Omega_1, \Omega_2$ then $\Omega_1 \times \Omega_2$ is a natural choice that makes the above embedding onto.

In this short note we show that if $\mathcal{K}(X, Y)$ is isometric to $C(\Omega)$ for a compact space $\Omega$ then $X^*$ is isometric to $C(\Omega_1)$ and $Y$ is isometric to $C(\Omega_2)$ for some compact sets $\Omega_1$ and $\Omega_2$. In the case of bounded operators we show that if $\partial X_1$ is non-empty, then that $\mathcal{L}(X, Y)$ is isometric to a $C(\Omega)$ will imply that $X$ is isometric to an abstract $L^1$-space and $Y$ is isometric to a $C(\Omega_1)$.

Our methods involve the $L^1$ predual theory, for which we shall refer to [2]. We also follow the notation of Lacey’s book. Our results are valid for both real and complex scalar fields.
2. Main result

Theorem. Let $X$ and $Y$ be Banach spaces. Suppose $\mathcal{K}(X,Y)$ is isometric to $C(\Omega)$ for some compact set $\Omega$. Then there are compact sets $\Omega_1$ and $\Omega_2$ such that $\Omega$ is homeomorphic to $\Omega_1 \times \Omega_2$ and $X^*$ is isometric to $C(\Omega_1)$ and $Y$ is isometric to $C(\Omega_2)$.

Proof. Fix $x_0 \in X_1$, $y_0 \in Y_1$, and $x_0^* \in X_1^*$, $y_0^* \in Y_1^*$ such that $x_0^*(x_0) = 1 = y_0^*(y_0)$. Note that $x^* \rightarrow x^* \otimes y_0(y \rightarrow x_0^* \otimes y)$ is an isometric embedding of $X^*(Y)$ into $\mathcal{K}(X,Y)$ and $T \rightarrow T^*(y_0^* \otimes y_0(T \rightarrow x_0^* \otimes T(x_0))$ is a norm one projection onto the range of this embedding. Since $C(\Omega)$ is a $L^1$-predual space and since the range of a norm one projection of a $L^1$-predual (see [5]), we conclude that both $X^*$ and $Y$ are $L^1$-predual spaces. Since any dual $L^1$-predual space is isometric to a continuous function space, we have that $X^*$ is isometric to $C(\Omega_1)$ for some compact (hyperstonean) space $\Omega_1$. Note that $\mathcal{K}(X,Y)$ can now be identified with $C(\Omega_1,Y)$ (space of $Y$-valued continuous functions). To show that $Y$ is isometric to a $C(\Omega_2)$ for a compact set $\Omega_2$, by Proposition 6.2 of [8], it is enough to show that $\partial_\alpha Y_1^*$ is $w^*$-closed and $\partial_\alpha Y_1$ is non-empty.

To see the former, let $\{y_\alpha^*\}$ be a net in $\partial_\alpha Y_1^*$ such that $y_\alpha^* \rightarrow y^*$ in the $w^*$-topology. For any $w \in \Omega_1$, $\delta(w) \otimes y_\alpha^* \in \partial_\alpha C(\Omega_1,Y)_1^*$ and $\delta(w) \otimes y_\alpha^* \rightarrow \delta(w) \otimes y^*$ in the $w^*$-topology of $C(\Omega_1,Y)^*$. As $\partial_\alpha C(\Omega_1,Y)_1^*$ is a $w^*$-closed set by hypothesis, we get that $\delta(w) \otimes y^* \in \partial_\alpha C(\Omega_1,Y)_1^*$. Therefore $y^* \in \partial_\alpha Y_1^*$. Thus $\partial_\alpha Y_1^*$ is a $w^*$-closed set. Let $g \in \partial_\alpha C(\Omega_1,Y)_1$ correspond to the constant function 1 in $C(\Omega_1,Y)_1$. Since $C(\Omega)$ has the extreme point intersection property (see [3] for this concept), we have that $C(\Omega_1,Y)_1$ has the extreme point intersection property. Thus by Theorem 3 of [6] (which should read as "$C(K,X)$ has the E.P.I.P iff $X$ has the E.P.I.P and the extreme points of $C(K,X)_1$ take extremal values"), we get that $g$ takes values in $\partial_\alpha Y_1$ and in particular $\partial_\alpha Y_1$ is non-empty. Thus $Y$ is isometric to $C(\Omega_2)$. Therefore we have that $C(\Omega_1,C(\Omega_2))$ is isometric to $C(\Omega)$. Hence by the classical Banach-Stone theorem we get that $\Omega$ is homeomorphic to $\Omega_1 \times \Omega_2$. $lacksquare$

Concerning the analogous question for the space of bounded operators we have only some partial answers. It is known (see [7], page 252) that if $X$ is an abstract $L$-space and $Y = C(K)$ for some extremally disconnected compact set $K$ then $\mathcal{L}(X,Y)$ is a $C(\Omega)$ space. On the other hand if $\mathcal{L}(X,Y)$ is a $C(\Omega)$ space, since $X^*$ and $Y$ embed into $\mathcal{L}(X,Y)$ (as in the case of $\mathcal{K}(X,Y)$) as ranges of norm one projections, one has that $X$ is an abstract $L$-space (see
and $Y$ is a $L^1\)-predual space.

**Proposition.** Suppose $X$ is a Banach space such that $\partial_x X_1$ is non-empty. If $\mathcal{L}(X, Y)$ is isometric to a $C(\Omega)$ space then $X$ is isometric to a $L$-space and $Y$ is isometric to $C(\Omega_1)$ for some compact set $\Omega_1$.

**Proof.** Let $x_0$ be an extreme point of $X_1$. Since $X$ is isometric to a $L$-space (from the preceding discussion), the one dimensional space line $\{x_0\}$ is the range of a $L$-projection in $X$. Thus by Proposition 6.3 of [3] we get that $Y$ is isometric to the range of a $M$-projection in $C(\Omega)$. Therefore by a well-known result in $M$-structure theory (see [1], page 3) it follows that $Y$ is isometric to $C(\Omega_1)$ for a clopen subset $\Omega_1$ of $\Omega$.

**Remark.** This argument shows that for any discrete set $\Gamma$ and for any Banach space $Y$, if $\mathcal{L}(\ell^\infty(\Gamma), Y)$ (which can also be isometrically identified as the $\ell^\infty$ direct sum of $[\Gamma]$-many copies of $Y$) is a $C(\Omega)$ space, then $Y$ is also of the same type. I do not know an example of a $L$-space $X$ with $\partial_x X_1$ empty and a compact set $K$ which is not extremally disconnected but $\mathcal{L}(X, C(K))$ is isometric to a $C(\Omega)$.

**References**


