PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 133, Number 6, Pages 1721–1722
S 0002-9939(04)07701-9
Article electronically published on December 20, 2004

REMARKS ON A RESULT OF KHALIL AND SALEH

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(Communicated by Joseph A. Ball)

Abstract. We give a short proof of a recent result that describes onto isometries of $L(X, Y)$ for certain pairs of Banach spaces X, Y.

1. Result

Definition 1 ([4]). Let X, Y be Banach spaces. The pair (X, Y) is called an ideal pair if both are reflexive, X^* has the approximation property, X and Y^* are strictly convex and $K(X, Y)$ is an M-ideal in $L(X, Y)$.

The main result of [4] describes onto isometries of $L(X, Y)$ for an ideal pair. In this short note we point out a geometric procedure for describing onto isometries of $L(X, Y)$, once a description of isometries of $K(X, Y)$ is known (Theorem 1.1 of [4]). Thus our approach yields a simpler proof of the above-described result and also enlarges the class of Banach spaces for which this result is valid.

Our idea is based on well-known results in the general structure of Banach spaces which are M-ideals when canonically embedded in their biduals (M-embedded spaces; see [3], Chapters III and VI).

First we note that the hypothesis of reflexivity and the assumption that compacts form an M-ideal implies that $L(X, Y)$ is indeed the bidual of $K(X, Y)$ and the inclusion map is the canonical embedding of the space in its bidual (see [2] and [5]). Thus we are in the situation of the so-called M-embedded spaces.

It is to be noted that if one were to only assume that $K(X, Y)$ is an M-ideal in its bidual, then this hypothesis implies that X is reflexive, Y is an M-embedded space and the additional hypothesis of the approximation property leads to $L(X, Y^{**})$ being the bidual of $K(X, Y)$; see [5].

Thus a description of isometries of $L(X, Y^{**})$ in this situation involves knowing information on isometries of the bidual of M-embedded spaces. Let Z be an M-embedded space. According to Proposition 2.2 of chapter III in [3], any onto isometry T of Z^{**} is of the form S^{**} for an onto isometry S of Z. This is the key ingredient of our remark (we recall the analogy with $Z = K(\ell^2)$).

Theorem 2. *Let (X, Y) be an ideal pair. Any onto isometry of $L(X, Y)$ is given by $T \to UTV$ for some onto isometries U and V of Y and X respectively.*

Received by the editors October 11, 2003 and, in revised form, February 6, 2004.
2000 *Mathematics Subject Classification.* Primary 46B20.
Key words and phrases. Isometries, M-embedded spaces.

Proof. By our earlier remarks any onto isometry of $L(X,Y)$ is the bitranspose of an onto isometry of $K(X,Y)$. Thus we are reduced to the study of onto isometries of $K(X,Y)$. As we now have a general way of lifting isometries of compacts, the conclusion follows from Theorem 1.1 of [4]. □

Remark 3. Under an appropriate assumption of the approximation property that ensures $K(X,Y) = X^* \otimes^\vee Y$, it can be seen that when X is reflexive and Y is an M-embedded space, X and Y^* are strictly convex, the arguments given during the proof of Theorem 1.1 in [4] go through giving a description of the onto isometries exactly as in Theorem 1.1. Thus when $K(X,Y)$ is an M-embedded space with appropriate assumptions of strict convexity and the approximation property, the isometries of the bidual, $L(X,Y^{**})$ have a description similar to the one given above. We also recall that by e) of Theorem III.4.6 of [3], any M-embedded space can be renormed so that the dual space is strictly convex and the space is still M-embedded.

We also take this opportunity to point out that Theorem 2.2 of [4] is a part of general folklore. To see this we note the well-known identification of $K(\ell^1)$ with $C(\beta(N), \ell^1)$ (similarly $L(\ell^1)$ can be identified with the ℓ^∞-direct sum of ℓ^1) and the general Banach-Stone theorem that describes onto isometries of these spaces ([1]).

Note added on 1-9-04. More information on isometries is available in my articles: 1) Some generalizations of Kadison's theorem: A survey, Extracta Mathematicae 19 (2004) and 2) Isometries of spaces of operators, preprint. It is shown here that Theorem 2.1 of [4] is incorrect and the assumption of strict convexity cannot be dropped on Y^* in Theorem 1.1 of [4].

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