LOCAL ISOMETRIES OF $\mathcal{L}(X, C(K))$

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ABSTRACT. In this paper we study the structure of local isometries on $\mathcal{L}(X, C(K))$. We show that when $K$ is first countable and $X$ is uniformly convex and the group of isometries of $X^*$ is algebraically reflexive, the range of a local isometry contains all compact operators. When $X$ is also uniformly smooth and the group of isometries of $X^*$ is algebraically reflexive, we show that a local isometry whose adjoint preserves extreme points is a $C(K)$-module map.

1. Introduction

Let $K$ be a compact Hausdorff space and $X$ a Banach space. By $K(X, C(K))$ and $\mathcal{L}(X, C(K))$ we denote the space of compact and bounded operators respectively. Let $\mathcal{G}(X)$ denote the group of isometries of $X$. Let $\Phi : X \to X$ be a linear map. $\Phi$ is said to be a local surjective isometry if for every $x \in X$ there exists a $\Psi_x \in \mathcal{G}(X)$ such that $\Phi(x) = \Psi_x(x)$. An interesting question is for what Banach spaces $X$ is such a $\Phi$ always surjective. This property is also known as algebraic reflexivity of the group of isometries. We refer to [12] Chapter 3 for a very comprehensive account of this problem and its variations. A natural setting for studying this problem is the class of Banach spaces for which a rich and complete description is available of the set $\mathcal{G}(X)$. See the recent monograph [6] for a description of the isometry group of various classical Banach spaces. Making use of the Banach-Stone theorems in the complex scalar field, it was shown in [11] that for a first countable compact set $K$, $\mathcal{G}(C(K))$ is algebraically reflexive. These questions for the case of the space of $X$-valued continuous maps on a first countable compact set $K$, equipped with the supremum norm, were considered in [7].

Among several positive answers given there, we recall (Theorem 7) that for a uniformly convex $X$ for which $\mathcal{G}(X)$ is algebraically reflexive, $\mathcal{G}(C(K, X))$ is algebraically reflexive.

Thus a natural question that arises is, when $K$ is a first countable space and $X$ is such that $\mathcal{G}(X^*)$ is algebraically reflexive, are the spaces $\mathcal{G}(K(X, C(K)))$ and $\mathcal{G}(\mathcal{L}(X, C(K)))$ algebraically reflexive?

We assume that $K$ is identified via the canonical homeomorphism, with the set of Dirac measures in $C(K)^*$ equipped with the weak*-topology. It is well known that the space $K(X, C(K))$ via the map $T \to T^*|K$ is onto isometric to the space...
C(K, X∗). The main tool used in [7] is the description of surjective isometries of K(X, C(K)) given by the study of vector-valued Banach-Stone theorems [2]. We recall that for any ρ : K → G(X∗) that is continuous when G(X∗) is equipped with the strong operator topology, and for any homeomorphism φ of K, f(k) → ρ(k)(f(φ(k))) describes a surjective isometry of C(K, X∗).

In this paper we study the structure of local surjective isometries of the space L(X, C(K)). Part of the motivation for this comes from the fact that using a theorem of Kadison [8] that describes G(L(ℓ2)), it was proved in [10] that G(L(ℓ2)) is algebraically reflexive. A main difficulty in the study of G(L(X, C(K))) is that no complete analogue of the Banach-Stone theorem is available for a general X and K. We mainly rely on the description given in [3] (see also [4] for some partial results).

A key idea of our approach is to consider situations where the restriction of a local isometry to K(X, C(K)) is again a local isometry and use the algebraic reflexivity of G(K(X, C(K)))). We use the identification of L(X, C(K)) with the space W∗C(K, X∗) of X∗-valued functions on K that are continuous when X∗ is equipped with the weak∗-topology, equipped with the supremum norm. We show that when K is first countable and X is a uniformly convex space such that G(X∗) is algebraically reflexive, the range of a local isometry Φ contains all compact operators. Further, if X is also uniformly smooth and Φ∗ preserves extreme points of the unit ball of W∗C(K, X∗)*, we show that Φ is a C(K)-module map in the sense that there is a homeomorphism ϕ of K such that Φ(gf) = g ∘ ϕΦ(f) for all g ∈ C(K) and f ∈ W∗C(K, X∗). We only consider the complex scalar field. Let S(X) = {x ∈ X : ∥x∥ = 1}.

2. Main results

We mainly rely on the following description of G(W∗C(K, X∗)), which is essentially in [5].

Theorem 1. Let K be a compact first countable space and suppose X∗ has the Namioka-Phelps property (i.e., weak∗ and norm topologies coincide on S(X∗)). Then any surjective isometry Ψ of W∗C(K, X∗) has the form Ψ(f)(k) = ρ(k)(f(ψ(k))) where ψ is a homeomorphism of K and ρ : K → G(X∗) is continuous when the latter space has the strong operator topology. Thus a surjective isometry of W∗C(K, X∗) leaves C(K, X∗) invariant.

Proof. Let Φ be a surjective isometry. It was proved in [14] that for spaces with the Namioka-Phelps property, the centralizer Z(X∗) is trivial. Thus it follows from Theorem 4 of [5] that there exists a homeomorphism ψ of K and a ρ : K → G(X∗) that is continuous when G(X∗) has the strong operator topology, such that Ψ(f)(k) = ρ(k)(f(ψ(k))) for k ∈ K and f ∈ W∗C(K, X∗). Since for any f ∈ C(K, X∗), ρ ∘ f ∈ C(K, X∗), Φ(C(K, X∗)) ⊂ C(K, X∗). □

Remark 2. It is worth recalling that K(ℓ2) has the Namioka-Phelps property [9] and any surjective isometry of the dual is weak∗-continuous.

For 1 < p ≠ 2 < ∞, ℓp satisfies the hypothesis imposed on our next set of results; see [5].

Proposition 3. Let K be a first countable compact Hausdorff space and let X be a uniformly smooth Banach space such that G(X∗) is algebraically reflexive. Let
\( \Phi : \mathcal{L}(X, C(K)) \rightarrow \mathcal{L}(X, C(K)) \) be a local surjective isometry. Then \( \text{range}(\Phi) \) contains all compact operators.

**Proof.** Since \( X^* \) is uniformly convex, it has the Namioka-Phelps property. Thus it follows from Theorem 1 that the restriction of any surjective isometry of \( \mathcal{L}(X, C(K)) \) is a surjective isometry of \( \mathcal{K}(X, C(K)) \). Therefore by our hypothesis \( \Phi \) is a local surjective isometry on \( \mathcal{K}(X, C(K)) \). Since \( \mathcal{G}(X^*) \) is algebraically reflexive it follows from Theorem 7 in \([7]\) that \( \mathcal{G}(X^* C(K, C(K))) \) is algebraically reflexive. Therefore \( \Phi \) is surjective on \( \mathcal{K}(X, C(K)) \). \( \square \)

**Remark 4.** It may be recalled that one of the key steps in the proof of algebraic reflexivity of \( \mathcal{G}(\ell^2) \) in \([10]\) is that the range of \( \Phi \) contains a rank one operator.

In the following theorem we once again use the identification of \( \mathcal{K}(X, C(K)) \) with \( C(K, X^*) \) and \( \mathcal{L}(X, C(K)) \) with \( W^* C(K, X^*) \).

**Theorem 5.** Let \( K \) be a metric space and \( X \) a uniformly smooth space such that \( \mathcal{G}(X^*) \) is algebraically reflexive. Let \( \Phi \) be a local surjective isometry of \( W^* C(K, X^*) \). For any \( f \in W^* C(K, X^*) \) there exists a sequence \( \{f_n\} \subseteq C(K, X^*) \) such that \( \Phi(f_n)(k) \rightarrow f(k) \) for all \( k \in K \).

**Proof.** Let \( \Phi : W^* C(K, X^*) \rightarrow W^* C(K, X^*) \) be a local surjective isometry. As before by Theorem 1 we have that \( \Phi(C(K, X^*)) \) is a local surjective isometry. From Theorem 7 in \([7]\) we have that \( \Phi|C(K, X^*) \) is surjective and again by Theorem 1, there exists a homeomorphism \( \phi \) and a weight function \( \rho \) such that \( \Phi(f)(k) = \rho(k)(\phi(f(k))) \) for all \( k \in K \) and for \( f \in C(K, X^*) \).

Now let \( f \in W^* C(K, X^*) \). Since \( K \) is a metric space and \( X^* \) is reflexive, it follows from the results in \([1]\) (see also \([10]\)) that there exists a sequence \( \{g_n\} \subseteq C(K, X^*) \) such that \( g_n(k) \rightarrow f(k) \) for every \( k \in K \). Let \( f_n(k) = \rho^{-1}(k)\phi^{-1}(k) \). Then \( \{f_n\} \subseteq C(K, X^*) \). We know that \( \Phi(f_n)(k) = \rho(k)(\phi(f_n(k))) \) for all \( n \) and \( k \). Thus \( \Phi(f_n)(k) = \rho(k)(\phi(f_n(k))) = g_n(k) \rightarrow f(k) \).

One of the main difficulties in adapting the arguments from \([7]\) to the case of \( W^* C(K, X^*) \) is the non-availability of a complete description of the extreme points of the dual unit ball of \( W^* C(K, X^*) \). We recall that \( \delta(k) \otimes x^* \), for \( k \in K \) and \( x^* \) an extreme point of the unit ball of \( X^* \), completely describes the extreme points of the unit ball of \( C(K, X^*) \). Note that for any \( X \)-valued function \( f \), \( (\delta(k) \otimes x^*)(f) = x^*(f(k)) \). In the following theorem we also assume that \( \Phi^* \) preserves extreme points of the dual unit ball. A similar assumption was made in an earlier context in \([13]\) to achieve surjectivity.

**Theorem 6.** Let \( K \) be a first countable space and let \( X \) be as in the above theorem. Suppose in addition that \( \Phi^* \) preserves extreme points of the dual unit ball and that \( X \) is also uniformly convex. Then \( \Phi \) is a \( C(K) \)-module map in the sense that there is an onto homeomorphism \( \phi \) of \( K \) such that \( \Phi(gf)(k) = g(\phi(k))\Phi(f)(k) \) for \( g \in C(K) \), \( f \in W^* C(K, X^*) \) and \( k \in K \).

**Proof.** As in the previous theorem we get the structure of \( \Phi|C(K, X^*) \), which gives the homeomorphism \( \phi \).

Let \( f \in W^* C(K, X^*) \), \( g \in C(K) \) and \( k \in k \). We will verify the module identity at a unit vector \( x_0 \). It follows from Theorem 0.2 in \([13]\) that as \( X \) is uniformly convex, \( x_0 \) is also a denting point and hence \( \delta(k) \otimes x_0 \) is an extreme point of the unit ball of \( W^* C(K, X^*) \).
Note by the structure of $\Phi|C(K, X^*)$, $\Phi^*(\delta(k) \otimes x_0) = \delta(\phi(k)) \otimes \rho(k)(x_0)$. Now by our hypothesis $\Phi^*(\delta(k) \otimes x_0)$ is an extreme point of the unit ball of $W^*C(K, X^*)$. Note that since $\{\delta(k) \otimes x : k \in K, \|x\| = 1\}$ is a norming set for $W^*C(K, X^*)$, the unit ball of $W^*C(K, X^*)^*$ is the weak* closed convex hull of $\{\delta(k) \otimes x : k \in K, \|x\| = 1\}$. Since $\Phi^*(\delta(k) \otimes x_0)$ is an extreme point, by Milman’s converse of the Krein-Milman theorem, we get a net $\{x_{\alpha}\} \subset S(X)$ and a net $\{k_{\alpha}\} \subset K$ such that $\delta(k_{\alpha}) \otimes x_0 \to \Phi^*(\delta(k) \otimes x_0)$ in the weak* topology of $W^*C(K, X^*)$. We assume w.l.o.g. that $k_{\alpha} \to k'$.

Note that if $h \in C(K)$ and $F \in C(K, X^*)$, then
\[
h(\phi(k))F(k)(\rho(k)(x_0)) = \Phi^*(\delta(k) \otimes x_0)(hF) = \lim h(k_{\alpha})\delta(k_{\alpha}) \otimes x_0(F) = h(k')\Phi^*(\delta(k) \otimes x_0)(F) = h(k')F(k)(\rho(k)(x_0)).
\]

Therefore we have $\phi(k) = k'$. Finally $\Phi^*(\delta(k) \otimes x_0)(gf) = \lim (\delta(k_{\alpha}) \otimes x_0)(gf) = g(\phi(k))\Phi^*(\delta(k) \otimes x_0)(f)$. \hfill \Box

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