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LOCAL ISOMETRIES OF $\mathcal{L}(X, C(K))$

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Abstract. In this paper we study the structure of local isometries on $\mathcal{L}(X, C(K))$. We show that when K is first countable and X is uniformly convex and the group of isometries of X^* is algebraically reflexive, the range of a local isometry contains all compact operators. When X is also uniformly smooth and the group of isometries of X^* is algebraically reflexive, we show that a local isometry whose adjoint preserves extreme points is a $C(K)$-module map.

1. Introduction

Let K be a compact Hausdorff space and X a Banach space. By $\mathcal{K}(X, C(K))$ and $\mathcal{L}(X, C(K))$ we denote the space of compact and bounded operators respectively. Let $\mathcal{G}(X)$ denote the group of isometries of X. Let $\Phi : X \to X$ be a linear map. Φ is said to be a local surjective isometry if for every $x \in X$ there exists a $\Psi_x \in \mathcal{G}(X)$ such that $\Phi(x) = \Psi_x(x)$. An interesting question is for what Banach spaces X is such a Φ always surjective. This property is also known as algebraic reflexivity of the group of isometries. We refer to [12] Chapter 3 for a very comprehensive account of this problem and its variations. A natural setting for studying this problem is the class of Banach spaces for which a rich and complete description is available of the set $\mathcal{G}(X)$. See the recent monograph [6] for a description of the isometry group of various classical Banach spaces. Making use of the Banach-Stone theorems in the complex scalar filed, it was shown in [11] that for a first countable compact set K, $\mathcal{G}(C(K))$ is algebraically reflexive. These questions for the case of the space of X-valued continuous maps on a first countable compact set K, equipped with the supremum norm, were considered in [7].

Among several positive answers given there, we recall (Theorem 7) that for a uniformly convex X for which $\mathcal{G}(X)$ is algebraically reflexive, $\mathcal{G}(C(K, X))$ is algebraically reflexive.

Thus a natural question that arises is, when K is a first countable space and X is such that $\mathcal{G}(X^*)$ is algebraically reflexive, are the spaces $\mathcal{G}(\mathcal{K}(X, C(K))$ and $\mathcal{G}(\mathcal{L}(X, C(K))$ algebraically reflexive?

We assume that K is identified via the canonical homeomorphism, with the set of Dirac measures in $C(K)^*$ equipped with the weak*-topology. It is well known that the space $\mathcal{K}(X, C(K))$ via the map $T \to T^*|K$ is onto isometric to the space

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$C(K, X^*)$. The main tool used in [7] is the description of surjective isometries of $\mathcal{K}(X, C(K))$ given by the study of vector-valued Banach-Stone theorems [2]. We recall that for any $\rho : K \to \mathcal{G}(X^*)$ that is continuous when $\mathcal{G}(X^*)$ is equipped with the strong operator topology, and for any homeomorphism ϕ of K, $f(k) \to \rho(k)(f(\phi(k))$ describes a surjective isometry of $C(K, X^*)$.

In this paper we study the structure of local surjective isometries of the space $\mathcal{L}(X, C(K))$. Part of the motivation for this comes from the fact that using a theorem of Kadison [8] that describes $\mathcal{G}(\mathcal{L}(\ell^2))$, it was proved in [10] that $\mathcal{G}(\mathcal{L}(\ell^2))$ is algebraically reflexive . A main difficulty in the study of $\mathcal{G}(\mathcal{L}(X, C(K))$ is that no complete analogue of the Banach-Stone theorem is available for a general X and K. We mainly rely on the description given in [5] (see also [4] for some partial results).

A key idea of our approach is to consider situations where the restriction of a local isometry to $\mathcal{K}(X, C(K))$ is again a local isometry and use the algebraic reflexivity of $\mathcal{G}(\mathcal{K}(X, C(K)))$. We use the identification of $\mathcal{L}(X, C(K))$ with the space $W^*C(K, X^*)$ of X^*-valued functions on K that are continuous when X^* is equipped with the weak*-topology, equipped with the supremum norm. We show that when K is first countable and X is a uniformly convex space such that $\mathcal{G}(X^*)$ is algebraically reflexive, the range of a local isometry Φ contains all compact operators. Further, if X is also uniformly smooth and Φ^* preserves extreme points of the unit ball of $W^*C(K, X^*)^*$, we show that Φ is a $C(K)$-module map in the sense that there is a homeomorphism ϕ of K such that $\Phi(gf) = g \circ \phi \Phi(f)$ for all $g \in C(K)$ and $f \in W^*C(K, X^*)$. We only consider the complex scalar field. Let $S(X) = \{x \in X : \|x\| = 1\}$.

2. Main results

We mainly rely on the following description of $\mathcal{G}(W^*C(K, X^*))$, which is essentially in [5].

Theorem 1. *Let K be a compact first countable space and suppose X^* has the Namioka-Phelps property (i.e., weak* and norm topologies coincide on $S(X^*)$). Then any surjective isometry Ψ of $W^*C(K, X^*)$ has the form $\Psi(f)(k) = \rho(k)(f(\psi(k))$ where ψ is a homeomorphism of K and $\rho : K \to \mathcal{G}(X^*)$ is continuous when the latter space has the strong operator topology. Thus a surjective isometry of $W^*C(K, X^*)$ leaves $C(K, X^*)$ invariant.*

Proof. Let Φ be a surjective isometry. It was proved in [14] that for spaces with the Namioka-Phelps property, the centralizer $Z(X^*)$ is trivial. Thus it follows from Theorem 4 of [5] that there exists a homeomorphism ψ of K and a $\rho : K \to \mathcal{G}(X^*)$ that is continuous when $\mathcal{G}(X^*)$ has the strong operator topology, such that $\Psi(f)(k) = \rho(k)(f(\psi(k))$ for $k \in K$ and $f \in W^*C(K, X^*)$. Since for any $f \in C(K, X^*)$, $\rho \circ f \in C(K, X^*)$, $\Phi(C(K, X^*)) \subset C(K, X^*)$. □

Remark 2. It is worth recalling that $\mathcal{K}(\ell^2)$ has the Namioka-Phelps property [9] and any surjective isometry of the dual is weak*-continuous.

For $1 < p \neq 2 < \infty$, ℓ^p satisfies the hypothesis imposed on our next set of results; see [3].

Proposition 3. *Let K be a first countable compact Hausdorff space and let X be a uniformly smooth Banach space such that $\mathcal{G}(X^*)$ is algebraically reflexive. Let*

$\Phi : \mathcal{L}(X, C(K)) \to \mathcal{L}(X, C(K))$ be a local surjective isometry. Then range(Φ) contains all compact operators.

Proof. Since X^* is uniformly convex, it has the Namioka-Phelps property. Thus it follows from Theorem 1 that the restriction of any surjective isometry of $\mathcal{L}(X, C(K))$ is a surjective isometry of $\mathcal{K}(X, C(K))$. Therefore by our hypothesis Φ is a local surjective isometry on $\mathcal{K}(X, C(K))$. Since $\mathcal{G}(X^*)$ is algebraically reflexive it follows from Theorem 7 in [7] that $\mathcal{G}(\mathcal{K}(X, C(K)))$ is algebraically reflexive. Therefore Φ is surjective on $\mathcal{K}(X, C(K))$. □

Remark 4. It may be recalled that one of the key steps in the proof of algebraic reflexivity of $\mathcal{G}(\mathcal{L}(\ell^2))$ in [10] is that the range of Φ contains a rank one operator.

In the following theorem we once again use the identification of $\mathcal{K}(X, C(K))$ with $C(K, X^*)$ and $\mathcal{L}(X, C(K))$ with $W^*C(K, X^*)$.

Theorem 5. *Let K be a metric space and X a uniformly smooth space such that $\mathcal{G}(X^*)$ is algebraically reflexive. Let Φ be a local surjective isometry of $W^*C(K, X^*)$. For any $f \in W^*C(K, X^*)$ there exists a sequence $\{f_n\}_{n\geq 1} \subset C(K, X^*)$ such that $\Phi(f_n)(k) \to f(k)$ for all $k \in K$.*

Proof. Let $\Phi : W^*C(K, X^*) \to W^*C(K, X^*)$ be a local surjective isometry. As before by Theorem 1 we have that $\Phi|C(K, X^*)$ is a local surjective isometry. From Theorem 7 in [7] we have that $\Phi|C(K, X^*)$ is surjective and again by Theorem 1, there exists a homeomorphism ϕ and a weight function ρ such that $\Phi(f)(k) = \rho(k)(f(\phi(k))$ for all $k \in K$ and for $f \in C(K, X^*)$.

Now let $f \in W^*C(K, X^*)$. Since K is a metric space and X^* is reflexive, it follows from the results in [1] (see also [16]) that there exists a sequence $\{g_n\}_{n\geq 1} \subset C(K, X^*)$ such that $g_n(k) \to f(k)$ for every $k \in K$. Let $f_n(k) = \rho^{-1}(k)(g_n(\phi^{-1}(k))$. Then $\{f_n\}_{n\geq 1} \subset C(K, X^*)$. We know that $\Phi(f_n)(k) = \rho(k)(f_n(\phi(k)))$ for all n and k. Thus $\Phi(f_n)(k) = \rho(k)(f_n(\phi(k))) = g_n(k) \to f(k)$. □

One of the main difficulties in adapting the arguments from [7] to the case of $W^*C(K, X^*)$ is the non-availability of a complete description of the extreme points of the dual unit ball of $W^*C(K, X^*)$. We recall that $\delta(k) \otimes x^*$, for $k \in K$ and x^* an extreme point of the unit ball of X^*, completely describes the extreme points of the unit ball of $C(K, X)^*$. Note that for any X-valued function f, $(\delta(k) \otimes x^*)(f) = x^*(f(k))$. In the following theorem we also assume that Φ^* preserves extreme points of the dual unit ball. A similar assumption was made in an earlier context in [15] to achieve surjectivity.

Theorem 6. *Let K be a first countable space and let X be as in the above theorem. Suppose in addition that Φ^* preserves extreme points of the dual unit ball and that X is also uniformly convex. Then Φ is a $C(K)$-module map in the sense that there is an onto homeomorphism ϕ of K such that $\Phi(gf)(k) = g(\phi(k))\Phi(f)(k)$ for $g \in C(K)$, $f \in W^*C(K, X^*)$ and $k \in K$.*

Proof. As in the previous theorem we get the structure of $\Phi|C(K, X^*)$, which gives the homeomorphism ϕ.

Let $f \in W^*C(K, X^*)$, $g \in C(K)$ and $k \in k$. We will verify the module identity at a unit vector x_0. It follows from Theorem 0.2 in [13] that as X is uniformly convex, x_0 is also a denting point and hence $\delta(k) \otimes x_0$ is an extreme point of the unit ball of $(W^*C(K, X^*))^*$.

Note by the structure of $\Phi|C(K,X^*)$, $\Phi^*(\delta(k)\otimes x_0) = \delta(\phi(k))\otimes\rho(k)(x_0)$. Now by our hypothesis $\Phi^*(\delta(k)\otimes x_0)$ is an extreme point of the unit ball of $W^*C(K,X^*)$. Note that since $\{\delta(k)\otimes x : k\in K\ ,\ \|x\|=1\}$ is a norming set for $W^*C(K,X^*)$, the unit ball of $W^*C(K,X^*)^*$ is the weak* closed convex hull of $\{\delta(k)\otimes x : k\in K\ ,\ \|x\|=1\}$. Since $\Phi^*(\delta(k)\otimes x_0)$ is an extreme point, by Milman's converse of the Krein-Milman theorem, we get a net $\{x_\alpha\}\subset S(X)$ and a net $\{k_\alpha\}\subset K$ such that $\delta(k_\alpha)\otimes x_\alpha \to \Phi^*(\delta(k)\otimes x_0)$ in the weak* topology of $W^*C(K,X^*)$. We assume w.l.o.g. that $k_\alpha\to k'$.

Note that if $h\in C(K)$ and $F\in C(K,X^*)$, then

$$\begin{aligned} h(\phi(k))F(k)(\rho(k)(x_0)) &= \Phi^*(\delta(k)\otimes x_0)(hF) = \lim\ h(k_\alpha)(\delta(k_\alpha)\otimes x_0)(F)\\ &= h(k')\Phi^*(\delta(k)\otimes x_0)(F) = h(k')F(k)(\rho(k)(x_0)). \end{aligned}$$

Therefore we have $\phi(k)=k'$. Finally $\Phi^*(\delta(k)\otimes x_0)(gf) = \lim\ (\delta(k_\alpha)\otimes x_\alpha)(gf) = g(\phi(k))\Phi^*(\delta(k)\otimes x_0)(f)$. □

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