ISOMETRIES OF $A_C(K)$

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Abstract. We completely describe isometries of $A_C(K)$, when $K$ is a compact Choquet simplex, using facially continuous functions on the extreme boundary.

1. Introduction. Let $K$ be a compact convex set in a locally convex space and denote by $E(K)$ the set of extreme points of $K$ and by $A_C(K)$ the continuous complex-valued affine functions on $K$, equipped with the supremum norm.

We first describe a class of isometries for $A_C(K)$ when $K$ is any compact convex set and give a sufficient condition on an isometry, in terms of facially continuous functions on $E(K)$, so that the isometry in question is in the prescribed class and then deduce that when $K$ is a Choquet simplex, the class of isometries considered, completely describes the isometries of $A_C(K)$.

2. Notations and definitions. For the concepts and results of convexity theory used here we cite [1].

A set $D \subset E(K)$ is said to be facially closed if there exists a closed split face $F$ of $K$ such that $E(F) = D$. The sets $D$ form the closed sets of a topology on $E(K)$ called the facial topology.

Let $C$ denote the complex plane and $\Gamma$, the unit circle in $C$. For a probability measure $\mu$, let $r(\mu)$ denote the resultant of $\mu$ and $\text{Supp}\mu$ denote the topological support of $\mu$.

3. Description of isometries. Following the notations of [1], we denote by $Z(A_C(K))$ the set of elements $b \in A_C(K)$ such that for every $a \in A_C(K)$ there exists $c \in A_C(K)$ satisfying

$$c(x) = a(x) \cdot b(x) \quad \forall x \in E(K).$$

Since for any $b \in Z(A_C(K))$, real and imaginary parts of $b$ are in $Z(A(K))$, using Corollary II.7.4 and Theorem II.7.10 of [1], we can easily see that for $b \in A_C(K)$, $b$ is in $Z(A_C(K))$ if and only if $b \mid E(K) \to C$ is continuous in the facial topology.

Let $Q : K \to K$ be an onto affine homeomorphism and let $a_0 \in Z(A_C(K))$ be such that $|a_0| = 1$ on $E(K)$. Define $\Phi : A_C(K) \to A_C(K)$ by $\Phi(a) = c$, where $c$ is the unique element of $A_C(K)$ such that $c(x) = a(Q(x)) \cdot a_0(x) \forall x \in E(K)$.

It is easy to see that $\Phi$ is an onto isometry and $\Phi(1) = a_0$.
Theorem 3.1. Let $\Phi: A_c(K) \to A_c(K)$ be any onto isometry. Assume

$$\Phi(1) \in Z(A_c(K)).$$

Then there exists an affine homeomorphism $Q$ from $K$ onto $K$ such that

$$\Phi(a)(x) = a(Q(x))\Phi(1)(x) \quad \forall x \in E(K).$$

Proof. Define $\delta: K \to A(K)^*$ by $\delta(x)(a) = a(x) \quad \forall a \in A_c(K)$ and $x \in K$. It is well known that $\delta$ is an affine homeomorphism of $K$ onto $\{f \in A_c(K)^*: \|f\| = f(1) = 1\}$, with $w^*$-topology. Since $\Phi^*: A_c(K)^* \to A_c(K)^*$ is an onto isometry and a $w^*$-homeomorphism it is easy to see that $\Phi^*(\delta(E(K))) \subseteq \Gamma \cdot \delta(E(K))$.

Let $x \in E(K)$. Since $A_c(K)$ separates points of $K$ and $1 \in A_c(K)$, there exist unique $x' \in E(K)$ and $t \in \Gamma$, such that $\delta(x) = t \cdot \delta(x')$. Moreover

$$\delta(x)(1) = t \cdot \delta(x').$$

Hence $\delta(x)(1) = 1$ on $E(K)$. Let $\Phi(1) = u + iv$, $u, v \in A(K)$ (real-valued functions in $A_c(K)$). Then since $Z(A_c(K))$ is selfadjoint, $\Phi(1) = u - iv$ is in $Z(A_c(K))$. Define now $T: A_c(K) \to A_c(K)$ by

$$T(a)(x) = \Phi(a)(x) \cdot \overline{\Phi(1)(x)} \quad \forall x \in E(K).$$

Since $|\Phi(1)| = 1$ on $E(K)$, it follows from the remarks in the beginning of this section that $T$ is a well-defined, onto isometry. Moreover, $T(1) = 1$. It is easy to see that $T^*$ maps $\delta(K)$ onto $\delta(K)$ and $Q = \delta^{-1} \circ T^* \circ \delta$ is an affine homeomorphism of $K$ onto $K$. That $\Phi(a)(x) = a(Q(x)) \cdot \Phi(1)(x) \forall x \in E(K)$ follows from (*) and the definition of $T$.

Definition (Effros). Say a closed set $D \subseteq K$ is a dilated set if for any maximal measure $\mu$ with $r(\mu) \in D$, $\text{Supp} \mu \subseteq D$.

Proposition 3.2. Let $K$ be a compact Choquet simplex and let $a_0 \in A_c(K)$ and $|a_0| = 1$ on $E(K)$. Then $a_0 \in Z(A_c(K))$.

Proof. In view of the results quoted in the beginning of this section it is sufficient to show that $a_0 |E(K)$ is facially continuous.

For a closed set $B \subseteq T$, let $B' = \{x \in \overline{E(K)}: a_0(x) \in B\}$. We claim that the closed set $B'$ is a dilated set. Let $\mu$ be a maximal probability measure with $x_0 = r(\mu) \in B'$. Since

$$1 = |a_0(x_0)| = \int_{E(K)} a_0 \, d\mu \leq \int_{E(K)} |a_0| \, d\mu \leq 1,$$

we get that $a_0 \equiv a_0(x_0)$ on $\text{Supp} \mu$ and hence $\text{Supp} \mu \subseteq B'$.

It now follows from a result of [2] that $F$, the closed convex hull of $B'$, is a split face and hence $\{x \in E(K): a_0(x) \in B\} = F \cap E(K)$ is a facially closed set.

Remark. When $K$ is a simplex, $a \in A_c(K)$ is an extreme point of the closed unit ball of $A_c(K)$ iff $|a| = 1$ on $E(K)$ iff $a \in Z(A_c(K))$ and is an extreme point of the closed unit ball of $Z(A_c(K))$.

Corollary 3.3. If $K$ is a compact Choquet simplex then for any onto isometry $\Phi$ of $A_c(K)$, $\exists$ an affine homeomorphism $Q$ of $K$ such that

$$\Phi(a)(x) = a(Q(x)) \cdot \Phi(1)(x) \quad \forall x \in E(K).$$
Proof. We have observed in the proof of Theorem 3.1 that $|\Phi(1)| = 1$ on $E(K)$, hence the conclusion follows from Corollary 3.2 and Theorem 3.1.

Remark. These results generalize the classical Banach-Stone theorem dealing with the isometries of $C_c(X)$, where $X$ is a compact Hausdorff space; also generalized is the work of A. J. Lazar [3] on isometries of $A(K)$.

4. Example. We end by giving a simple example of a nonsimplicial compact convex set $K$ and an isometry $\Phi$ of $A_c(K)$ which is not of the form described in Theorem 3.1.

Let $K$ be the unit square in $\mathbb{R}^2$ centred at $(0,0)$, so
\[
E(K) = \{(x, y) : |x| = 1 = |y|\} \cdot K
\]
has no proper split faces and hence $Z(A_c(K)) = \{\alpha \cdot 1 : \alpha \in \mathbb{C}\}$. Any $f \in A_c(K)$ is of the form $f(x, y) = ax + by + c$ where $a, b, c \in \mathbb{C}$. Define $\Phi(f)(x, y) = cx + by + a$. Now $\|f\| = \max |a \pm b \pm c|$ and $\|\Phi(f)\| = \max |c \pm b \pm a|$ hence $\Phi$ is an isometry. It is obvious that $\Phi$ is onto. But $\Phi(1) = x$, a nonconstant. Hence $\Phi$ is not of the form in Theorem 3.1.

Acknowledgement. I thank my supervisor Professor A. K. Roy and the referee for simplifying the proof of Theorem 3.1.

References