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On a new geometric property for Banach spaces

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Abstract. In this paper we study a geometric property for Banach spaces called condition $(*)$, introduced by de Reyna *et al* in [3]. A Banach space has this property if for any weakly null sequence $\{x_n\}$ of unit vectors in X, if $\{x_n^*\}$ is any sequence of unit vectors in X^* that attain their norm at x_n's, then $x_n^* \xrightarrow{w^*} 0$. We show that a Banach space satisfies condition $(*)$ for all equivalent norms iff the space has the Schur property. We also study two related geometric conditions, one of which is useful in calculating the essential norm of an operator.

Keywords. Banach spaces; Schur property.

1. Introduction

In a recent work [3], de Reyna *et al* considered a new geometric property for a Banach space called "condition $(*)$". They have introduced it as a necessary condition for a Banach space X so that $C(K,X)$, the space of X-valued continuous functions is an M-ideal in $WC(K,X)$ the space of X-valued functions on K that are continuous when X has the weak topology (i.e. there is a projection P: $WC\ (K,X)^* \to WC(K,X)^*$ whose range is $C(K,X)^{\perp}$ and $\|P\Lambda\| + \|\Lambda - P\Lambda\| = \|\Lambda\| \forall \Lambda \in WC(K,X)^*$ for the ordinal space $K=[0,\omega]$.

Motivated by some results in M-structure theory (see [10], [11]), this author has proved in [12] that condition $(*)$ is also a necessary condition for $C(\beta\mathbb{N},X)$ to be an M-ideal in $WC(\beta\mathbb{N},X)$. It is known that this condition is not a sufficient condition for continuous functions to be a M-ideal (see [3]).

Among other non-trivial examples of Banach spaces with condition $(*)$ are the ℓ^p-spaces for $1<p<\infty$ and spaces with property (m_∞), (see [7]) the Bloch space as well as c_0 direct sum of spaces with the Schur property (see [3]). Any Banach space with the Schur approximation property or its modified version (see [3], [12]) satisfies condition $(*)$. Clearly condition $(*)$ is hereditary and is a separably determined property.

It is clear that any Banach space that satisfies the Schur property (i.e. weak and norm sequential convergence coincide) has $(*)$. We show that if a Banach space is such that it satisfies condition $(*)$ in all equivalent renormings, then X has the Schur property. This comes as a consequence of our result on the stability of this condition under ℓ^1-direct sums. We next show that this condition is invariant under c_0-direct sums. If X has the Dunford and Pettis property then again condition $(*)$ for X^* implies the Schur property on X^*.

We next consider a weaker form of condition $(*)$ which we shall call as condition $(*)$ with extreme points. For a Banach space that has this property, zero is a ω^*-accumulation point of $\partial_e X_1^*$, the set of extreme points of the dual unit ball. Motivated by the example $c_0\ (\Gamma)$ (Γ is a discrete set) we show that a Banach space X

whose dual is isometric to a $L^1(\mu)$ has this weaker condition iff it is isometric to $c_0(\Gamma)$ for a discrete set Γ. This is also a hereditary property but we do not know if it is a separably determined one. However we show that if X is a Asplund space, this weaker form is also a seperably determined property.

For a Banach space X, let $\mathscr{L}(X)$, $\mathscr{K}(X)$ denote the space of bounded and compact linear operators respectively. In [2], Axler *et al* consider Banach spaces for which the essential norm of an operator $\|T\|_e = d(T, \mathscr{K}(X))$ equals that of the adjoint. Motivated by a formula for the essential norm obtained by Werner, in the case when compact operators form an M-ideal in the space of bounded operators [14], we introduce a stronger form of condition (*) to show that if X is a reflexive Banach space whose dual has this property and Y is a separable space such that $\mathscr{K}(X, Y)$ is an M-ideal in $\mathscr{L}(X, Y)$, then $\|T\|_e = \|T^*\|_e$ $\forall T \in \mathscr{L}(X, Y)$.

Since any space with the Schur property satisfies all these properties, unless otherwise specified we consider only space that do not have the Schur property.

All the Banach spaces considered are over the real scalar field. For basic Banach space theory we shall refer to [4, 5] and for L^1-predual theory [8] and for results about M–structure of Banach spaces the fundamental paper [1] and the latest monograph [6].

2. Main results

Our first result shows that ℓ^p-spaces $(1 < p < \infty)$ and any Banach space that has property (m_∞) satisfies condition (*).

Recall from [7] that a Banach space has "property (m_p)" if $\overline{\lim}\|x + x_n\| = \|(\|x\|, \overline{\lim}\|x_n\|)\|_p$ whenever $x_n \to 0$ weakly.

PROPOSITION 1

Any Banach space that has property (m_∞) and ℓ^p for $1 < p < \infty$, satisfies condition (*).

Proof. Let $\|x_n\| = 1$, $x_n \xrightarrow{w} 0$ and $1 = x_n^*(x_n) = \|x_n^*\|$. Let $\|x\| = 1$. We shall show that $x_n^*(x) \to 0$. Passing through a subsequence if necessary, assume that $x_n^*(x) \to \alpha$.

$$|1 \pm \alpha| = \lim_n |x_n^*(x_n) \pm x_n^*(x)|$$

$$\leqslant \overline{\lim}\|x_n \pm x\| = 1.$$

Hence $\alpha = 0$. That ℓ^p satisfies condition (*) can be seen easily using the facts that the x_n^*'s are uniquely determined in terms of the x_n's and the weak convergence is pointwise convergence on the unit ball.

For the case $p = 1$ the following proposition shows that if μ is not purely atomic, $L^1(\mu)$ fails (*).

PROPOSITION 2

Any Banach space X satisfying condition (*) has trivial L-structure i.e. there is no nontrivial projection P in X such that $\|Px\| + \|x - Px\| = \|x\|$ $\forall x \in X$.

Proof. As part of our general assumption we have that X is an infinite dimensional space failing the Schur property. Suppose $X = M \oplus N$ for nontrivial M and N such that $\|m+n\| = \|m\| + \|n\|$ $\forall m \in M$, $n \in N$.

Since X fails the Schur property, we assume w.l.o.g. that there exists a sequence $\{m_k\}$ in M with $\|m_k\| = 1$, $m_k \xrightarrow{w} 0$. Choose $\|m_k^*\| = 1 = m_k^*(m_k)$, $m_k^* \in M^*$. Since N is nontrivial we can choose $\{n^*\}$ a unit vector in N^* and n in N such that $n^*(n) = 1$.

Now consider the sequence $\{m_k^* + n^*\}$ in $X^*(= M^* \oplus N^*$ equipped with the maximum norm).

For any k, $\|(1-(1/k))m_k + (1/k)n\| = 1$ and $(m_k^* + n^*)$ $((1-(1/k))m_k + (1/k)n) = 1-(1/k)+(1/k) = 1$. Also $(1-(1/k))m_k + (1/k)n \xrightarrow{\omega} 0$. Since X has property (*), we get, $m_k^* + n^* \xrightarrow{\omega^*} 0$, a contradiction.

Therefore there is no nontrivial L-projection in X.

COROLLARY

An infinite dimensional Banach space has the Schur property iff it satisfies condition (*) for each equivalent norm.

Proof. Suppose X satisfies condition (*) for each equivalent norm.

Let $X = M \oplus N$ where M and N are two proper closed subspaces.

Now $\|x\|' = \|m\| + \|n\|$ where $x = m+n$, $m \in M$, $n \in N$ is an equivalent norm and since $(X, \|\ \|')$ has a non-trivial L-projection, we conclude that X has the Schur property.

We next consider the stability of condition (*) under c_0-direct sums.

PROPOSITION 3

If $\{X_i\}$ is a family of Banach spaces satisfying condition (*) then so does $Y = \oplus_{c_0} X_i$.

Proof. Let us note that if each X_i has the Schur property then Y satisfies condition (*) (see [3]).

Let $y_n \in Y$, $\|y_n\| = 1$ and $y_n \xrightarrow{w} 0$. Clearly $y_n(i) \xrightarrow{w} 0$ $\forall i$.

Let $y_n^* \in Y^* = \oplus_1 X_i^*, \|y_n^*\| = 1 = y_n^* \cdot (y_n)$. It is enough to show that $y_n^*(i) \xrightarrow{w^*} 0$ $\forall i$.

Now since $\sum_i \|y_n^*(i)\| = 1 = \sum y_n^*(i)(y_n(i))$,

we have

$$\sum_i \|y_n^*(i)\| \frac{y_n^*(i)}{\|y_n^*(i)\|}(y_n(i)) = 1 \ \forall n.$$

Since

$\|y_n\| = 1$ $\forall n$, we get

$$\frac{y_n^*(i)}{\|y_n^*(i)\|}(y_n(i)) = 1$$

whenever $y_n^*(i) \neq 0$ $\forall i$ and this implies $\|y_n(i)\| = 1$. However since $y_n(i) \xrightarrow{\omega} 0$ we get $y_n^*(i) \xrightarrow{\omega^*} 0$ as $n \to \infty$.

Therefore Y satisfies condition (*).

PROPOSITION 4

Suppose Y has the Dunford and Pettis property, then Y^* has condition $(*)$ iff it has the Schur property.

Proof. Suppose Y^* fails the Schur property. Using the Bishop–Phelps theorem [4], we may assume that $\exists y_n \in Y$, $\|y_n\| = 1$ and $\|y_n^*\| = 1 = y_n^*(y_n)$, $y_n^* \in Y^*$, $y_n^* \xrightarrow{w} 0$. Since Y^* has condition $(*)$, $y_n \xrightarrow{w} 0$. Also since Y has the Dunford and Pettis property, $y_n^*(y_n) \longrightarrow 0$, a contradiction. Hence, Y^* has the Schur property.

Remark. This should be compared with Corollary 2 of [12], where we have proved that if X has the MSAP then any subspace of X that is isomorphic to a dual space has the Schur property.

We now introduce a weaker form of condition $(*)$ which we shall call as condition $(*)$ with extreme points.

(A) DEFINITION

A Banach space X satisfies condtion $(*)$ with extreme points if for any weakly null sequence $\{x_n\}$ of unit vectors in X such that $x_n^*(x_n) = 1$, $x_n^* \in \partial_e X_1^*$ implies $x_n^* \xrightarrow{w^*} 0$.

Since by our general assumption on X, there is at least one sequence $\{x_n\}$ of unit vectors with $x_n \xrightarrow{w} 0$. By choosing $x_n^* \in \partial_e X_1^*$, $1 = x_n^*(x_n)$ we conclude that 0 is in the w^*-closure of $\partial_e X_1^*$. Thus c, the space of convergent sequences fails this property.

Since for any closed subspace $M \subset X$, any $f \in \partial_e M_1^*$ can be extended to a $g \in \partial_e X_1^*$, this is a hereditary property.

Theorem 1. *Suppose X is a Asplund space. If all separable subspace of X have condition $(*)$ with extreme points then X has the same property.*

Proof. Let $x_n \in X$, $x_n^* \in \partial_e X_n^*$, $\|x_n\| = 1 = x_n^*(x_n)$, $x_n \xrightarrow{w} 0$.

Fix a $x_0 \in X$. Let $Y = \overline{\text{Span}}\{x_n\}_{x \geqslant 0}$ and $Y' = \overline{\text{Span}}\{x_n^*\}_{n \geqslant 1}$.

It follows from the results in [13] that there exists a separable subspace Z of X containing Y and a linear operator T: $Z^* \to X^*$ such that $T(Z^*) \supseteq Y'$ and $T(z^*)$ is a norm preserving extension of z^* for each $z^* \in Z^*$.

Fix n and let $F_n = \{z^* \in Z_1^* \colon T(z^*) = x_n^*\}$. Since T is linear and x_n^* is an extreme point, F_n is an extreme subset of Z_1^*. Since X is a Asplund space it follows that (see [4], Chapter 6) there is a $z_n^* \in \partial_e Z_1^* \cap F_n$.

Since Z is separable it follows from the hypothesis that $z_n^* \xrightarrow{w^*} 0$ in Z^*. As $x_0 \in Z$, we get $x_n^*(x_0) \to 0$. Since x_0 is arbitrary we conclude that $x_n^* \to 0$. Hence X has condition $(*)$ with extreme points.

Theorem 2. *Let X be an L^1-predual space. Suppose X satisfies condition $(*)$ with extreme points then X is isometric to $c_0(\Gamma)$ for some discrete set Γ.*

Proof. We shall first prove the theorem when X is a separable Banach space. Since c, the space of convergent sequences, fails $(*)$ with extreme points and since this is a hereditary property, clearly X has no isometric copy of c. Hence X^* is isometric to ℓ^1 (see [8], p. 226). Let $\partial_e X_1^*$ denote the set of extreme points of the dual unit ball X_1^* of X. We can choose a sequence $\{x_n^*\}$ of linearly independent vectors such that

$$\partial_e X_1^* = \pm\{x_n^*\}_{n\geq 1}.$$

If we can show that $x_n^* \xrightarrow{w^*} 0$, then it would follow that X is a c_0-space.

Since any linear functional defined on the span of finitely many x_n^*'s has a norm preserving extension to an element of X (see [1], Corollary 5.5), inductively define a sequence $\{x_n\} \subset X$ such that $x_n^*(x_n) = 1 = \|x_n\|$, $x_n^*(x_m) = 0$ for $m > n$. Clearly then for each $x_n^* \in \partial_e X_1^*$,

$$x_n^*(x_m) \to 0 \quad \text{as} \quad m \to \infty.$$

Hence, by Rainwater's theorem [5], $x_n \to 0$ weakly. Therefore by condition (*) we have that $x_n^* \xrightarrow{w^*} 0$. Since any infinite w^*-convergent sequence in $\partial_e X_1^*$ is going to give an isometric copy of c in X (see [9]), it is clear that $\partial_e X_1^*$ is discrete. Therefore X is isometric to c_0.

To see the general case, let $Y \subset X$ be any separable subspace. Then there exist a separable L^1-predual space Z such that $Y \subset Z \subset X$ (see [8], Chapter 7). From what we saw above, Z is isometric to c_0. At this stage instead of using purely L^1-predual theory techniques, we shall use the following argument using results about Banach spaces that are M-ideals in their biduals to conclude that X is isometric to c_0 (Γ).

Since c_0 is a Banach space that is an M-ideal in its bidual and since this property is hereditary, we can conclude from the arguments above that X is such that every separable subspace of it is an M-ideal in its bidual. Therefore it follows from Proposition 2.8 of [10] that X is an M-ideal in its bidual and being an L^1-predual it therefore is isometric to $c_0(\Gamma)$ (see [6] Chapter 3).

Remark. The motivation for the above theorem comes from a result of [10] where the authors prove that if X is a Banach space such that the space of compact operators $\mathscr{K}(\ell^1, X)$ forms an M-ideal in $\mathscr{L}(\ell^1, X)$, the space of bounded operators, then X is an M-ideal in its bidual and in particular the only L^1-predual space in the above class is $c_0(\Gamma)$. This author has observed in [12] that for such an X, $C(\beta\mathbb{N}, X)$ is an M-ideal in $WC(\beta\mathbb{N}, X)$ and further this latter class is a strictly smaller class. Hence it is natural to ask if $c_0(\Gamma)$ is the only L^1-predual in the latter class as well.

c_0 is an M-ideal in its bidual ℓ^∞ and ℓ^∞ fails condition (*) with extreme points. Also $0 \notin \overline{\partial_e(\ell^1)_1^\infty}^{w}$. The ideas of Theorem 1 can be used to prove the following result.

PROPOSITION 5

Suppose X is an infinite dimensional Banach space which is an M-ideal in its bidual. If $0 \notin \overline{\partial_e X_1^*}^{w}$ (closure in the weak topology) then X^{**} fails condition (*) with extreme points.

Proof. This is clear when X is a reflexive space. So assume that $\exists x_n^{**} \in X^{**}, \|x_n^{**}\| = 1$, $x_n^{**} \xrightarrow{w} 0$ and X is non-reflexive. Note that since X has an isomorphic copy of c_0, X and hence X^{**} cannot have the Schur property. It is also known (see [6], Chapter 3) that such an X is a Asplund space. So, using this fact and the Bishop–Phelps theorem and imitating the arguments given before we can choose,

$y_n^* \in \partial_e X_1^*$ such that $y_n^{**}(y_n^*) = \|y_n^{**}\| = 1$, $y_n^{**} \in X^{**}$ and $y_n^{**} \xrightarrow{w} 0$. Now if X^{**} has condition $(*)$, we get that $y_n^* \xrightarrow{w} 0$, a contradiction.

Delving into a different aspect of the geometry of $c_0(\Gamma)$ we get the next result.

Theorem 3. *Suppose X is such that $\partial_e X_1^* \cup \{0\}$ is ω^*-closed. Suppose X has either* (a) *Dunford and Pettis property and ω^* and weak topologies agree on $S(X^*) = \{x^* : \|x^*\| = 1\}$ or* (b) *ω^* and norm topologies agree on $S(X^*)$, then X has condition* $(*)$ *with extreme points.*

Proof. It is easy to see that under either of the conditions, every separable subspace has a separable dual. Hence by Theorem 6 on page 230 of [5], X_1^* is ω^*-sequentially compact.

If X has (a), let $x_n^* \in \partial_e X_1^*$, $x_n^*(x_n) = \|x_n\| = 1$ and $x_n \xrightarrow{w} 0$. By hypothesis if $x_n^* \xrightarrow{w^*} x^* \neq 0$, then $x^* \in \partial_e X_1^*$. Hence $x_n^* \xrightarrow{w} x^*$. But this contradicts the fact that X has the Dunford and Pettis property. Therefore $x_n^* \xrightarrow{w^*} 0$. If X has (b) then again if $x_n^* \xrightarrow{\|\|} x^*$, $x^* \in \partial_e X_1^*$ we get a contradiction to the fact that $x_n^*(x_n) = 1$ so that $x_n^* \xrightarrow{w^*} 0$.

We conclude with an application to operator theory of a stronger form of $(*)$ which is satisfied by the (m_∞) spaces.

For any Banach space X, Y let $\mathscr{K}(X, Y)$ and $\mathscr{L}(X, Y)$ denote the space of compact and bounded operators respectively. For any $T \in \mathscr{L}(X, Y)$, the essential norm of T, $\|T\|_e$ is defined by $\|T\|_e = d(T, \mathscr{K}(X, Y))$.

It can be deduced from the observations made by Axler *et al* [2] that

1. $\|T\|_e \geqslant \|T^*\|_e$,
2. $\|T\|_e = \|T^*\|_e$ if Y is a dual space.

The following formula for the essential norm when X is a Banach space such that $\mathscr{K}(X, Y)$ is an M-ideal in $\mathscr{L}(X, Y)$ was given by Werner in [14].

$$\|T\|_e = \max\{\omega(T), w^*(T)\},$$

where

$$w(T) = \sup\{\overline{\lim}_\alpha \|T(x_\alpha)\| : \{x_\alpha\} \text{ is a net in } X, \|x_\alpha\| = 1\ \forall \alpha \text{ and } x_\alpha \xrightarrow{w} 0\},$$

$$w^*(T) = \sup\{\overline{\lim} \|T^*(y_\alpha^*)\| : \{y_\alpha^*\} \text{ is a net in } Y^*, \|y_\alpha^*\| = 1\ \forall \alpha \text{ and } y_\alpha^* \xrightarrow{w} 0.\}$$

and at least one of the involved suprema is actually attained.

First we prove a general proposition regarding these weights $w(T), w^*(T)$.

PROPOSITION 6

Suppose $\|T\|_e = \overline{\lim}_\alpha \|T(x_\alpha)\|$ for a net $\{x_\alpha\}$ of unit vectors in X such that $x_\alpha \xrightarrow{w} 0$, then $\|T\|_e = \|T^*\|_e$.

Proof. For any $S \in \mathscr{K}(Y^*, X^*)$, since S^* is a compact operator, $S^*(x_\alpha) \longrightarrow 0$ in the norm.

Thus $\|T^* - S\| = \|T^{**} - S^*\| \geq \overline{\lim} \|T(x_\alpha)\| = \|T\|_e$. Hence $\|T\|_e \leq \|T^*\|_e$ and therefore $\|T\|_e = \|T^*\|_e$

Now we are ready to formulate a stronger form of condition (*). As before X is a Banach space failing the Schur property.

(B) DEFINITION

If $\{x_n\}$ is a sequence of vectors in the unit ball of X such that $x_n \xrightarrow{\omega} 0$ and $\overline{\lim}_n x_n^*(x_n) = 1$ for a sequence of unit vectors $\{x_n^*\}$ in X^*, then $x_n^* \xrightarrow{\omega^*} 0$.

Theorem 4. *Let X be a reflexive Banach space such that X^* satisfies the above property and Y be a separable Banach space such that $\mathscr{K}(X,Y)$ is an M-ideal in $\mathscr{L}(X,Y)$. Then $\|T\|_e = \|T^*\|_e\ \forall T \in \mathscr{L}(X,Y)$.*

Proof. Since $\mathscr{K}(X,Y)$ is an M-ideal in $\mathscr{L}(X,Y)$ we know that

$$\|T\|_e = \overline{\lim}_\alpha \|T(x_\alpha)\| \quad \text{for a net of unit vectors} \quad x_\alpha \xrightarrow{w} 0$$

or

$$\|T\|_e = \overline{\lim}_\alpha \|T^*(y_\alpha^*)\| \quad \text{for a net of unit vectors} \quad y_\alpha^* \xrightarrow{w^*} 0.$$

In view of the above Proposition, our theorem is proved if the first case happens. In the second case, again using the hypothesis that $\mathscr{K}(X,Y)$ is an M-ideal in $\mathscr{L}(X,Y)$, $\|T\|_e = \|T-S\|$ for some $S \in \mathscr{K}(X,Y)$. So we may assume w.l.o.g that $\|T\| = 1 = \|T\|_e$.

Since X is reflexive and Y is separable we get

$$1 = \overline{\lim}_n T^*(y_n^*)(x_n) = \overline{\lim}_n y_n^*(T(x_n))$$

for a sequence of unit vectors $\{y_n^*\}$ in Y with $y_n^* \xrightarrow{w^*} 0$ (see the proof of lemma 7 in [14]) and sequence $\{x_n\}$ of unit vectors in X.

Since $\|T\| = 1$, $T^*(y_n^*)$ is a sequence in the unit ball of X. Also $T^*(y_n^*) \xrightarrow{w} 0$. By our hypothesis on X we get that $x_n \xrightarrow{w} 0$.

Also

$$\overline{\lim} \|T(x_n)\| = 1.$$

Hence by applying the above Proposition again we get $\|T\|_e = \|T^*\|_e$.

Remark. It is easy to see that a suitable version of (B) can be a imposed on Y to derive the same conclusion when $\mathscr{K}(X,Y)$ is an M-ideal in $\mathscr{L}(X,Y)$.

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