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Riesz isomorphisms of tensor products of order unit Banach spaces

T S S R K RAO

Stat.-Math. Unit, Indian Statistical Institute, R.V. College Post, Bangalore 560 059, India
E-mail: tss@isibang.ac.in

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Abstract. In this paper we formulate and prove an order unit Banach space version of a Banach–Stone theorem type theorem for Riesz isomorphisms of the space of vector-valued continuous functions. Similar results were obtained recently for the case of lattice-valued continuous functions in [5] and [6].

Keywords. Order unit Banach space; Banach–Stone theorem; Riesz isomorphism; Choquet simplex; injective tensor products.

1. Introduction

Let X, Y be compact Hausdorff spaces and E a Banach lattice and F be an abstract M-space with unit. Let $\pi\colon C(X, E) \to C(Y, F)$ be a Riesz isomorphism (i.e., order-preserving linear bijection) such that $0 \notin f(X)$ if and only if $0 \notin \pi(f)(Y)$ for each $f \in C(X, E)$. Ercan and Önal have proved in [6] that E is Riesz isomorphic to F and X is homeomorphic to Y. This has been extended to the case of Banach lattices in [5]. Such results are called Banach–Stone theorems after the classical result of Banach and Stone, in the scalar-valued case. Identifying the space $C(X, E)$ with the injective tensor product space $C(X) \otimes_\epsilon E$ the above theorem can be interpreted as imposing additional conditions on the Riesz isomorphism of injective tensor product to ensure that the component spaces are again Riesz isomorphic. It is well-known that an additional condition is necessary on π to ensure this conclusion (see Corollary 7.3 of [3] and the Remark following it). Also see [4] for topological conditions which make the component spaces homeomorphic.

Let E be an Archimedean ordered vector space with an order unit e that is also a Banach space under the order unit norm. Such an E is called an order unit Banach space (see [2] and [7]). Let K be a compact convex set. We follow the notation and terminology of [2] and [1], as these monographs relate to both lattice theory and convexity theory. Let $A(K, E)$ denote the space of affine E-valued continuous functions, equipped with the supremum norm. We equip this space with the point-wise ordering. When E is the scalar field, we denote the corresponding space by $A(K)$. It is easy to see that it is an order unit Banach space. We recall that K is a Choquet simplex if and only if $A(K)$ has the Riesz decomposition property. It is also well-known that when K is a Choquet simplex, $A(K)$ has the metric approximation property and the space $A(K, E)$ can be identified with $A(K) \otimes_\epsilon E$ (see [9]). Also when K is a Choquet simplex with the set of extreme points $\partial_e K$, closed (the so-called Bauer simplex), $A(K)$ is isometric (via the restriction map) to $C(\partial_e K)$. Thus it is interesting to consider questions similar to those answered in [6] for the family of Choquet simplexes and order unit Banach spaces. We recall that $S = \{e^* \in E^*\colon e^*(e) = 1 = \|e^*\|\}$ is called the state space of E. An extreme convex set $F \subset K$ is called a face. When K is a Choquet simplex, for any closed face $F \subset K$, there exists a face $F' \subset K$ such that K is the convex hull of F and F' and the decomposition is unique. Such a F' is called a complementary face. We note that in the case of $C(X)$, the Choquet simplex K corresponds to it is the set of probability measures with the weak*-topology and X is identified with the set of extreme points via the Dirac map $x \to \delta(x)$. Thus closed sets correspond to the set of extreme points of a closed face F of K and open sets correspond to the set of extreme points of the complementary face F'.

We also prove a Banach–Stone theorem when E is a M-space with unit and F is an order unit space.

Our argument relies on tensor product theory of convex sets as developed in [9] and [8].

2. Main result

The following theorem (Theorem 2.2) extends the main result of [6]. In order to prove this we recall the Banach–Stone theorem for $A(K)$ spaces, when K is a simplex, due to Lazar [8]. See [10] for this formulation. We recall that Riesz isomorphism preserves the identity and hence is an isometry. We note that analogous to the continuous function space, multiplication of affine functions on the extreme boundary of a simplex is well-defined.

Theorem 1. *Let K_1, K_2 be simplexes. Let $\pi\colon A(K_1) \to A(K_2)$ be an isometry. Then $|\pi(1)| \equiv 1$ on $\partial_e K_2$ and there exists an affine homeomorphism $\phi\colon K_2 \to K_1$ such that $\pi(a) = \pi(1)\, a \circ \phi$ on $\partial_e K_2$.*

In particular if π is a Riesz isomorphism, then $\pi(1) = 1$.

In the following theorem we impose an additional condition, in a symmetric way, on the component spaces to derive conclusion similar to the one in [6]. We recall that for any closed face F of a convex set, its complementary face is denoted by F' (see Chapter 2.6 of [1]). The notation involved in the statement here will be clear during the proof of the theorem.

Theorem 2. *Let K_1, K_2 be Choquet simplexes. Let E be an order unit Banach space and let F be an order unit Banach space with the Riesz decomposition property. Let $\pi\colon A(K_1, E) \to A(K_2, F)$ be a Riesz isomorphism. Suppose $0 \notin a(\partial_e K_1) \Leftrightarrow 0 \notin \pi(a)(\partial_e K_2)$ for every $a \in A(K_1, E)$. Suppose for any affine continuous surjection ψ from the tensor product of K_2 and the state space of F onto K_1 and for any complementary face H' with $\psi(H') = K_1$, there is a closed face $G \subset H'$ such that $\psi(G) = K_1$. A similar condition is imposed on K_1, the state space of E and K_2. Then K_1 is affine homeomorphic to K_2 and E is order unit isometric to F.*

Proof. Since E is an order unit Banach space, by a theorem of Kadison (Theorem II.1.8 in [1]), it follows that there is a compact convex set K_3 such that E is Riesz isomorphic to $A(K_3)$. Similarly since F also has the Riesz decomposition property it follows from Proposition II.3.3 in [1] that F is order unit isometric to $A(K_4)$ for a simplex K_4. We also note that K_3 and K_4 are the state spaces of E and F respectively.

Let $BA(K_1 \times K_3)$ denote the space of continuous functions that are affine in each variable. Define $\psi\colon A(K_1, A(K_3)) \to BA(K_1 \times K_3)$ by $\psi(a)(k_1, k_3) = a(k_1)(k_3)$. It is easy to see that ψ is an order unit isometry. Since K_1 is a simplex, it follows from Corollary 2.6 of [9] that $A(K_1, A(K_3))$ is Riesz isomorphic to $A(K_1 \otimes K_3)$.

On the other hand, it can be directly verified that $A(K_2, A(K_4)) = A(K_2 \otimes K_4)$ has the Riesz decomposition property so that $K_2 \otimes K_4$ is a simplex. Thus $K_1 \otimes K_3$ is a simplex and hence by Proposition 2.10 of [9] we have that K_3 is also a simplex.

Since a Riesz isomorphism is also an isometry, by Theorem 2.1, we may assume without loss of generality that there is an affine homeomorphism ϕ: $K_2 \otimes K_4 \to K_1 \otimes K_3$ such that $\pi(a) = a \circ \phi$. We also recall from Theorem 1.2 of [9] that any extreme point s of $K_2 \otimes K_4$ is of the form $s = k_2 \otimes k_4$ for extreme points k_2 and k_4. We further note that treating $\{k_4\}$ as a face of K_4 and hence a simplex, we have clearly shown that $K_2 \otimes \{k_4\}$ is a face of $K_2 \otimes K_4$. Thus if we can show that there is a unique $k_3 \in \partial_e K_3$ such that $\phi(K_2 \otimes \{k_4\}) = K_1 \otimes \{k_3\}$. Then it will follow that K_2 is affine homeomorphic to K_1 . Similar argument will show that K_4 is affine homeomorphic to K_3 so that F is order unit isometric to E.

In what follows by P_i we denote projection from the simplex tensor product to the corresponding component space. We first show that for any $k_4 \in \partial_e K_4$, $P_1\phi(K_2 \otimes \{k_4\}) = K_1$. Clearly it is enough to show that $\partial_e K_1 \subset P_1\phi(K_2 \otimes \{k_4\})$. Suppose an extreme point $k_1 \notin P_1\phi(K_2 \otimes \{k_4\})$. Since $\{k_1\} \otimes K_3$ and $\phi(K_2 \otimes \{k_4\})$ are disjoint closed faces of a simplex, it follows from Lemma 3.1.3 of [2] that there exists a $a \in A(K_1 \otimes K_3)$ such that $a = 1$ on $\phi(K_2 \otimes \{k_4\})$ and $a = 0$ on $\{k_1\} \otimes K_3$. Thus we have $0 \notin \pi(a)(\partial_e K_2)$ but $0 \in a(\partial_e K_1)$. This contradiction shows that $P_1\phi(K_2 \otimes \{k_4\}) = K_1$.

Now in order to complete the proof along the lines of arguments given during the proof of Lemma 5 in [6] we need to show the following. Since $\{k_4\}$ is a split face of the simplex K_4, by Theorem II.6.22 of [1], there exists a complementary face F' such that $K_4 = \mathrm{CO}(\{k_4\} \cup F')$ (CO denotes the convex hull). We now claim that $P_1\phi(K_2 \otimes F') \neq K_1$.

Suppose $P_1\phi(K_2 \otimes F') = K_1$. Note that $P_1\phi$: $K_2 \otimes K_4 \to K_1$ is an affine continuous onto map. Now by the condition we have assumed on the state spaces, there is a closed face $G \subset K_2 \otimes F'$ such that $P_1\phi(G) = K_1$. Now $\phi(G)$ and $\phi(K_2 \otimes \{k_4\})$ are disjoint closed faces of $K_1 \otimes K_3$. Thus by a separation argument identical to the one given above we get a contradiction. Thus $P_1\phi(K_2 \otimes F') \neq K_1$. This completes the proof. ■

Remark 3. It follows from Lemma 4 of [6] that in the case of continuous function spaces, the additional condition imposed on the state spaces in the above theorem is always satisfied. We do not know if this additional condition always holds in the category of simplex spaces.

Our last result deals with another variation on this theme.

Theorem 4. *Let E is a M space with unit and F be an order unit space. Let $\pi: C(X, E) \to C(Y, F)$ be a Riesz isomorphism. Suppose $0 \notin f(X) \Leftrightarrow 0 \notin \pi(f)(Y)$ for every $f \in C(X, E)$. Then F is a M-space with unit and the component spaces are Riesz isomorphic.*

Proof. By a result of Kakutani [8], E is Riesz isomorphic to $C(Z)$ for a compact set Z, so that $C(X, E) = C(X \times Z)$ is Riesz isomorphic to $C(Y, F)$. It is easy to see that $f \to 1 \otimes f$ is a Riesz isomorphic embedding of F in $C(Y, F)$ and for a fixed $y_0 \in Y$, $g \to \delta(y_0) \circ g$ is a positive, order unit preserving projection of norm one on $C(Y, F)$ onto this embedding. Thus F is Riesz isomorphic to the range of a positive, order unit preserving projection of norm one in the continuous function space $C(X \times Z)$. Let e_0 be the order unit in E. It is easy to see that by the uniqueness of order unit, the order unit in F gets mapped to $1 \otimes e_0$, which is the order unit of $C(X, E)$. Thus F is a M-space with unit. Now the conclusion follows from the main result of [6]. ■

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