The Space of Compact Operators as an $M$–Ideal in its Bidual

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INTRODUCTION.

A Banach space $X$ is said to be an $M$–ideal in its bidual if the canonical decomposition $X^{***} = X^* \oplus X^*$ is an $\ell^1$–direct sum. These spaces enjoy some remarkable topological properties. For example, for any such $X$, $X^*$ has the Radon Nikodym property [9] and $X$ has the Pelczyński property ($\mathcal{N}$) [7] and $X$ is weakly compactly generated [5].

Harmand and Lima [9] have proved that for a reflexive Banach space $X$, if $\mathcal{N}(X)$ the space of compact operators is an $M$–ideal in $\mathcal{L}(X)$ the space of bounded operators then $\mathcal{L}(X)$ is indeed the bidual of $\mathcal{N}(X)$ and hence $\mathcal{N}(X)$ is an $M$–ideal in its bidual. This result has recently been extended in [4] to obtain the same conclusion for $\mathcal{N}(X,Y)$ when $X$ and $Y$ are reflexive Banach spaces and $\mathcal{N}(X,Y)$ is an $M$–ideal in $\mathcal{L}(X,Y)$.

In this paper we exhibit several classes of Banach spaces for which $\mathcal{N}(X,Y)$ is an $M$–ideal in its bidual so that $\mathcal{N}(X,Y)$ enjoys the nice topological properties some of which have been mentioned above. See also [14].

We refer the reader to [2] for relevant definitions and results of $M$–structure theory that we will be using here and the forth coming monograph [10] and its exhaustive bibliography for examples and properties of Banach spaces that are $M$–ideals in their biduals.

We shall be repeatedly making use of the following theorem where part A) has been proved in [9] and part B) very recently in [12].

Theorem. Let $X$ be a Banach space.

A) If $X$ is an $M$–ideal in its bidual then for any closed subspace $Y \subset X$, $Y$ is an $M$–ideal in its bidual.

B) If $X$ is such that every separable Banach subspace of $X$ is an $M$–ideal in
its bidual then $X$ is an $M$–ideal in its bidual.

**Main Results.**

Since $X^*$ and $Y$ are isometric to subspaces of $\mathcal{K}(X,Y)$, by A) of the above theorem we see that for $\mathcal{K}(X,Y)$ to be an $M$–ideal in its bidual it is necessary that both $X^*$ and $Y$ be $M$–ideals in their biduals and appealing to Corollary 3.7 of [9], as was done in [9] we conclude that it is necessary that $X$ is reflexive and $Y$ is an $M$–ideal in its bidual.

We first look at the situation when $X$ and $Y$ are reflexive and present an argument that gives a simple geometric proof of the main result of [4].

**Proposition 1.** Suppose $X$ and $Y$ are reflexive Banach spaces and $\mathcal{K}(X,Y)$ is an $M$–ideal in $\mathcal{L}(X,Y)$ then $\mathcal{L}(X,Y)$ is the bidual of $\mathcal{K}(X,Y)$.

**Proof.** By hypothesis we have

$$\mathcal{L}(X,Y)^* = \mathcal{K}(X,Y)^* \oplus_1 \mathcal{K}(X,Y)^{**}.$$  

However since functionals in the unit ball of $\mathcal{K}(X,Y)^*$ determine the norm of any operator we conclude that the canonical embedding of $\mathcal{L}(X,Y)$ into $\mathcal{K}(X,Y)^{**}$ is an isometry. That this isometry is onto follows from the results of Feder and Saphar [6].

**Theorem 1.** Suppose that $X$ and $Y$ are reflexive Banach spaces and $\mathcal{K}(X,Y)$ is an $M$–ideal in $\mathcal{L}(X,Y)$ and suppose further $X$ has the compact approximation property then for any closed subspace $Z \subset Y$, $\mathcal{K}(X,Z)$ is an $M$–ideal in $\mathcal{L}(X,Z)$ and dually if $Y$ has the compact approximation property then for any closed subspace $M \subset X$, $\mathcal{K}(X/M,Y)$ is an $M$–ideal in $\mathcal{L}(X/M,Y)$.

**Proof.** Since $X$ and $Y$ are reflexive it follows from the results of [6] that

$$\mathcal{K}(X,Y) \subset \mathcal{K}(X,Y)^{**} \subset \mathcal{L}(X,Y).$$

From the hypothesis we known that $\mathcal{K}(X,Y)$ is an $M$–ideal in its bidual.

Since $\mathcal{K}(X,Z) \subset \mathcal{K}(X,Y)$ we conclude that $\mathcal{K}(X,Z)$ is an $M$–ideal in its bidual. Now since $X$ has the compact approximation property, invoking Corollary 1.3 of [8] we get that $\mathcal{K}(X,Z)^{**} = \mathcal{L}(X,Z)$ and hence $\mathcal{K}(X,Z)$ is an $M$–ideal in $\mathcal{L}(X,Z)$.

To see the dual statement we observe first that since $Y$ is reflexive, $Y^*$ has the compact approximation property and the map $T \mapsto T^*$ is an onto isometry.
from the operator spaces $\mathcal{K}(X/M, Y)$ ($\mathcal{L}(X/M, Y)$) and $\mathcal{K}(Y^*,M^*)$ ($\mathcal{L}(Y^*,M^*)$) therefore the conclusion follows from the first part of this theorem and this observation.

**Corollary.** Let $X$ be reflexive and $\mathcal{N}(X)$ an $M$–ideal in $\mathcal{L}(X)$ then for any $Z \subset X$, $\mathcal{N}(X,Z)$ is an $M$–ideal in $\mathcal{L}(X,Z)$ and $\mathcal{N}(X|Z,X)$ is an $M$–ideal in $\mathcal{L}(X|Z,X)$.

**Proof.** It follows from Lemma 5.1 of [9] that $X$ has the compact approximation property.

**Remark.** It should be noted that these conclusion can also be drawn from a more general approach involving properties of compact operator spaces as $M$–ideals, as was done in Proposition 2.9 of [12].

From now on we assume that $Y$ is a non–reflexive space that is an $M$–ideal in its bidual and $X$ is a reflexive Banach space. Note that we still have from the results of Feder and Saphar [6]

$$\mathcal{N}(X,Y)^{**} \subset \mathcal{L}(X,Y^{**}).$$

Let us also note here that $\mathcal{L}(X,Y^{**})$ is isometric to $\mathcal{L}(Y^*,X^*)$ by the map $T \mapsto T^*|Y^*$ (this is true for any Banach spaces $X$ and $Y$).

**Proposition 2.** Let $Y$ be such that for all Banach space $Z$, $\mathcal{N}(Z,Y)$ is an $M$–ideal in $\mathcal{L}(Z,Y)$ then for any reflexive Banach space $X$, $\mathcal{N}(X,Y)$ is an $M$–ideal in its bidual.

**Proof.** The class of Banach spaces $Y$ described above is the so called $M_m$ spaces studied in [13], [10] ($Y$ is non–reflexive when it is infinite dimensional). It follows from the special compact approximation of the identity enjoyed by these spaces (see [10] Chapter 6) that for any such $Y$, $\mathcal{N}(Z,Y)$ is also an $M$–ideal in $\mathcal{L}(Z,Y^{**})$.

Hence when $X$ is a reflexive Banach space from the results of Feder and Saphar alluded to before we have

$$\mathcal{N}(X,Y) \subset \mathcal{N}(X,Y)^{**} \subset \mathcal{L}(X,Y^{**})$$

and hence $\mathcal{N}(X,Y)$ is an $M$–ideal in its bidual.

**Remark.** It is known that the class of $M_m$ spaces is not closed under subspaces, however if $Y \in M_m$ and $Z \subset Y$ is a closed subspace then since $\mathcal{N}(X,Z)$...
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we conclude that \( \mathcal{K}(X,Z) \) is an \( M \)-ideal in its bidual for such a \( Z \) and for any reflexive Banach space \( X \).

The authors in [12] study a class of Banach spaces closely related to the \( M_\infty \) spaces. These are Banach spaces \( Y \) with the property that \( \mathcal{K}(\ell^1, Y) \) is an \( M \)-ideal in \( \mathcal{L}(\ell^1, Y) \). Our final result concerns this class.

**Theorem 2.** Let \( Y \) be a Banach space such that \( Y \) has the compact metric approximation property and \( \mathcal{K}(\ell^1, Y) \) is an \( M \)-ideal in \( \mathcal{L}(\ell^1, Y) \) then for any reflexive Banach space \( X \), \( \mathcal{K}(X,Y) \) is an \( M \)-ideal in its bidual.

**Proof.** In view of B) of the Theorem quoted above, we only need to show that every separable subspace \( S \) of \( \mathcal{K}(X,Y) \) is an \( M \)-ideal in its bidual. Let \( S \subset \mathcal{K}(X,Y) \), \( S \) a separable subspace. W.l.o.g. assume that \( S \subset \mathcal{K}(X,Z) \) where \( Z \subset Y \) and \( Z \) is a separable Banach space. Since the space \( Y \) is an \( M \)-ideal in its bidual ((a) of Theorem 2.12 [12]) it is weakly compactly generated and hence by a result of Amir and Lindenstrauss [1], there is a separable subspace \( Z' \) of \( Y \) which is 1-complemented in \( Y \) such that

\[ Z \subset Z' \subset Y. \]

Note that \( Z' \) has now the metric compact approximation property and \( \mathcal{K}(\ell^1, Z') \) is an \( M \)-ideal in \( \mathcal{L}(\ell^1, Z') \), (see [11]). Therefore by c) Theorem 2.12 [12] we get that \( Z' \) is in the class \( M_\infty \). Hence by the remark made above we conclude that \( \mathcal{K}(X,Z) \) is an \( M \)-ideal in its bidual.

There is a natural way of generating more examples of this class we mention without proof that if \( \{ Y_\alpha \} \) is a family of Banach spaces such that \( \mathcal{K}(X, Y_\alpha) \) is an \( M \)-ideal in its bidual then \( \mathcal{K}(X, \bigoplus \alpha Y_\alpha) \) is an \( M \)-ideal in its bidual.

From what we saw above for reflexive spaces with the compact approximation property, the space of compact operators is an \( M \)-ideal in the bidual is equivalent to the space of compact operator being an \( M \)-ideal in the space of bounded operator. It is well known (see [10]) that for \( X=L^p[0,1], \ p \neq 2, \ K(X) \) is not an \( M \)-ideal in \( L(X) \) and hence \( K(X) \) is not an \( M \)-ideal in its bidual. So by taking \( Y=X \otimes_{\alpha} c_0 \) we get a non-reflexive Banach space that is an \( M \)-ideal in its bidual for which \( K(X,Y) \) is not an \( M \)-ideal in its bidual (I am greatful to Dirk Werner for this remark).

Since the injective tensor product \( X \otimes Y \) of two \( M_\infty \)-spaces \( X \) and \( Y \) is again an \( M_\infty \)-space ([10], Chapter 6), if \( Y \) is as in Theorem 2 and \( X \) a subspace
of an $M_n$-space or a reflexive space then arguments similar to the one given during the proof of Theorem 3 yield that $X \otimes_\varepsilon Y$ is an $M$-ideal in its bidual. The following question is open.

If $Y$ is a subspace of a $M_n$-space, is $X \otimes_\varepsilon Y$ an $M$-ideal in its bidual for any $X$ that is in $M$-ideal in its bidual?

REFERENCES