A SCHRÖDINGER—LIKE PROBABILITY AMPLITUDE DESCRIPTION OF AN ENSEMBLE OF A CLASS OF HAMILTONIAN DYNAMICAL SYSTEMS AND THE CONCEPT OF ENSEMBLE MODES.

RAM K. VARMA
Physical Research Laboratory, Ahmedabad 380 009, India.

ABSTRACT

It is pointed out that the behaviour of certain appropriate ensembles of a class of Hamiltonian systems can be described in terms of a set of probability amplitudes obeying Schrodinger—like equations. Such a description leads to a new concept of "ensemble modes" described by the different equations. Such "ensemble modes" have in fact been observed experimentally recently in a certain classical mechanical system and thus testify to the validity of the probability amplitude description for the classical deterministic system. The possibility of the existence of new modes of quantum behaviour corresponding to the new ensemble modes other than the Schrödinger mode is also pointed out.

INTRODUCTION

THE most important feature which characterises the quantum mechanical description is the assignment of a probability amplitude to every possible "path" in which an event can occur. The "superposition principle" has been accorded the position of a "fundamental principle" and it is thereby required that the equations of motion for the probability amplitude be linear.

It has been widely believed that the quantum mechanical description is intrinsically probabilistic and, therefore, irreducible in terms of, say, a statistical description of a possible underlying deterministic substructure. This is the tenet of the so-called Copenhagen interpretation of quantum mechanics which has recently been widely questioned subsequent to the construction of a so-called "hidden variable" theory of quantum mechanics by Bohm[1] in 1952.

If one assumes that it is possible, in principle, to construct a deterministic substructure underlying the quantum mechanical description in terms of a probability amplitude, then it must be possible to derive the probability amplitude description for at least a certain class of deterministic systems. Attempts have continued in this direction. It is, however, not the purpose of this communication to discuss all such attempts, but merely to review a particular line of investigation that the author has followed in this direction and present it with a more general perspective than he has done so far.

The emphasis in this review will not, therefore, be on the construction of a deterministic theory for quantum mechanics in particular, but on the question whether the ensemble properties of a class of deterministic dynamical systems can be described in terms of a set of probability amplitudes obeying Schrödinger—like equations.

CHARGED PARTICLES IN MAGNETIC MIRROR TRAP

I shall begin by describing a particular dynamical system for which such a description has been found and whose surprising predictions have been experimentally verified. The dynamical system consists of a charged particle moving in an inhomogeneous magnetic field $\mathbf{B}(\mathbf{r})$. The equation describing the motion is:

$$m\frac{d\mathbf{v}}{dt} = \frac{e}{c}\mathbf{v}\times\mathbf{B}(\mathbf{r}) \qquad (1)$$

where $\mathbf{B}(\mathbf{r})$ represents the inhomogeneous magnetic field, e, the charge and m the mass of the particle. The motion, in general, consists of a gyration around the magnetic field and a motion parallel to it.

If the magnetic field, assumed to be static, is a slowly varying function of position, such that

$$\frac{\mathbf{v}_\perp}{\Omega} \frac{\mathrm{d}}{\mathrm{d}x} \ln B \ll 1$$

(where $\mathbf{v}_\perp$ is the component of velocity perpendicular to the magnetic field and $\Omega = eB/mc$ is the cyclotron frequency), then the quasi-periodic gyration around the magnetic field admits of an adiabatic action invariant[2].

$$\mu = 1/2 \; m\mathbf{v}_\perp^2/\Omega \tag{2}$$

If one makes use of the adiabatic invariance of μ an approximate equation for the motion of the particle along the magnetic field lines can be shown to be given by Northrop[2].

$$m\dot{\mathbf{v}}_\parallel = -\mu \nabla_\parallel \Omega \tag{3}$$

where $\mathbf{v}_\parallel$ is the component of the particle velocity parallel to the magnetic field locally. One sees from (3) that $\mu\Omega$ acts as a "potential" for the parallel motion. Since the approximate equation of motion (3) is based on the existence of the approximate (that is, adiabatic) invariance of μ, the potential $\mu\Omega$ is referred to as the *adiabatic* potential. A charged particle can thus be trapped adiabatically in the potential $\mu\Omega$, if it exhibits a well, that is a region of weak magnetic field bounded by regions of stronger magnetic field. The particle trapping will occur if the total energy E of the particle is less than the maximum of the potential hump $(\mu\Omega)_{\max}$, that is, $E < (\mu\Omega)_{\max}$.

The (adiabatic) trapping of a charged particles which results from (3) is, of course, approximate in as much as μ is only an adiabatic invariant. Since violations of the invariance will, in general, occur during the motion, the particles trapped adiabatically at an initial time (that is with $E < \mu\Omega_{\max}$) may leak out of the well at some later time. Such a leakage of particles from these adiabatic magnetic traps has indeed been observed experimentally[3].

Viewed from the frame work of the adiabatic theory the leakage of the adiabatically trapped particles as a consequence purely of the exact motion (and not due to collisions) appears to be analogous to the quantum tunnelling of classically trapped particles. It appears tempting to inquire whether the analogy is real and whether, it is possible to obtain a Schrödinger—like description for what is referred to as the nonadiabatic leakage of particles.

It must be emphasized, to avoid any possible confusion, that the exact motion of the particles in a given magnetic field is determined by (1) and can always be solved, using a computer. The nonadiabatic escape of particles from the adiabatic traps would indeed follow from the exact equation of motion, and such an escape for an individual particle would constitute no surprise. The interesting point, however, is whether from the framework of the adiabatic theory, the nonadiabatic loss can be considered as something analogous to the quantum tunnelling.

Furthermore, one can pose a rather practical problem of theoretically determining the life time against the non adiabatic escape of an ensemble of particles injected into the trap on a certain magnetic field line, and with a given energy E and adiabatic action invariant μ. Even though one knows the exact equation of motion governing the escape, it is a highly nontrivial problem to formulate properly, unless one resorts to an actual integration of the equation of motion for the multitude of initial conditions constituting the ensemble. The latter effort can however, hardly be considered as a theory for the determination of the life time.

The adiabatic theory, according to which a particle can be trapped in the adiabatic potential does provide a framework in terms of which a formulation of the above problem can be given: One can ask the question, what are the ensemble properties of motions governed by (1), but which lie in the neighbourhood of the adiabatic motion (governed by (3))? Since the adiabatic motion, taken strictly, describes trapping, the untrapped motion (no matter how long the life time) must belong to its neighbourhood. The problem was posed by the author in this manner and surprisingly the following set of Schrodinger—like equations were obtained[4] which describe the ensemble properties of exact motions belonging to the neighbourhood of the adiabatic motion.

$$\frac{i\mu}{n}\frac{\partial\Psi(n)}{\partial t} = -\left[\frac{\mu}{n}\right]^2 \frac{1}{2m}\frac{\partial^2\Psi(n)}{\partial x^2} + (\mu\Omega)\Psi(n).$$

$$n = 1,2,3\ldots\ldots \tag{4a}$$

$$G(x,t) = \sum_n \Psi^*(n)\Psi(n) \tag{4b}$$

A more rigorous recent derivation is given the Appendix.

We see that the adiabatic potential $(\mu\Omega)$ appears in the place where the potential of classical mechanics occurs in the Schrodinger equation, and the initial value of the action invariant μ, appears in the role of $\hbar$.

While (4a) are completely analogous to the Schrodinger equation of quantum mechanics, there is one major difference namely that we have here an infinite set of equations for $n = 1,2,3\ldots\ldots$, for what may be termed as the modes of the ensemble, or "ensemble-modes". This is a very significant new point. We thus see that the ensemble of particles is indeed describable in terms of (not one but) an infinite set of probability amplitudes $\Psi(n)$ and the total probability density $G(x,t)$ is given according (4b) as a sum of $\Psi^*(n)$ $\Psi(n)$ summed over all the modes.

Clearly these equations enable us to calculate the life times in the adiabatic traps in a manner similar to that in quantum mechanics, and the process of nonadiabatic loss can indeed be looked upon as analogous to quantum tunnelling, the tunnelling here of the (approximate) adiabatic potential. The life time can, of course, be calculated (for a particular form for $\Omega(x)$) using the different equations corresponding to $n = 1,2,3\ldots\ldots$ for the given values of energy E and the action μ. Different expressions for the life times will result corresponding to different values of n. This leads to the somewhat surprising conclusion and prediction that for an ensemble of particles specified by the same values of energy E and action μ, and differing only in the initial value of the Larmer phase ϕ, different groups of particles possess different life times τ_1, τ_2, $\tau_3\ldots$ corresponding to $n = 1,2,3\ldots\ldots$

If one assumes for Ω the form:

$$\Omega = \Omega_0 + (\Omega_{max} - \Omega_0)\cosh^2(\alpha x) \tag{5}$$

then the life times τ_n ($n = 1,2,3\ldots\ldots$) are obtained as

$$\tau_n = T_n \exp\left[(2m)^{1/2}\frac{2\pi n^?}{\alpha\sqrt{E}}\left\{\sin\theta\left(\frac{B_m}{B_0} - 1\right)^{1/2} - \cos\theta\right\}\right] \tag{6}$$

where θ is the initial value of the pitch angle of the particles defined through:

$$\mu = E\sin^2\theta/\Omega$$

and T_n is some effective bounce time in the trap. Experiments carried out at the Physical Research Laboratory[5-7] have definitely estab-

TABLE 1

Variation of the slopes m_1 and m_2 as a function of total energy E and magnetic field scale length α^{-1} at a pitch angle value of $\theta \simeq 35°$.

E (keV)	α^{-1} (cm)	m_1	m_2	m_2/m_1
2.2	8	$(5.56 \pm 0.18)10^{-3}$	$(11.9 \pm 0.5)10^{-3}$	2.14 ± 0.03
2.9	8	$(4.9 \pm 0,24)10^{-3}$	$(10.46 \pm 0.7)10^{-3}$	2.12 ± 0.04
3.7	8	$(4.54 \pm 0.3)10^{-3}$	$(9.95 \pm 0.68)10^{-3}$	2.15 ± 0.6
4.5	8	$(3.96 \pm 0.42)10^{-3}$	$(9.15 \pm 0.7)10^{-3}$	2.31 ± 0.1
2.9	11	$(6.16 \pm 0.18)10^{-3}$	$(12.62 \pm 0.62)10^{-3}$	2.05 ± 0.03
2.9	13	$(8.24 \pm 0.3)10^{-3}$	$(17.2 \pm 0.53)10^{-3}$	2.09 ± 0.038

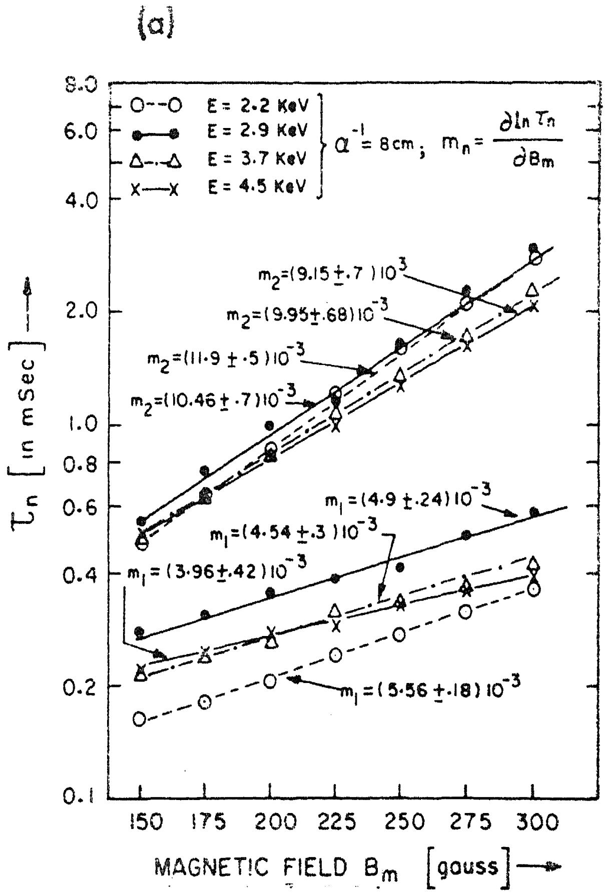
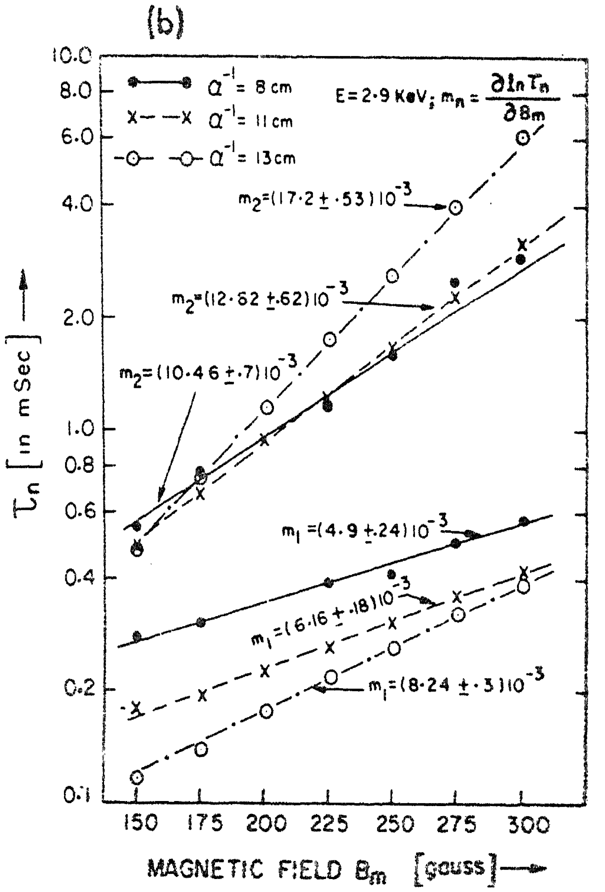


Figure 1a.



Figures 1a & b. Dependence of log τ_n on B_{max} (magnetic field at the mirror throat) (a) For different energies of the particles, but same magnetic field scale length $\alpha^{-1} = 8$ cm. (b) For different magnetic field scale lengths, but same energy $E = 2.9$ keV.

lished the existence of at least two distinct life times in the nonadiabatic decay of particles corresponding to $n = 1$ and 2. The life times τ_1 and τ_2 were, furthermore, found to vary with the various parameters in accordance with the expression (6). Of particular interest is the dependence on the integer n.

To study this dependence note that from (6)

$$\ln \tau_n = \ln \tau_n + n\beta B \tag{7}$$

where

$$\beta = (2m)^{1/2} \frac{2\pi}{\alpha\sqrt{E}} \frac{e}{mc}\left[\sin\theta\left(\frac{B_m}{B_0} - 1\right)^{1/2} - \cos\theta\right] \tag{8}$$

$\ln \tau_n$ then plotted as a function of B would give straight lines with slopes $n\beta$, as β is a constant with respect to B. Clearly from (7) the slope of $\ln \tau_2$ *vs.* B is expected, according to the theory, to be twice the slope of the $\ln \tau_1$ *vs* B line. Figures 1(a) and (b) give the experimental curves for $\ln \tau$ *vs* B for (a) different values of the energy, but the same magnetic field scale length $\alpha^{-1} = 8$ cm, and (b) same value of the energy 2.9 keV but different scale lengths $\alpha^{-1} = 8$, 11 and 13. All these results which are tabulated in table 1 correspond to the same value of the pitch angle, namely $\theta \simeq 35°$. Similar results have been obtained for another value of the pitch angle $\theta \simeq 33°$.

We see from table that the ratio of the slopes of $\ln \tau_2$ *vs* B and $\ln \tau_1$ *vs* B lines lie very close to the theoretically predicted value of 2 except for one value, 2.3 which is somewhat larger but not unreasonable. Experiments have also yielded in some cases three life times. One such case is shown in figure 2 where the slopes for the three life times are found to be in the ratio 1:2.3:3.2 against the theoretically expected ratio 1:2:3.

Apart from the dependence on the mode number n, the dependences on the energy E, the scale length α^{-1} and the pitch angle θ have also been found to agree well with the theoretical prediction.

SOME GENERAL REMARKS

We would like to draw 3 major conclusions from here:

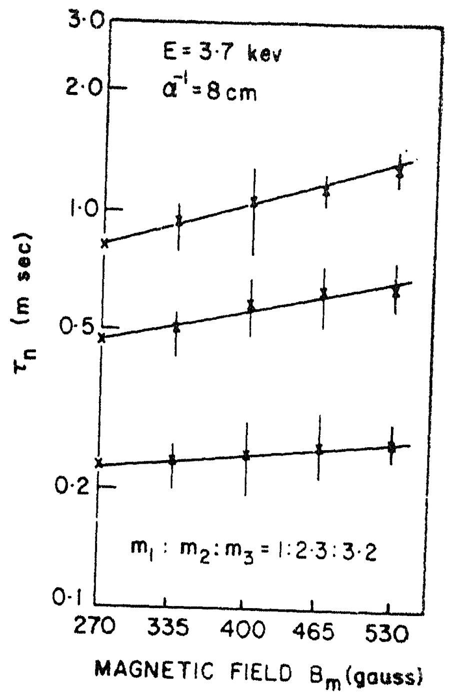


Figure 2. Three life times τ_1, τ_2, τ_3 plotted as $\log \tau_n$ vs B_{max}, the magnetic field at the mirror throat.

1. The behaviour of an ensemble of a certain Hamiltonian dynamical system can be meaningfully described in terms of a set of probability amplitudes obeying Schrodinger—like equations. This shows that a Schrodinger—like description in terms of probability amplitudes is not the sole preserve of quantum mechanics as has been generally believed.
2. The description, as governed by the set of equations (4a,& b), represents in fact, a generalized Schrodinger description. It leads to a new concept of "ensemble modes" of behaviour corresponding to the mode numbers $n = 1,2,3, \ldots\ldots$ In the case discussed above such modes are manifested through the observation of multiple life times. This is an important property of the ensemble of the dynamical system and indeed represents a surprising behaviour. It is surprising since the different members of the ensemble which are apparently independent of each other appear to group together to exhibit the model behaviour which is generally indicative of the existence of correlations. It may be emphasized that the concept of "ensemble modes" is distinct from that of the collective modes.
3. It would appear from the considerations that led to the Eqs. (4a & b) that this kind of generalized Schrodinger—like description may hold more generally for a class of Hamiltonian systems typified by the physical example (of the motion of a charged particle in an inhomogenous magnetic field) given above. Another example of a system belonging to such a class has been considered by the author[8] in his attempt to give a deterministic model for quantum mechanics, where a similar system of generalized set of Schrodinger—like equations have been obtained.

The situations where such a description obtains are characterized by the existence of quasi-cyclic variable q corresponding to which an adiabatic action invariant exists. The elimination of the corresponding "velocity" $\dot{q}$ then leads to a Routhian, which is a Lagrangian for the remaining degrees of freedom and with a (fictitious) potential which is a function of the remaining active coordinates. This Routhian, then describes the approximate motion (termed the "adiabatic" motion) as if the adiabatic invariant were an exact constant of motion. The exact motion (without the use of the adiabatic invariant), however, would show departures from the approximate (adiabatic) motion.

If we now construct an ensemble which is specified by different initial values of the quasicyclic variable q, but the same values for the rest of the initial data, then the Schrodinger like equations discussed here describe the behaviour of this ensemble of motions in the neighbourhood of the adiabatic motion.

APPLICATION TO QUANTUM MECHANICS

In view of the existence of such a description of a class of certain deterministic dynamical systems it appears natural to look for a deterministic model for quantum mechanics along these lines. As is already mentioned, the author has explored such a model recently[8] where a quasi-cyclic angular coordinate is introduced as an angle canonical to the quantum of action h. The members of the ensemble of quantum mechanics then correspond to the different initial values of the quasi-cyclic angular coordinate.

I do not wish to enter into a discussion here as to whether this model constitutes a valid representation of quantum mechanical systems. Many important questions remain to be explored and answered in this connection. I only like to point out an important aspect of the model namely that it has all along employed classical probability to describe the ensemble which has eventually led to the Schrodinger—like description in terms of a set of probability amplitudes. This is in contrast to the work of Jauch[9] and his collaborators who find it necessary to introduce the concept of quantum probability (on a non-Boolean lattice) as distinct from classical probability to describe quantum events. It is of considerable interest to investigate whether the other new modes (the non-Schrodinger modes, $n = 2,3$ etc.) of propagation of probability that we have obtained must be present (even if in small amounts) to supplement the Schrodinger mode $n = 1$ in order for the classical probability to adequately describe quantum events.

The other interesting and important aspect of our deterministic model is, of course, the fact that it does not just reproduce the Schrödinger equation but predicts new modes for the behaviour of quantum systems. As we have discussed above the physical reality of such modes has already been established through the observation of multiple life times for the case of charged particles in magnetic mirror traps[5-7]. One could similarly look for multiple life times (of quantum particles in finite potential wells) which are predicted by our equations. We are currently planning such experiments. If observed, these multiple life times would point towards the existence of a deterministic substructure underlying the probabilistic quantum mechanical description.

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