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Spherical Means on the Heisenberg Group and a Restriction Theorem for the Symplectic Fourier Transform

S. Thangavelu

Dedicated to Prof. E. M. Stein on his 60th birthday

1. Introduction

Spherical mean value operators on a compact Riemannian manifold M have been extensively studied by Sunada in a series of papers [14], [15] and [16]. He has studied the eigenvalue problem $L_r f = \alpha f$ associated with the spherical mean value operator L_r. The question about the eigenvalues $\alpha = 1$ and $\alpha = -1$ are related to the ergodicity and mixing properties of the geodesic random walk of step size r on the manifold M. In a recent article [7] Pati-Shahshahani-Sitaram have investigated the eigenvalue problem in the case when M is a compact symmetric space. In this case they are able to identify the eigenvalues completely in terms of the elementary spherical functions associated to M. They provide alternate proofs of some results of Sunada regarding the eigenvalues 1 and -1. Let us briefly recall their result.

Let $M = G/K$ be a compact symmetric space. Then the spherical mean value operator L_r can be identified with a convolution operator $L_r f = f * \nu_r$ where ν_r is a certain probability measure which can be viewed as a K-biinvariant measure on G. Let $\hat{G}_1$ denote the collection of all pairwise inequivalent, irreducible, unitary representations of class 1 of G. For each $\pi \in \hat{G}_1$ there is an elementary spherical function ϕ_π associated with it. The main result of [7] can now be stated as follows.

Theorem 0. *All the eigenvalues of the operator L_r are of the form*

$$\psi_\pi(r) = \int_G \phi_\pi(x)\, d\nu_r$$

where $\pi \in \hat{G}_1$.

The aim of this paper is to study spherical mean value operators on the reduced Heisenberg group H^n/Γ. Here H^n is the Heisenberg group and Γ is the subgroup $\{(0, 2\pi k): k \in \mathbb{Z}\}$ of H^n. The Heisenberg group H^n is a nilpotent Lie group whose underlying manifold is $\mathbb{C}^n \times \mathbb{R}$. The coordinates on H^n are (z, t) where $z = x + iy$ with $x, y \in \mathbb{R}^n$ and $t \in \mathbb{R}$. The group law is defined by

$$(z,t)(w,s) = \left(z + w, t + s + \frac{1}{2}\operatorname{Im} z\bar{w}\right).$$

The Haar measure in H^n is the Lebesgue measure $dz\,dt$. The group H^n/Γ is a nilpotent Lie group with compact centre. A function on H^n is said to be radial or rotation invariant if it is invariant under rotations in the variable z.

By a spherical mean value operator we mean an operator of the form $T_\mu f = f * \mu$ where μ is a rotation invariant compactly supported probability measure on H^n/Γ. We are able to identify all the eigenvalues of the operator T_μ. For each $k = 0, 1, 2, \ldots$ and $\lambda \neq 0$ there are certain radial functions e_k^λ on the Heisenberg group H^n which can be thought of as the elementary spherical functions for the Heisenberg group. As in the case of the compact symmetric space, the eigenvalues are then given by the averages of e_k^λ with respect to μ.

Theorem 1. *Assume that μ has no mass at the centre of H^n/Γ. Then all the eigenvalues of the operator T_μ are given by*

$$\alpha_k(j) = \int_{H^n/\Gamma} e_k^j(z,t)\, d\mu,$$

where j is an integer. Further, any function of the form $f * e_k^{-j}$ satisfies

$$T_\mu(f * e_k^{-j}) = \alpha_k(j)(f * e_k^{-j}).$$

We can make more precise statements regarding the eigenvalues if we take $\mu = \mu_{r,t}$ where $\mu_{r,t}$ is the normalized Lebesgue (surface) measure on the sphere $S_{r,t} = \{(z,t): |z| = r\}$ in H^n/Γ. Let $M_{r,t}$ stand for T_μ when $\mu = \mu_{r,t}$. The elementary spherical functions $e_k^j(z, s)$ are radial functions of z and slightly abusing the notation we write $e_k^j(r, s)$ in place of $e_k^j(z, s)$ when $|z| = r$.

Theorem 2.
(i) *All the eigenvalues of the operator $M_{r,t}$ are given by $\alpha_k(j) = e_k^j(r, t)$.*
(ii) $\alpha = 1$ *and* $\alpha = -1$ *are not eigenvalues of the operator $M_{r,t}$ for any $r > 0$.*

By writing down the Fourier series of $f * \mu_{r,t}$ we can see that it involves operators of the form $g \times \mu_r$ where $g \times \mu_r$ is the twisted convolution of g with the surface measure on the sphere $|z| = r$ in $\mathbb{C}^n$. The spectral properties of the operator $T_r g = (2\pi)^n g \times \mu_r$ are worth studying and we have the following theorem.

Theorem 3.

(i) *All the eigenvalues of the operator T_r are given by*

$$\alpha_k = \frac{k!\,(n-1)!}{(k+n-1)!}\phi_k(r)$$

where ϕ_k are the Laguerre functions of type $(n-1)$.
(ii) *For each k the eigenspace corresponding to the eigenvalue α_k is infinite dimensional; hence the operator T_r is not compact.*
(iii) $\alpha = 1$ *and* $\alpha = -1$ *are not eigenvalues of T_r and $\alpha = 0$ is an eigenvalue if and only if $\phi_k(r) = 0$ for some k.*

The operators T_r also arise naturally in connection with certain restriction operators R_r for the symplectic Fourier transform on $\mathbb{R}^{2n}$. In Section 5 we will show that we can write

$$f(z) = (4\pi)^{-2n}\omega_{2n}\int_0^\infty R_r f(z) r^{2n-1}\,dr$$

where R_r *are the restriction operators. These restriction operators* R_r are related to T_r by $R_r f = (2\pi)^{-n} T_r(\mathfrak{F}_s f)$ where $\mathfrak{F}_s f$ is the symplectic Fourier transform of f. Using the above relation and the spectral properties of T_r we are able to prove the following theorem regarding the mapping properties of the restriction operators R_r.

Theorem 4. *Assume that $n \geqslant 3$. Then the following are true.*

(i) $\|R_r f\|_{p'} \leqslant C_r \|f\|_p$, *for* $1 \leqslant p \leqslant 2$,

(ii) $\|R_r f\|_p \leqslant C_r \|f\|_p$, *for* $\dfrac{2n}{n+1} \leqslant p \leqslant 2$,

(iii) $\|R_r f\|_q \leqslant C_r \|f\|_p$, *for* $1 \leqslant p \leqslant \frac{2n}{n+1}$, $\quad q = \frac{n-1}{n+1} p'$.

To prove this theorem we need to use some mapping properties of the projection operators associated with special Hermite expansions. We will also show that the operators R_r are regularising in the sense that they take $L^2(\mathbb{C}^n)$ into $\mathcal{W}^s(\mathbb{C}^n)$ where $\mathcal{W}^s(\mathbb{C}^n)$ are the twisted Sobolev spaces to be defined in the sequel. The plan of the paper is as follows. In the next section we will define the functións e_k^λ and show that they have all the properties satisfied by the elementary spherical functions. In Section 3 we will prove Theorem 1. The spectral properties of T_r will be taken up in Section 4 and finally the restriction operators will be studied in Section 5.

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2. Elementary Spherical Functions on the Heisenberg Group

Let us briefly recall the definition and properties of elementary spherical functions. Let G be a semisimple, noncompact, connected Lie group with finite centre and K a maximal compact subgroup. Let $C_c(K\backslash G/K)$ denote the space of continuous functions with compact support on G which satisfy $f(k_1 g k_2) = f(g)$ for all k_1, k_2 in K. Such functions are called spherical or K-biinvariant. Then $C_c(K\backslash G/K)$ forms a commutative Banach algebra under convolution. An elementary spherical function ϕ is then defined to be a K-biinvariant continuous function with $\phi(e) = 1$ such that $f \to f * \phi(e)$ defines an algebra homomorphism of $C_c(K\backslash G/K)$.

The elementary spherical functions are characterised by the following properties (see [3]).

(i) They are eigenfunctions of the convolution operator:

$$f * \phi = \hat{\phi}(f)\phi,$$

where

$$\hat{\phi}(f) = \int_G f(x^{-1})\phi(x)\,dx.$$

(ii) They are eigenfunctions for a large class of left invariant differential operators on G.

(iii) They satisfy

$$\int_K \phi(xky)\,dk = \phi(x)\phi(y).$$

Let us now consider the case of the Heisenberg group H^n. The role of the K-biinvariant functions will be played by the radial functions on H^n. If $L^1_{\text{rad}}(H^n)$ stand for the subspace of $L^1(H^n)$ containing all the radial functions then it is well known that L^1_{rad} is a commutative Banach algebra under convolution (see Hulanicki-Ricci [5]). This will play the role of $C_c(K\backslash G/K)$. On the Heisenberg group we have the following $(2n+1)$ left invariant vector fields X_j, Y_j, T:

$$X_j = \frac{\partial}{\partial x_j} - \frac{1}{2}y_j\frac{\partial}{\partial t}, \qquad Y_j = \frac{\partial}{\partial y_j} + \frac{1}{2}x_j\frac{\partial}{\partial t}, \qquad T = \frac{\partial}{\partial t}.$$

The sublaplacian on the Heisenberg group is defined by

$$\mathcal{L} = -\sum_{j=1}^{n}(X_j^2 + Y_j^2).$$

Let ϕ_k be the Laguerre functions of type $(n-1)$ defined by the generating function identity

$$\sum_{k=0}^{\infty} r^k\phi_k(z) = (1-r)^{-n}e^{-(1/4)(1+r)/(1-r)|z|^2}. \tag{2.1}$$

For any nonzero real number λ we set $\phi_k^\lambda(z) = \phi_k(|\lambda|^{1/2}z)$ and define e_k^λ by

$$e_k^\lambda(z,t) = \frac{k!\,(n-1)!}{(k+n-1)!}e^{i\lambda t}\phi_k^\lambda(z). \tag{2.2}$$

It follows from the properties of the Laguerre functions (see Szego [17]) that $e_k^\lambda(0,0) = 1$. We claim that these functions satisfy the following properties.

Theorem 2.1.

(i) *For any polynomial p with constant coefficients one has*

$$p(\mathcal{L})\,e_k^\lambda = p(2|\lambda|(2k+n))\,e_k^\lambda. \tag{2.3}$$

(ii) *For any radial function f on H^n one has*

$$f * e_k^\lambda = (2\pi)^n R_k(-\lambda, f)\,e_k^\lambda \tag{2.4}$$

where $R_k(\lambda, f)$ is defined by the formula

$$R_k(\lambda,f) = (2\pi)^{-n}\frac{k!\,(n-1)!}{(k+n-1)!}\int_{\mathbb{C}^n}\tilde{f}(z,\lambda)\phi_k^\lambda(z)\,dz, \tag{2.5}$$

$\tilde{f}(z, \lambda)$ *being the inverse Fourier transform*

$$\tilde{f}(z, \lambda) = \int_{-\infty}^{\infty} e^{i\lambda t} f(z, t)\, dt.$$

(iii) *For any* (w, s) *in* H^n *with* $|w| = r$ *one has the identity*

$$\int_{|w'|=1} e_k^{\lambda}((z, t) \cdot (-rw', -s))\, d\sigma(w') = e_k^{\lambda}(z, t)\, e_k^{-\lambda}(w, s)$$

where $d\sigma$ *is the normalized surface measure on* $|w'| = 1$.

Thus we see that the functions e_k^{λ} have all the properties satisfied by the elementary spherical functions on a semisimple Lie group. So, they can be rightly called the elementary spherical functions for the Heisenberg group. The above properties of the function e_k^{λ} are fairly wellknown in the literature though not stated in the above form (see *e.g.* Stempak [12] and Strichartz [13]). Nevertheless, we will give a proof of the above theorem here.

To prove the theorem we need to recall several facts about the twisted convolution and the Weyl transform (see Folland [2], Mauceri [6] and Peetre [8]). The twisted convolution of two functions f and g defined on $\mathbb{C}^n$ is defined to be

$$f \times g(z) = \int_{\mathbb{C}^n} f(z - w) g(w)\, e^{i/2 \operatorname{Im} z\bar{w}}\, dw. \tag{2.7}$$

The Weyl transform of a function f is the bounded operator $W(f)$ acting on $L^2(\mathbb{R}^n)$ given by

$$W(f)\phi(\xi) = \int_{\mathbb{C}^n} f(z) W(z)\phi(\xi)\, dz \tag{2.8}$$

where $\phi \in L^2(\mathbb{R}^n)$ and $W(z)$ is the operator valued function

$$W(z)\phi(\xi) = e^{i(x\xi + xy/2)}\phi(\xi + y). \tag{2.9}$$

The relation between the Weyl transform and the twisted convolution is given by $W(f \times g) = W(f)W(g)$.

The Hermite functions $\Phi_\alpha(x)$ play an important role in the harmonic analysis on the Heisenberg group (see Folland [2]). These are eigenfunctions of the Hermite operator $H = (-\Delta + |x|^2)$, $H\Phi_\alpha = (2|\alpha| + n)\Phi_\alpha$. Let P_k be the orthogonal projection of $L^2(\mathbb{R}^n)$ onto the k^{th} eigenspace spanned by $\{\Phi_\alpha : |\alpha| = k\}$. We also need certain properties of the special Hermite functions. Let us define

$$\Phi_{\alpha\beta}(z) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\xi}\Phi_\alpha\left(\xi + \frac{y}{2}\right)\Phi_\beta\left(\xi - \frac{y}{2}\right) d\xi. \tag{2.10}$$

Then it is well known that they form a complete orthonormal system for $L^2(\mathbb{C}^n)$ (see Strichartz [13]). Let $\pi(\alpha, \beta)$ denote the operator defined by

$$\pi(\alpha, \beta)\phi = (\phi, \Phi_\alpha)\Phi_\beta .$$

Then one has the following proposition (see Folland [2] and Peetre [8]).

Proposition 2.1.

(i) $W(\bar{\Phi}_{\alpha\beta}) = (2\pi)^{n/2}\pi(\alpha, \beta)$ *and consequently* $\Phi_{\alpha\beta} \times \Phi_{\nu\delta} = 0$ *if* $\beta \neq \nu$ *and* $\Phi_{\alpha\beta} \times \Phi_{\beta\delta} = (2\pi)^{n/2}\Phi_{\alpha\delta}$.

(ii) $W(\phi_k) = (2\pi)^n P_k$ *and consequently* $\phi_j \times \phi_k = (2\pi)^n \delta_{jk}\phi_k$ *where* δ_{jk} *is the Kronecker* δ.

By abusing the notation slightly let us write $\phi_k(r)$ in place of $\phi_k(z)$ when $|z| = r$. Then the functions

$$\psi_k(r) = \left(\frac{2^{1-n}k!}{(k+n-1)!}\right)^{1/2} \phi_k(r)$$

form a complete orthonormal system in $L^2(\mathbb{R}_+, r^{2n-1}\,dr)$. If f is a radial function on $\mathbb{C}^n$ then we can expand f interms of $\psi_k(r)$ obtaining

$$f(z) = \sum_{k=0}^{\infty} R_k(f)\phi_k \tag{2.11}$$

where

$$R_k(f) = (2\pi)^{-n}\frac{k!\,(n-1)!}{(k+n-1)!}\int_{\mathbb{C}^n} f(z)\phi_k(z)\,dz.$$

This proves that when f is a radial function one has $f \times \phi_k = (2\pi)^n R_k(f)\phi_k$.

Now we are ready to prove Theorem 2.1. The assertion (i) is already proved in Strichartz [13] and so we will not prove it here. For (ii) an easy calculation reveals that

$$f * e_k^\lambda(z, t) = \frac{k!\,(n-1)!}{(k+n-1)!}e^{i\lambda t}\int_{\mathbb{C}^n} \tilde{f}(z-w, -\lambda)\phi_k^\lambda(w)\,e^{-i(\lambda/2)\operatorname{Im} z\bar{w}}\,dw.$$

It is therefore enough to show that the above integral is equal to $(2\pi)^n R_k(-\lambda, f)$ $\phi_k^\lambda(z)$. By rescaling we can assume that $\lambda = -1$. But then we need to show that

$$\int_{\mathbb{C}^n} \tilde{f}(z-w, 1)\phi_k(w)\,e^{i/2\operatorname{Im} z\bar{w}}\,dw = (2\pi)^n R_k(1, f)\phi_k(z)$$

which follows from the above remark as f is radial.

The proof of the assertion (iii) is similar. We have

$$\int_{|w'|=1} e_k^\lambda\left(z - w, t - s - \frac{1}{2}\operatorname{Im} z\bar{w}\right) d\sigma(w')$$
$$= \frac{k!\,(n-1)!}{(k+n-1)!} e^{i\lambda(t-s)} \int_{|w'|=1} \phi_k^\lambda(z - w)\, e^{-i(\lambda/2)\operatorname{Im} z\bar{w}}\, d\sigma(w').$$

Again we can assume that $\lambda = -1$. The function

$$F_k(z, w) = \int_{|w'|=1} \phi_k(z - w)\, e^{i/2 \operatorname{Im} z\bar{w}}\, d\sigma(w')$$

is a radial function of w and hence in view of (2.11)

$$F_k(z, w) = \sum_{j=0}^{\infty} R_j(F_k)\phi_j(w).$$

But

$$R_j(F_k) = (2\pi)^{-n} \frac{k!\,(n-1)!}{(k+n-1)!} \int_{\mathbb{C}^n} F_k(z, w)\phi_j(w)\, dw$$
$$= (2\pi)^{-n} \frac{k!\,(n-1)!}{(k+n-1)!} \phi_k \times \phi_j(z).$$

This proves that

$$F_k(z, w) = \frac{k!\,(n-1)!}{(k+n-1)!} \phi_k(z)\phi_k(w).$$

We have proved that

$$\int_{|w'|=1} e_k^\lambda\left(z - w, t - s - \frac{1}{2}\operatorname{Im} z\bar{w}\right) d\sigma(w') = e_k^\lambda(z, t)\, e_k^{-\lambda}(w, s).$$

Hence the theorem.

We would like to end this section with the following remark. Recently Benson-Jenkins-Ratcliff [1] has studied «spherical functions» on the Heisenberg group. Let K be a compact group of automorphisms of H^n such that the convolution algebra L_K^1 of K-invariant functions is commutative. A bounded, continuous K-invariant function φ such that $f \to \int f\varphi$ is an algebra homomorphism on L_K^1 is called a K-spherical function. In [1] the authors have studied the K-spherical functions for various different K. When $K = U(n)$, the K-spherical functions include our e_k^λ. (We are indebted to G. B. Folland and the referee for bringing the above work to our attention.)

3. Spherical Mean Value Operator on the Heisenberg Group

Let H^n be the n dimensional Heisenberg group defined in the previous section. Let Γ be the discrete subgroup $\{(0,0,2\pi k): k \in \mathbb{Z}\}$. Then the quotient group H^n/Γ is called the reduced Heisenberg group. For $g \in SO(2n, \mathbb{R})$ we define a rotation

$$\bar{g}: H^n/\Gamma \to H^n/\Gamma \quad \text{by} \quad \bar{g}(z,t) = (gz, t).$$

By a radial measure we mean a measure μ such that for every $g \in SO(2n, \mathbb{R})$ and every Borel set $S \subset H^n/\Gamma$ one has $\mu(S) = \mu(g^{-1}S)$. Let μ be such a rotation invariant probability measure with compact support. Then the operator $T_\mu f = (f * \mu)$ is called a spherical mean value operator. In the following theorem we identify all eigenvalues of T_μ as averages of the elementary spherical functions e_k^j as claimed in the introduction.

Theorem 3.1. *Assume that μ has no mass at the centre of H^n/Γ. Then all the eigenvalues of the operator T_μ are given by $\alpha_k(j)$ where*

$$\alpha_k(j) = \int_{H^n/\Gamma} e_k^j(z,t)\, d\mu.$$

Any function of the form $f * e_k^{-j}$ is an eigenfunction corresponding to $\alpha_k(j)$.

To prove this theorem we need to recall several results about the Fourier transform on the Heisenberg group (a good reference is Geller [4]). For each real $\lambda \neq 0$ we have an irreducible representation $\pi_\lambda(z,t)$ acting on $L^2(\mathbb{R}^n)$. It is defined by

$$\pi_\lambda(z,t)\phi(\xi) = e^{i\lambda t} e^{i\lambda(x\cdot\xi + 1/2x\cdot y)}\phi(\xi + y). \tag{3.1}$$

The Fourier transform of a function f on H^n is the operator valued function $\hat{f}(\lambda)$ defined by

$$\hat{f}(\lambda) = \int_{H^n} f(z,t)\pi_\lambda(z,t)\, dz\, dt.$$

When f is a radial function $\hat{f}(\lambda)$ is given by the formula

$$\hat{f}(\lambda) = (2\pi)^n \sum_{k=0}^{\infty} R_k(\lambda, f) P_k(\lambda). \tag{3.2}$$

Here $R_k(\lambda, f)$ are as defined in (2.5) and $P_k(\lambda)$ are the projections of $L^2(\mathbb{R}^n)$ onto the space spanned by $\{|\lambda|^{n/4}\Phi_\alpha(|\lambda|^{1/2}x): |\alpha| = k\}$. In the case of H^n/Γ, π_λ is a representation only if λ is an integer say $\lambda = j$.

Let χ_k be the function defined by

$$\chi_k(z, t) = (2\pi)^{-n} \sum_{j=-\infty}^{\infty} e^{-ijt} e^{-j^2/2} \phi_k^j(z) |j|^n. \tag{3.3}$$

Then it is easy to see that $\chi_k \in L^2(H^n)$ and $\hat{\chi}_k(j) = e^{-j^2/2} P_k(j)$. We can now calculate the convolution $\mu * \chi_k$.

Lemma 3.1. *Let μ be a rotation invariant probability measure and let χ_k be as above. Then $(\mu * \chi_k)^\wedge(j) = \alpha_k(j)\hat{\chi}_k(j)$.*

PROOF. Since $\mu * \chi_k$ is a radial function we can calculate the Fourier transform using formula (3.2). A calculation shows that

$$\int e^{ijt} \mu * \chi_k(z, t)\, dt = (2\pi)^{-n} |j|^n e^{-j^2/2} \int_{\mathbb{C}^n} \phi_k^j(z - w) e^{-i(j/2)\operatorname{Im} z\bar{w}} d\mu^j(w)$$

where $\mu^j(w)$ is the j-th Fourier coefficient of μ in the t variable. It follows that

$$R_k(j, \mu * \chi_k) = (2\pi)^{-n} \frac{k!\,(n-1)!}{(k+n-1)!} e^{-j^2/2} \int_{\mathbb{C}^n} \phi_k^j(w)\, d\mu^j(w)$$

and $R_i(j, \mu * \chi_k) = 0$ for $i \neq k$. This shows that

$$(\mu * \chi_k)^\wedge(j) = e^{-j^2/2} \Big(\int_{H^n/\Gamma} e_k^j(z, t)\, d\mu \Big) P_k(j).$$

This completes the proof of Lemma 3.1.

We can now prove the first part of Theorem 3.1. We claim that there exists k and $j \neq 0$ such that $(f * \chi_k)^\wedge(j) \neq 0$. Assuming the claim for a moment we will prove the theorem. Let $f * \mu = \alpha f$ for a non zero f in $L^2(H^n/\Gamma)$. Then in view of the lemma $\alpha_k(j)(f * \chi_k)^\wedge(j) = (f * \mu * \chi_k)^\wedge(j) = \alpha(f * \chi_k)^\wedge(j)$. This proves that $\alpha = \alpha_k(j)$ as $(f * \chi_k)^\wedge(j) \neq 0$.

We will now prove the claim. If $(f * \chi_k)^\wedge(j) = 0$ for all k and $j \neq 0$ then calculating the Fourier coefficients of $f * \chi_k$ one can see that all the Fourier coefficients of f except the zero-th one are zero and consequently

$$f(z, t) = Af(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z, t)\, dt.$$

As $f \neq 0$, $Af \neq 0$ and $f * \mu = \alpha f$ becomes $Af * \mu_0 = \alpha Af$ where now the convolution is on $\mathbb{R}^{2n}$ and μ_0 is the compactly supported measure

$$d\mu_0(z) = \int_0^{2\pi} d\mu(z, t).$$

But then Af has to be zero which is a contradiction.

To prove the second part of the theorem we need to recall Choquet's theorem. Let K^n stand for the set of all rotation invariant probability measures on H^n/Γ. Let $\mathrm{ext}(K^n)$ stand for the set of all extreme points of K^n. Then one has $\mathrm{ext}(K^n) = E \cup \Delta_n$ where $E = \{\mu_{r,t}\}$ and $\Delta_n = \{\delta_t : t \in \mathbb{R}\}$. Here $\mu_{r,t}$, $r > 0$ is the normalized Lebesgue measure on the sphere $S_{r,t} = \{(z,t) : |z| = r\}$ in H^n/Γ and δ_t are the Dirac measures. Given $\mu \in K^n$, according to Choquet's theorem there is a measure M such that

$$\mu(B) = \int_{E \cup \Delta_n} \sigma(B)\, dM(\sigma). \tag{3.4}$$

If μ has no mass at the centre of H^n/Γ then $M(\Delta_n) = 0$ (see Stempak [11]) and we have

$$\mu(B) = \int_E \sigma(B)\, dM(\sigma). \tag{3.5}$$

Let us consider $f * e_k^{-j} * \mu$. In view of (3.5) we have

$$f * e_k^{-j} * \mu = \int_E (f * e_k^{-j} * \sigma)\, dM(\sigma). \tag{3.6}$$

When $\sigma = \mu_{r,t}$ we can easily calculate $e_k^{-j} * \sigma$. In fact,

$$e_k^{-j} * \mu_{r,t}(z,s) = \frac{k!\,(n-1)!}{(k+n-1)!} e^{ij(t-s)} \int_{\mathbb{C}^n} e^{i(j/2)\operatorname{Im} z\bar{w}} \phi_k^j(z-w)\, d\mu_r.$$

Recall that μ_r are the normalized surface measures on the sphere $|z| = r$. Since the functions $|j|^{n/2}\psi_k(|j|^{1/2}r)$ where $\psi_k(r)$ are defined in the previous section form an orthonormal basis for $L^2(\mathbb{R}_+, r^{2n-1}\,dr)$, we can expand the radial function

$$G_k(r) = \int_{\mathbb{C}^n} e^{i(j/2)\operatorname{Im} z\bar{w}} \phi_k^j(z-w)\, d\mu_r$$

in terms of them. In view of the relations $\phi_j \times \phi_k = (2\pi)^n \delta_{jk} \phi_k$ one calculates that $e_k^{-j} * \mu_{r,t} = e_k^j(w,t)\, e_k^{-j}(z,s)$ where $|w| = r$. This means that

$$f * e_k^{-j} * \mu_{r,t} = e_k^j(w,t)(f * e_k^{-j}) = \mu_{r,t}(e_k^j) f * e_k^{-j}$$

where

$$\mu_{r,t}(e_k^j) = \int e_k^j(z,s)\, d\mu_{r,t} = e_k^j(w,t), \qquad |w| = r.$$

Putting this back in (3.6) we get

$$f * e_k^{-j} * \mu = \left(\int_E \sigma(e_k^j)\, dM(\sigma)\right)(f * e_k^{-j}) = \mu(e_k^j) f * e_k^{-j}.$$

This completes the proof of Theorem 3.1.

When $\mu = \mu_{r,t}$ it is immediate that $\alpha_k(j) = e^j_k(r, t)$. This proves part (i) of Theorem 2. To prove that $\alpha = \pm 1$ are not eigenvalues of the operator $T_\mu = M_{r,t}$ we will prove in the next section (see Proposition 4.2) that for all $r > 0$ one has

$$\frac{k!\,(n-1)!}{(k+n-1)!}|\phi_k(r)| < 1.$$

This will then complete the proof of Theorem 2.

4. Spetral Properties of the Operator $T_r f = (2\pi)^n f \times \mu_r$

As we have seen in the introduction the study of the spherical mean value operator on the quotient group H^n/Γ involves operators of the form $f \to f \times \mu_r$. These operators are interesting in their own right and we will show in the next section that they are connected to the restriction operators for the symplectic Fourier transform. We study the spectral properties of the operators $T_r f = (2\pi)^n f \times \mu_r$ where μ_r is the normalized surface measure on the sphere $|z| = r$. For the operators T_r we have the following alternate description.

Theorem 4.1.

$$T_r f(z) = \sum_{k=0}^{\infty} \frac{k!\,(n-1)!}{(k+n-1)!}\phi_k(r) f \times \phi_k(z). \tag{4.1}$$

This Theorem is an immediate consequence of the following Proposition in view of the relation $W(f \times g) = W(f)W(g)$.

Proposition 4.1.

$$W(\mu_r) = \sum_{k=0}^{\infty} \frac{k!\,(n-1)!}{(k+n-1)!}\phi_k(r)P_k. \tag{4.2}$$

PROOF. Let $p_\epsilon(z)$ be the Poisson Kernel defined by

$$p_\epsilon(z) = \Gamma\left(\frac{2n+1}{2}\right)\pi^{-(2n+1)/2}\epsilon(\epsilon^2 + |z|^2)^{-(2n+1)/2}.$$

If F is a continuous function vanishing at infinity then we know that

$$\int_{\mathbb{C}^n} F(z)\,d\mu_r = \lim_{\epsilon\to 0}\int_{\mathbb{C}^n} p_\epsilon * \mu_r(z)F(z)\,dz.$$

Given functions ϕ and ψ in $L^2(\mathbb{R}^n)$ let us define $F(z)$ by $F(z) = (W(z)\phi, \psi)$. Since

$$(W(z)\phi, \psi) = e^{ixy/2} \int_{\mathbb{R}^n} e^{ix\xi}\phi(\xi + y)\bar{\psi}(\xi)\, d\xi$$

it is clear that $F(z)$ is a continuous function vanishing at infinity. Hence we have

$$\begin{aligned}(W(\mu_r)\phi, \psi) &= \int_{\mathbb{C}^n} (W(z)\phi, \psi)\, d\mu_r \\ &= \lim_{\epsilon \to 0} \int_{\mathbb{C}^n} p_\epsilon * \mu_r(z) F(z)\, dz \\ &= \lim_{\epsilon \to 0} (W(p_\epsilon * \mu_r)\phi, \psi).\end{aligned}$$

Replacing ϕ by $P_k\phi$ we get

$$(W(\mu_r)P_k\phi, \psi) = \lim_{\epsilon \to 0} (W(p_\epsilon * \mu_r)P_k\phi, \psi).$$

Since $p_\epsilon * \mu_r$ is a radial function, its Weyl transform is given by

$$W(p_\epsilon * \mu_r) = (2\pi)^n \sum_{k=0}^{\infty} R_k(p_\epsilon * \mu_r)P_k$$

and consequently

$$(W(\mu_r)P_k\phi, \psi) = \lim_{\epsilon \to 0} (2\pi)^n R_k(p_\epsilon * \mu_r)(P_k\phi, \psi).$$

Let us now calculate $R_k(p_\epsilon * \mu_r)$. We have

$$\int_{\mathbb{C}^n} p_\epsilon * \mu_r(z)\phi_k(z)\, dz = \int_{\mathbb{C}^n} p_\epsilon * \phi_k(z)\, d\mu_r(z).$$

As $p_\epsilon * \phi_k(z) \to \phi_k(z)$ uniformly as $\epsilon \to 0$ one gets that

$$\lim_{\epsilon \to 0} R_k(p_\epsilon * \mu_r) = (2\pi)^{-n} \frac{k!\,(n-1)!}{(k+n-1)!} \int_{\mathbb{C}^n} \phi_k(z)\, d\mu_r(z).$$

This proves that

$$(W(\mu_r)P_k\phi, \psi) = \frac{k!\,(n-1)!}{(k+n-1)!} \phi_k(r)(P_k\phi, \psi).$$

Hence the proposition.

In view of Theorem 4.1 it is easy to see that T_r is a bounded self adjoint operator. In fact, the functions ϕ_k satisfy the estimate

$$|\phi_k(r)| < \frac{k!\,(n-1)!}{(k+n-1)!}$$

(see Proposition 4.2) and $f \times \phi_k$ are orthogonal projections associated to the special Hermite expansions (see [18]) and hence T_r is a bounded self adjoint operator on $L^2(\mathbb{C}^n)$. It is also clear that all the eigenvalues of T_r are given by

$$\alpha_k = \frac{k!\,(n-1)!}{(k+n-1)!}\phi_k(r)$$

and any function of the form $f \times \phi_k$ is an eigenfunction corresponding to the eigenvalue α_k.

In view of the relations $W(\bar{\Phi}_{\alpha\beta}) = (2\pi)^{n/2}\pi(\alpha,\beta)$ and $W(\phi_k) = (2\pi)^n P_k$ one checks that $\bar{\Phi}_{\alpha\beta} \times \phi_k = (2\pi)^n \bar{\Phi}_{\alpha\beta}$ provided $|\alpha| = k$. This means that the functions $\bar{\Phi}_{\alpha\beta}$ are eigenfunctions of T_r with eigenvalue α_k. As $\Phi_{\alpha\beta}$ are lineary independent this shows that the eigenspace corresponding to each α_k is infinite dimensional. Thus parts (i) and (ii) of Theorem 3 are proved. Part (iii) follows from the next proposition. The result of the proposition is not new but the novelty lies in the proof.

Proposition 4.2. *For any k and $r > 0$,*

$$|\phi_k(r)| < \frac{(k+n-1)!}{k!\,(n-1)!} = \phi_k(0). \tag{4.3}$$

PROOF. The proof is based on the following fact. If the Fourier transform of a function f is positive then $|f(x)| < f(0)$ for $x \neq 0$. We will show that the Fourier transform of the function $L_k^{n-1}(1/2\,|x|^2)\,e^{-1/4|x|^2}$ on $\mathbb{R}^n$ is positive. This means that

$$\left| L_k^{n-1}\left(\frac{1}{2}|x|^2\right) e^{-1/4|x|^2} \right| = |\phi_k(r)| < \frac{(k+n-1)!}{k!\,(n-1)!}.$$

To do this we calculate the kernel $K(x,y)$ of the projection P_k in different ways.

From the very definition one has the formula

$$K(x,y) = \sum_{|\alpha|=k} \Phi_\alpha(x)\Phi_\alpha(y).$$

On the other hand, as $W(\phi_k) = (2\pi)^n P_k$, $K(x,y)$ is also given by

$$K(x,y) = \int_{\mathbb{R}^n} \phi_k(\xi, y-x)\, e^{1/2\, i\xi(x+y)}\, d\xi$$

where we have written

$$\phi_k(\xi, \eta) = \phi_k\left(\frac{1}{2}(|\xi|^2 + |\eta|^2)\right).$$

Therefore, setting $x = y$ we get

$$(2\pi)^n \sum_{|\alpha| = k} (\Phi_\alpha(x))^2 = \int_{\mathbb{R}^n} e^{ix\xi} L_k^{n-1}\left(\frac{1}{2}|\xi|^2\right) e^{-1/4|\xi|^2} d\xi.$$

This proves that the Fourier transform of the Laguerre function is non-negative.

We will conclude this section with a result analogous to a theorem of Ragozin on the convolution of rotation invariant measures on $\mathbb{R}^n$. In [9] Ragozin proved that it μ is the surface measure on the unit sphere in $\mathbb{R}^n$, $n \geqslant 2$ then $\mu * \mu$ is absolutely continuous with respect to the Lebesgue measure. Here we will prove a similar result for the twisted convolution. Moreover, we will identify the density explicitly.

Proposition 4.3. *Assume that $n \geqslant 2$. Then $\mu_r \times \mu_r$ is absolutely continuous with respect to the Lebesgue measure. The density is given by*

$$J(z) = (2\pi)^{-2n} \sum_{k=0}^{\infty} \left(\frac{k!\,(n-1)!}{(k+n-1)!}\right)^2 (\phi_k(r))^2 \phi_k(z) \tag{4.4}$$

where the series converges uniformly on every compact subset of the form $0 < a \leqslant |z| \leqslant b$.

PROOF. In view of Theorem 4.1 one has

$$f \times \mu_r(z) = (2\pi)^{-n} \sum_{k=0}^{\infty} \frac{k!\,(n-1)!}{(k+n-1)!} \phi_k(r) f \times \phi_k(z).$$

This in turn gives us

$$f \times \mu_r \times \mu_r(z) = (2\pi)^{-n} \sum_{k=0}^{\infty} \left(\frac{k!\,(n-1)!}{(k+n-1)!}\right)^2 \phi_k(r)^2 f \times \phi_k(z).$$

This shows that $\mu_r \times \mu_r$ is given by $J(z)\,dz$. If $a \leqslant |z| \leqslant b$ then one has the asymptotic estimate (see Szego [17]),

$$\begin{aligned} &L_k^{n-1}(r^2) e^{-r^2/2} r^{n-1} \\ &\quad = K^{-(n-1)/2} \frac{(k+n-1)!}{k!} J_{n-1}(2r\sqrt{K}) + O(k^{(n-1)/2 - 3/4}) \end{aligned} \tag{4.5}$$

where $K = k + n/2$ and the bound holds uniformly in $a \leqslant r \leqslant b$. As $n \geqslant 2$, this shows that the series defining $J(z)$ converges uniformly on compact sets of the form $0 < a \leqslant |z| \leqslant b$.

5. A Restriction Theorem for the Symplectic Fourier Transform on $\mathbb{R}^{2n}$

The operator $T_r f = (2\pi)^n f * \mu_r$ is related to a restriction operator R_r for the symplectic Fourier transform as we are going to see now. Before that let us briefly recall the usual restriction operators for the Fourier transform on $\mathbb{R}^n$. If we define

$$Q_r f(x) = (2\pi)^{-n} \int_{|w|=1} e^{irxw} \hat{f}(rw)\, d\sigma(w), \tag{5.1}$$

then the Fourier inversion formula can be written as

$$f(x) = \int_0^\infty Q_r f(x) r^{n-1}\, dr. \tag{5.2}$$

The operators f going to $Q_r f$ are called the restriction operators for the Fourier transform. It is well known that

$$\|Q_r f\|_{p'} \leqslant C_r \|f\|_p, \qquad 1 \leqslant p \leqslant \frac{2(n+1)}{n+3}. \tag{5.3}$$

It is also known that such an estimate is not possible when $p > \dfrac{2n}{n+1}\cdot$ As a consequence of (5.3) one can prove the Stein-Tomas [19] restriction theorem

$$\left(\int_{|w|=1} |\hat{f}(w)|^2\, d\sigma \right)^{1/2} \leqslant C \|f\|_p, \qquad 1 \leqslant p \leqslant \frac{2(n+1)}{n+3} \tag{5.4}$$

which justifies the name restriction operators.

Let us now consider the symplectic Fourier transform on $\mathbb{R}^{2n}$. Identifying $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ the symplectic Fourier transform is defined as

$$\mathfrak{F}_s f(z) = \int_{\mathbb{C}^n} f(w)\, e^{-i/2 \operatorname{Im} z\bar{w}}\, dw$$

and the inversion formula is given by

$$f(z) = (4\pi)^{-2n} \int_{\mathbb{C}^n} \mathfrak{F}_s f(w)\, e^{-i/2 \operatorname{Im} z\bar{w}}\, dw.$$

We can rewrite the inversion formula as

$$f(z) = (4\pi)^{-2n} \int_{\mathbb{C}^n} \mathfrak{F}_s f(z - w)\, e^{i/2 \operatorname{Im} z\bar{w}}\, dw.$$

Let ω_{2n} be the surface area of $|z| = 1$. We can now write

$$f(z) = (4\pi)^{-2n}\omega_{2n}\int_0^\infty r^{2n-1}\,dr\int_{|w|=r}\mathfrak{F}_s f(z-w)\,e^{i/2\operatorname{Im} z\bar{w}}\,d\mu_r$$

where μ_r is the normalized surface measure on $|w| = r$. If we define $R_r f$ by

$$R_r f(z) = \int_{|w|=r}\mathfrak{F}_s f(z-w)\,e^{i/2\operatorname{Im} z\bar{w}}\,d\mu_r = \mathfrak{F}_s f \times \mu_r \tag{5.5}$$

then we have obtained the inversion in the form

$$f(z) = (4\pi)^{-2n}\omega_{2n}\int_0^\infty R_r f(z) r^{2n-1}\,dr. \tag{5.6}$$

This is the analogue of (5.2) and that is the reason why we call them restriction operators. Unlike the operators $Q_r f$, these $R_r f$ are no longer eigenfunctions of the Laplacian.

The relation between T_r and R_r is now clear: $R_r f = (2\pi)^{-n}T_r(\mathfrak{F}_s f)$. In view of this we have the alternate formula

$$R_r f = (2\pi)^{-n}\sum_{k=0}^{\infty}\frac{k!\,(n-1)!}{(k+n-1)!}\phi_k(r)(\mathfrak{F}_s f \times \phi_k). \tag{5.7}$$

Using the bounds for the functions ϕ_k one immediately gets

$$\|R_r f\|_2 \leqslant C\|\mathfrak{F}_s f\|_2 \leqslant C\|f\|_2.$$

If we interpolate with the trivial estimate

$$\|R_r f\|_\infty = \|\mathfrak{F}_s f \times \mu_r\|_\infty \leqslant C\|\mathfrak{F}_s f\|_\infty \leqslant C\|f\|_1$$

we obtain the following boundedness result.

Proposition 5.1. *For* $1 \leqslant p \leqslant 2$, *one has*

$$\|R_r f\|_{p'} \leqslant C\|f\|_p. \tag{5.8}$$

Using the asymptotic properties of the Laguerre functions $\phi_k(r)$ we can prove the following regularity theorem. To state the result we introduce the twisted Sobolev spaces $\mathcal{W}^s(\mathbb{C}^n)$. On $\mathbb{C}^n$ consider the $2n$ vector fields

$$Z_j = \frac{\partial}{\partial z_j} + \frac{1}{4}\bar{z}_j, \qquad \bar{Z}_j = \frac{\partial}{\partial \bar{z}_j} - \frac{1}{4}z_j$$

and the operator

$$L = -\frac{1}{2}\sum_{j=1}^{n}(Z_j\bar{Z}_j + \bar{Z}_j Z_j).$$

The Hermite operator H and L are related by $W(Lf) = W(f)H$. Using this formula we can define L^s by $W(L^s f) = W(f)H^s$. We then define the twisted Sobolev spaces by

$$\mathcal{W}^s(\mathbb{C}^n) = \{f \in L^2(\mathbb{C}^n): L^s f \in L^2(\mathbb{C}^n)\}. \tag{5.9}$$

With this definition we can now prove the following theorem.

Theorem 5.1. *If $s \leqslant (2n-1)/4$ then R_r maps $L^2(\mathbb{C}^n)$ continuously into $\mathcal{W}^s(\mathbb{C}^n)$.*

PROOF. Since $L^s(\phi_k) = (2k+n)^s \phi_k$ we have

$$L^s(R_r f) = L^s(\mathfrak{F}_s f \times \mu_r) = (2\pi)^{-n} \sum_{k=0}^{\infty} \frac{k!\,(n-1)!}{(k+n-1)!} \phi_k(r)(2k+n)^s(\mathfrak{F}_s f \times \phi_k)$$

(where we have used the relation $L^s(f \times g) = (f \times L^s g)$). In view of the estimate (4.5) it is clear that $(2k+n)^s \dfrac{k!\,(n-1)!}{(k+n-1)!} \phi_k(r)$ is bounded as long as $s \leqslant (2n-1)/4$ uniformly in k. This proves that $L^s(R_r f)$ belongs to $\mathcal{W}^s(\mathbb{C}^n)$ and $\|R_r f\|_{\mathcal{W}^s} \leqslant C\|f\|_2$.

We will now proceed to prove parts (ii) and (iii) of Theorem 4 of the introduction. We have already shown that $\|R_r f\|_{p'} \leqslant C\|f\|_p$ for $1 \leqslant p \leqslant 2$. The assertion (ii) that $\|R_r f\|_p \leqslant C\|f\|_p$ for $2n/(n+1) \leqslant p \leqslant 2$ will follow once we show that the following is true.

Proposition 5.2. *Assume that $n \geqslant 3$. Then*

$$\|f \times \mu_r\|_{2n/(n+1)} \leqslant C\|f\|_{2n/(n-1)}. \tag{5.10}$$

To see that the assertion (ii) *follows from* (5.10) *we observe that*

$$\begin{aligned}\|R_r f\|_{2n/(n+1)} &= \|\mathfrak{F}_s f \times \mu_r\|_{2n/(n+1)} \\ &\leqslant C\|\mathfrak{F}_s f\|_{2n/(n-1)} \leqslant C\|f\|_{2n/(n+1)}.\end{aligned}$$

An interpolation with $\|R_r f\|_2 \leqslant C\|f\|_2$ proves the assertion.

To prove Proposition 5.2 we need the following estimate for the projections $f \to f \times \phi_k$.

Proposition 5.3.

$$\|f \times \phi_k\|_{2n/(n-1)} \leqslant C\|f\|_{2n/(n+1)}. \tag{5.11}$$

As the projections f going to $f \times \phi_k$ are self adjoint we also have

$$\|f \times \phi_k\|_{2n/(n+1)} \leqslant C\|f\|_{2n/(n-1)}.$$

Using this one immediately gets

$$\begin{aligned}\|f \times \mu_r\|_{2n/(n+1)} &\leqslant C \sum_{k=0}^{\infty} \frac{k!\,(n-1)!}{(k+n-1)!} |\phi_k(r)|\, \|f \times \phi_k\|_{2n/(n+1)} \\ &\leqslant C\left(\sum_{k=0}^{\infty} (2k+n)^{-(n/2)+(1/4)}\right)\|f\|_{2n/(n-1)}.\end{aligned}$$

As $n \geqslant 3$ the series converges and this proves Proposition 5.2. So it remains to prove Proposition 5.3.

We have proved this proposition in [18]. We will briefly indicate the proof for the sake of completeness. The definition of the Laguerre polynomials L_k^α can be extended even for complex values of α, $\operatorname{Re}\alpha > -1/2$. We then consider the functions

$$\psi_k^\alpha(z) = \frac{\Gamma(k+1)\Gamma(\alpha+1)}{\Gamma(k+\alpha+1)} L_k^\alpha\left(\frac{1}{2}|z|^2\right) e^{-1/4|z|^2}$$

and define a family of operators $G_k^\alpha f = f \times \psi_k^{n\alpha}$. One verifies that this is an admissible analytic family of operators. By Stein's interpolation theorem [10] the estimate (5.11) will follow from the two estimates

$$\|G_k^{i\tau} f\|_\infty \leqslant C(1+|\tau|)^{1/2}\|f\|_1.$$
$$\|G_k^{1+i\tau} f\|_2 \leqslant C(1+|\tau|)^n k^{-n}\|f\|_2.$$

These estimates can be proved using certain bounds for the Laguerre function ψ_k^α. We refer to [18] for details.

We will now complete the proof of Theorem 4 by proving the assertion (iii) namely,

$$\|R_r f\|_q \leqslant C\|f\|_p \quad \text{for} \quad 1 \leqslant p \leqslant \frac{2n}{n+1} \quad \text{where} \quad q = \frac{n-1}{n+1}p'.$$

When $p = 2n/(n+1)$, $q = p$ and we already have the inequality $\|R_r f\|_{2n/(n+1)} \leqslant C\|f\|_{2n/(n+1)}$. Interpolating with the estimate $\|R_r f\|_\infty \leqslant C\|f\|_1$ we complete the proof.

We conclude the paper with the following remarks. The estimate $\|R_r f\|_q \leqslant C\|f\|_p$ was established in the interval $1 \leqslant p \leqslant 2n/(n+1)$. By increasing q we can extend the interval of validity. For example, by interpolating with the estimate $\|R_r f\|_{4n/(2n+1)} \leqslant C\|f\|_{4n/(2n+1)}$ we can prove

$$\|R_r f\|_q \leqslant C\|f\|_p, \qquad q = \frac{2n-1}{2n+1}p'$$

in the range $1 \leqslant p \leqslant \frac{4n}{2n+1}\cdot$ Similarly by decreasing the interval $1 \leqslant p \leqslant \frac{2n}{n+1}$ we can obtain estimates valid with $q = \gamma p'$ where $\gamma < \frac{n-1}{n+1}\cdot$ Another remark we would like to make is regarding the assumption $n \geqslant 3$. It would be interesting to see if the Theorem 4 remains true for $n = 1$ and $n = 2$ also.

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S. Thangavelu
Tata Institute of Fundamental Research
Indian Institute of Science Campus
Bangalore 560 012, INDIA