SUMMABILITY OF HERMITE EXPANSIONS. II

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ABSTRACT. We study the summability of n-dimensional Hermite expansions where $n > 1$. We prove that the critical index for the Riesz summability is $(n - 1)/2$. We also prove analogues of the Fejér-Lebesgue theorem and Riemann's localisation principle when the index α of the Riesz means is $> (3n - 2)/6$.

1. INTRODUCTION

In this paper we study the summability of the n-dimensional Hermite expansions. Let φ_m denote the mth Hermite function. The n-dimensional Hermite functions Φ_ν are defined as follows. For every multi-index $\nu = (\nu_1, \nu_2, \dots, \nu_n)$ we set

$$\Phi_\nu(x) = \varphi_{\nu_1}(x_1)\varphi_{\nu_2}(x_2)\cdots\varphi_{\nu_n}(x_n).$$

The functions $\Phi_\nu(x)$ are eigenfunctions of the operator $(-\Delta + |x|^2)$ with eigenvalues $(2|\nu| + n)$ where Δ is the n-dimensional Laplacian. The family $\{\Phi_\nu(x)\}$ defines an orthonormal system for $L^2(R^n)$. Given any L^p function f we can define its generalised Fourier coefficients by

$$(1.1) \qquad f^\wedge(\nu) = \int f(x)\Phi_\nu(x)\,dx.$$

Thus to each function f we have the associated Hermite expansion $f(x) = \sum f^\wedge(\nu)\Phi_\nu(x)$ where the sum is extended over all multi-indices ν. We can write the above series in the following way. For any nonnegative integer k let us define the functions $\Phi_k(x, y)$ and operators P_k by

$$(1.2) \qquad \Phi_k(x, y) = \sum_{|\nu|=k} \Phi_\nu(x)\Phi_\nu(y),$$

$$(1.3) \qquad P_k f(x) = \int \Phi_k(x, y)f(y)\,dy.$$

Then $P_k f$ is the orthogonal projection of f onto the subspace spanned by $\{\Phi_\nu(y): |\nu| = k\}$. The Hermite expansion then reads

$$(1.4) \qquad f(x) = \sum_{k\geq 0} P_k f(x).$$

Since we are interested in the summability of the above series we define the Riesz and Cesàro means of the expansion as in the one-dimensional case. For $\alpha > 0$ and $R > 0$ the Rth Riesz means of order α is defined by

$$(1.5) \qquad S_R(\alpha)f(x) = \sum_{K \leq R} \left(1 - \frac{K}{R}\right)^{\alpha} P_k f(x)$$

where $K = 2k + n$. The Rth Cesàro means of order α are defined by

$$(1.6) \qquad C_R(\alpha)f(x) = (A_R(\alpha))^{-1} \sum_{k \leq R} A_{R-k}(\alpha) P_k f(x)$$

where $A_k(\alpha) = \Gamma(k + \alpha + 1)/\Gamma(k + 1)\Gamma(\alpha + 1)$. We are interested in the convergence of the above means to the function as R tends to infinity.

This problem was considered by Hulanicki and Jenkins in [1]. They proved that for large values of α the Riesz means converge to the function in the norm. They studied the summability of eigenfunction expansions on a nilmanifold and deduced the summability results for the Hermite series as a corollary. We studied the summability of the one-dimensional Hermite expansions in [3]. There it is proved that $1/6$ is the critical index for the Riesz summability. In analogy with the one-dimensional case one expects that the critical index will be $(3n - 2)/6$. But to our great surprise the critical index turns out to be $(n - 1)/2$ for $n > 1$. The following theorem is the main result of this paper.

Theorem 1. *Assume that $n \geq 2$, $\alpha > (n - 1)/2$ and f is in L^p, $1 \leq p \leq \infty$. Then the Riesz means $S_R(\alpha)f$ converges to f in the norm as R tends to infinity. If $\alpha < (n - 1)/2$, then there is an L^1 function f for which the Riesz means $S_R(\alpha)f$ will not converge in the norm.*

Thus we see that the behaviour of the n-dimensional Hermite series is more or less similar to the behaviour of the corresponding Fourier series when $n \geq 2$. This distinction between the one dimensional and the higher dimensions is explained to some extent by the behaviour of certain oscillatory integrals as we will see later. The estimates we are going to obtain for the Riesz kernel are not very good when $(3n - 2)/6 > \alpha > (n - 1)/2$. But for $\alpha > (3n - 2)/6$ the estimates are quite neat and as in the one-dimensional case we can prove the Fejér-Lebesgue theorem and Riemann's localisation principle.

Theorem 2. *Assume that $n \geq 2$, $\alpha > (3n - 2)/6$ and f is in L^p, $1 \leq p \leq \infty$. If both x and $-x$ are Lebesgue points for f then $S_R(\alpha)f(x)$ converges to $f(x)$ as R tends to infinity.*

Theorem 3. *Assume that $n \geq 2$, $\alpha > (3n - 2)/6$ and f is in L^p, $1 \leq p \leq \infty$. If f vanishes near the points x and $-x$ then $S_R(\alpha)f(x)$ converges to 0 as R tends to infinity.*

This paper is organised as follows. In the next section we show that $\alpha > (n - 1)/2$ is a necessary condition for the convergence of $S_R(\alpha)f$ for L^1 functions. In the third section we get an expression for the Riesz kernel in terms

of an oscillatory integral and estimate the same when $\alpha > (3n - 2)/6$. The estimation of the kernel when $\alpha > (n - 1)/2$ will be taken up in the fourth section. All these estimates are carried out for the three-dimensional case. Finally we indicate how to estimate the kernel in the general case. Once the kernel is estimated we can prove all the theorems as in the one-dimensional case. We refer the reader to [3].

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2. THE CRITICAL INDEX FOR THE RIESZ SUMMABILITY

In this section we will show that if $\alpha < (n - 1)/2$ then there exists an L^1 function f such that its Cesàro means will not converge to it. Since the Riesz and Cesàro means have identical behaviour this will prove that the Riesz means of order $\alpha < (n - 1)/2$ is not effective for L^1 functions. For the sake of simplicity we assume that $n = 3$. The arguments can be carried out for any dimensions without much difficulty.

Suppose that the Cesàro means of order α of an L^1 function f converges to f. Let $S_n f$ denote the partial sums of the expansion defined by

$$(2.1) \qquad S_n f(x) = \sum_{k \leq n} P_k f(x).$$

Proceeding as in the case of the one-dimensional case (see [3]) we can show that with $N = 2n + 3$

$$(2.2) \qquad \|P_n f(x)\|_1 \leq C N^\alpha \|f\|_1.$$

Recall that the projections P_n are given by

$$(2.3) \qquad P_n f(x) = \int \Phi_n(x, y) f(y)\, dy.$$

From (2.1) and (2.3) we immediately obtain

$$(2.4) \qquad \sup_x \int |\Phi_n(x, y)|\, dy \leq C N^\alpha.$$

We will prove the following theorem.

Theorem 2.1. *There is a constant C such that for large n we have the estimate*

$$(2.5) \qquad \sup_y \int |\Phi_n(x, y)|\, dx \geq C N.$$

Consequently, for some L^1 function f the Cesàro means of order α, $\alpha < 1$, will not converge.

To prove the theorem we need a good expression for the functions $\Phi_n(x, y)$. We are going to express $\Phi_n(x, y)$ in terms of the Mehler kernel $M_z(x, y)$. This

kernel is defined for $|z| < 1$ by $M_z(x, y) = \sum z^n \Phi_n(x, y)$. Since $M_z(x, y)$ is obtained by multiplying three copies of the one-dimensional Mehler kernel, we have the following formula:

$$(2.6) \qquad M_z(x, y) = \pi^{-3/2}(1 - z^2)^{-3/2} \exp\{B_z(x, y)\},$$

$$B_z(x, y) = -\tfrac{1}{2}a^2(1 + z^2)/(1 - z^2) + 2bz/(1 - z^2)$$

where $a^2 = (|x|^2 + |y|^2)$ and $b = x_1 y_1 + x_2 y_2 + x_3 y_3$. Let $G_r(t, x, y) = M_z(x, y)$ where $z = re^{-2it}$. Then we have the following expression for $\Phi_n(x, y)$:

$$r^n \Phi_n(x, y) = \frac{1}{\pi i} \int_0^\pi e^{2nit} G_r(t, x, y) \, dt.$$

With the notations $A_r(t) = (e^{2it} - r^2 e^{-2it})$ and

$$B_r(t) = \{-\tfrac{1}{2}a^2(1 + r^2 e^{-4it}) + 2bre^{-2it}\}/(1 - r^2 e^{-4it})$$

we can express $\Phi_n(x, y)$ in the following form:

$$\Phi_n(x, y) = \lim_{r \to 1}(-i\pi^{-5/2}) \int_0^\pi e^{iNt} A_r(t)^{-3/2} e^{B_r(t)} \, dt.$$

Let us set $\varphi(t) = -2b \operatorname{cosec} 2t + \tfrac{1}{2}a^2 \cot 2t$ and define the following three integrals:

$$(2.7) \qquad F_n(x, y) = N \int (\sin 2t)^{1/2}\{a^2 - 2b \cos 2t\}^{-1} e^{iNt} e^{i\varphi(t)} \, dt,$$

$$(2.8) \qquad G_n(x, y) = \int (\sin 2t)^{-1/2} \cos 2t \{a^2 - 2b \cos 2t\}^{-1} e^{iNt} e^{i\varphi(t)} \, dt,$$

$$(2.9) \qquad H_n(x, y) = \int (\sin 2t)^{-3/2} \varphi''(t)\{\varphi'(t)\}^{-2} e^{iNt} e^{i\varphi(t)} \, dt.$$

All three integrals are extended from 0 to π. We are now ready to prove the following lemma.

Lemma 2.1. *Given x and y such that $|x - y| > 0$ and $|x + y| > 0$ we have the expression*

$$\Phi_n(x, y) = C_1 e^{i\pi/4} F_n(x, y) + C_2 G_n(x, y) + C_3 H_n(x, y)$$

where C_i are constants, C_1 is real and F_n, G_n and H_n are the integrals defined above.

Proof. We are going to pass to the limit in the equation defining $\Phi_n(x, y)$. As r tends to 1, we see that $A_r(t)$ tends to $\sin 2t$ and so at first sight it looks as though we cannot take the limit under the integral sign. But things are not so bad as we will see shortly.

Let us write $C_r(t) = (e^{2it} + r^2 e^{-2it})$, $D_r(t) = (r^{-1} e^{2it} - r e^{-2it})$ and $E_r(t) = (a^2 - b r e^{-2it} - b r^{-1} e^{2it})$. A simple calculation shows that $A_r'(t) = 2i C_r(t)$, $B_r'(t) = 4i r^2 A_r(t)^{-2} E_r(t)$ so that

$$(2.10) \qquad A_r(t)^{-3/2} \{B_r'(t)\}^{-1} = (4ir^2)^{-1} A_r(t)^{1/2} (E_r(t))^{-1}.$$

Now,

$$\frac{d}{dt}\{A_r(t)^{-3/2}\{B_r'(t)\}^{-1}\} = (4r^2)^{-1} A_r(t)^{-1/2} C_r(t)(E_r(t))^{-1}$$
$$+ (2r^2)^{-1} b A_r(t)^{1/2} D_r(t)(E_r(t))^{-2}.$$

Therefore, we can write

$$A_r(t)^{-3/2} \exp B_r(t) = \partial_t \{A_r(t)^{-3/2} \exp B_r(t)/B_r'(t)\}$$
$$- \tfrac{1}{4} r^{-2} A_r(t)^{-1/2} \exp B_r(t) C_r(t) E_r(t)^{-1}$$
$$- \tfrac{1}{2} r^{-2} b A_r(t)^{1/2} D_r(t) \exp B_r(t) E_r(t)^{-2}.$$

In view of this equation we have

$$(2.11) \qquad \int e^{iNt} A_r(t)^{-3/2} \exp\{B_r(t)\}\, dt = F_{r,n}(x,y) - G_{r,n}(x,y) - H_{r,n}(x,y),$$

$$(2.12) \qquad F_{r,n}(x,y) = \int e^{iNt} \frac{d}{dt}\{A_r(t)^{-3/2} \exp B_r(t)(B_r'(t))^{-1}\}\, dt,$$

$$(2.13) \qquad G_{r,n}(x,y) = (4r^2)^{-1} \int e^{iNt} \exp\{B_r(t)\} A_r(t)^{-1/2} C_r(t)(E_r(t))^{-1}\, dt,$$

$$(2.14) \qquad H_{r,n}(x,y) = (2r^2)^{-1} b \int e^{iNt} \exp\{B_r(t)\} A_r(t)^{1/2} D_r(t)(E_r(t))^{-2}\, dt.$$

Now we need certain estimates for A_r, B_r and E_r. The following estimates are easily obtained:

$$(2.15) \qquad |A_r(t)| \geq c \sin 2t, \quad |\exp\{B_r(t)\}| \leq 1 \quad \text{and} \quad |E_1(t)| \geq (a^2 \pm 2b).$$

Since we are assuming that $(a^2 \pm 2b) > 0$, we can find an $r_0 = r_0(x,y)$ such that for $r_0 \leq r \leq 1$ the lower bound $|E_r(t)| \geq \tfrac{1}{2}(a^2 \pm 2b)$ holds. With these estimates it is clear that we can pass to the limit under the integral sign in (2.13) and (2.14) getting the terms $G_n(x,y)$ and $H_n(x,y)$. In $F_{r,n}(x,y)$ we first integrate by parts and then pass to the limit. Noting that the boundary terms tend to 0 we obtain $F_n(x,y)$. This completes the proof of the lemma.

We now proceed to get estimates for the L^1 norm of $\Phi_n(x,y)$. Upper bounds for the L^1 norms of the terms $G_n(x,y)$ and $H_n(x,y)$ can be easily obtained. We have the following lemma.

Lemma 2.2. *There is a constant* C *independent of* y *and* n *such that*

(2.16) $\displaystyle\int |G_n(x,y)|\,dx \leq CN^{1/2}$ *and* $\displaystyle\int |H_n(x,y)|\,dx \leq CN^{1/2}$.

Proof. We will prove the estimate for $G_n(x,y)$. Estimating $H_n(x,y)$ is similar. Replacing x and y by $N^{1/2}x$ and $N^{1/2}y$ we need to show that

$$\int |g_n(x,y)|\,dx \leq C$$

where $g_n(x,y)$ is given by the integral

(2.17) $\displaystyle g_n(x,y) = \int (\sin 2t)^{-1/2} \cos 2t \{a^2 - 2b\cos 2t\}^{-1} e^{iN\psi(t)}\,dt$.

Estimate (2.17) will follow once we show that the following estimate holds for g_n:

(2.18) $|g_n(x,y)| \leq C\{|x-y|^{-2}(1+|x-y|)^{-2} + |x+y|^{-2}(1+|x+y|)^{-2}\}$.

Splitting g_n into four parts it is easy to see that we need to consider only the integral

$$I = \int_0^{\pi/4} (\sin 2t)^{-1/2} \cos 2t (a^2 - 2b\cos 2t)^{-1} e^{iN\{t+\varphi(t)\}}\,dt.$$

For this integral it is immediate that $|I| \leq C|x-y|^{-2}$. The following estimates are easily checked:

(2.19) $|\varphi'(t)\sin^2 2t| \geq \tfrac{1}{2}|x-y|^2$, $|\varphi''(t)\sin 2t| \leq 4|\varphi'(t)|$.

Integrating by parts and using the above estimates we obtain $|I_1| \leq C|x-y|^{-4}$. Combining the two estimates we get the required estimate.

To prove Theorem 2.1 we have to show that there is a y_0 such that

$$\int |F_n(x,y_0)|\,dx \geq CN.$$

Since there is rotational symmetry, we can assume that $y = (y_1,0,0)$. We are going to apply the method of stationary phase to the integral defining F_n. We restrict ourselves to the region $x_1 \geq 0, y_1 \geq 0$, and $|x|^2+|y|^2 \leq N$. Consider the function $\psi(t) = Nt - b\,\mathrm{cosec}\,2t + \tfrac{1}{2}a^2\cot 2t$. In the interval $0 \leq t \leq \pi/4$ there is only one stationary point t_1 for the function ψ given by $\cos 2t_1 = N^{-1}b + m$, where m is defined by $m^2 = 1 - N^{-1}a^2 + N^{-2}b^2$. Likewise, for the function $\psi^*(t) = Nt + b\,\mathrm{cosec}\,2t + \tfrac{1}{2}a^2\cot 2t$ there is only one stationary point t_2 given by $\cos 2t_2 = -N^{-1}b + m$. Let us write $f(t) = (\sin 2t)^{-1}\{\cos\psi(t) - \sin\psi(t)\}$ and $f^*(t) = (\sin 2t)^{-1}\{\cos\psi^*(t) - \sin\psi^*(t)\}$. With this notation we can now prove the following lemma.

Lemma 2.3. *Assume that* $x_1, y_1 \geq 0$, *and* $|x|^2 + |y|^2 \leq N$. *Then we have, with some constant* c,

(2.20) $\mathrm{Re}\,F_n(x,y) = cN^{-1/2}m^{-1/2}\{f(t_1) + (-1)^{n+1}f^*(t_2)\} + O(N^{-1})$.

Proof. To make the calculations easy let us replace x and y by $N^{1/2}x$ and $N^{1/2}y$, respectively and consider the following integral:

$$I = \int_0^\pi (\sin 2t)^{1/2} \{a^2 - 2b \cos 2t\}^{-1} e^{iN\rho(t)} \, dt.$$

We write this integral as the sum of four integrals $I = I_1 + (i)^N I_2 + (i)^{N+1} I_3 + (i)^3 I_4$ where $I_4 = I_1$, $I_3 = I_2$ and I_2 and I_1 are defined as follows:

$$I_1 = \int_0^{\pi/4} (\sin 2t)^{1/2} \{a^2 - 2b \cos 2t\}^{-1} e^{iN\rho(t)} \, dt,$$

$$I_2 = \int_0^{\pi/4} (\sin 2t)^{1/2} \{a^2 - 2b \cos 2t\}^{-1} e^{iN\rho^*(t)} \, dt.$$

In the above the functions ρ and ρ^* are the functions given by $\rho(t) = t - b \operatorname{cosec} 2t + \frac{1}{2} a^2 \cot 2t$ and $\rho^*(t) = t + b \operatorname{cosec} 2t + \frac{1}{2} a^2 \cot 2t$. Applying the method of stationary phase to the integral I_1 we get the following expression:

$$(2.21) \qquad I_1 = (2\pi)^{1/2} N^{-1/2} m^{-1/2} (\sin 2t_1)^{-1} \exp\{iN\rho(t_1) + \pi/4\} + O(N^{-1}).$$

Since $I_4 = I_1$ we obtain after going back to the original variables

$$(2.22) \qquad \operatorname{Re}\{e^{i\pi/4}(I_1 - iI_4)\} = cN^{-1/2} m^{-1/2} f(t_1) + O(N^{-1}).$$

Similarly for the other two integrals we obtain

$$(2.23) \qquad \operatorname{Re}\{e^{i\pi/4}(i)^N(I_2 + iI_3)\} = c(-1)^{n+1} N^{-1/2} m^{-1/2} f^*(t_2) + O(N^{-1}).$$

This completes the proof of the lemma.

We now consider the main term in the expression of $\operatorname{Re} F_n(x,y)$. The first term has the lower bound

$$|N^{-1/2} m^{-1/2} f(t_1)| \geq \tfrac{1}{2} N^{-1/2} m^{-1/2} (\sin 2t_1)^{-1} |\cos 2\psi(t_1)|$$

and the second term is bounded by $2N^{-1/2} m^{-1/2} (\sin 2t_2)^{-1}$. Next we proceed to get a lower bound for $|\cos 2\psi(t_1)|$. Also we need an estimate for the ratio of $\sin 2t_1$ and $\sin 2t_2$. To estimate the cosine term we use the following lemma due to Muckenhoupt [2].

Lemma 2.4 (Muckenhoupt). *Let L be an integer greater than 20 and let I be a set of L consecutive integers. If for k in I, $\frac{1}{3}L \leq g(k+1) - g(k) \leq \pi/4$ and $g(k+1) - g(k)$ is monotone decreasing in k, then for at least $2/3$ of the integers k in I we have $|\cos g(k)| \geq 1/(200)$.*

Proof. The intervals in which $|\cos x| < 1/(200)$ have length $< 2\sin^{-1} 1/(200)$ which is less than $1/(90)$. The last of these intervals that contains any of the $g(k)$'s then contains at most $[L/30] + 1$ of them where $[\ ]$ denotes the greatest integer. Before this each interval where $|\cos x|$ is greater than or equal to $1/(200)$ will contain at least three times as many $g(k)$'s as the preceding interval where $|\cos x| < 1/(200)$ because of the upper bound on $g(k+1) - g(k)$

and its monotonicity. Then for at most $[L/30]+1+\frac{1}{4}\{L-[L/30]-1\}$ of the k's in I is $|\cos g(k)| < 1/200$. Since $\frac{1}{L}[L/30] \leq 1/30$, this number is bounded by $L(1/40+1/4+3/4L)$ which is less than $\frac{1}{3}L$ since $L > 20$. This completes the proof of the lemma.

Lemma 2.5. *Let* $\varepsilon = (2500)^{-1}$ *and assume that* $y_1^2 \geq 4(2500)^3$. *Assume that* y_1 *and* x_1 *satisfy the following three conditions:*

$$(2.24) \qquad\qquad y_1^2 \leq x_1^2 \leq y_1^2 + \tfrac{1}{2}\varepsilon^2 y_1^2,$$

$$(2.25) \qquad\qquad \tfrac{1}{2}N(1 - \varepsilon^2) \leq y_1^2 \leq \tfrac{1}{2}N(1 - \tfrac{1}{2}\varepsilon^2),$$

$$(2.26) \qquad\qquad \tfrac{1}{4}(1 - \varepsilon^2)^{-1}\varepsilon^2 y_1^2 \leq r^2 \leq \tfrac{1}{2}(1 - \tfrac{1}{2}\varepsilon^2)^{-1}\varepsilon^2 y_1^2,$$

where we have put $r = |x'|$. *Let* $I(y_1)$ *denote the set of all consecutive integers* N *satisfying*

$$(2.27) \qquad\qquad 2(1 - \tfrac{1}{2}\varepsilon^2)^{-1}y_1^2 \leq N \leq 2(1 - \varepsilon^2)^{-1}y_1^2.$$

Then for at least $2/3$ *of the integers* N *in* $I(y_1)$ *we have* $|\cos 2\psi(t_1)| > 1/200$ *where* t_1 *is the point defined in Lemma* 2.3.

Proof. We want to check the conditions of Lemma 2.4. Observe that under the hypothesis of the lemma we have $|x|^2 + |y|^2 \leq N$. Let us set $f(N) = b/N + \{1 - a^2/N + b^2/N^2\}^{1/2}$. It is easy to check that $f'(N) > 0$ and hence f is an increasing function of N. Let $u = N^{-1/2}x$ and $v = N^{-1/2}y$ so that $f(N) = \sigma + \mu$ where $\sigma = u_1 v_1$, $\tau^2 = u_1^2 + v_1^2$ and $\mu^2 = 1 - \tau^2 + \sigma^2$. We then have

$$(2.28) \qquad\qquad v_1^2 \leq u_1^2 \leq v_1^2 + \tfrac{1}{2}\varepsilon^2 v_1^2,$$

$$(2.29) \qquad\qquad \tfrac{1}{2}(1 - \varepsilon^2) \leq v_1^2 \leq \tfrac{1}{2}(1 - \tfrac{1}{2}\varepsilon^2),$$

$$(2.30) \qquad\qquad \tfrac{1}{4}(1 - \varepsilon^2)^{-1}\varepsilon^2 v_1^2 \leq \rho^2 \leq \tfrac{1}{2}(1 - \tfrac{1}{2}\varepsilon^2)^{-1}\varepsilon^2 v_1^2,$$

where $\rho = |u'|$. From these it is clear that $2\sigma \geq (1 - \varepsilon^2)$ and $1 - \tau^2 \leq \varepsilon^2$. Therefore,

$$(2.31) \qquad\qquad f(N) = \sigma + \mu \geq 2\sigma \geq (1 - \varepsilon^2) \geq 2^{-1/2}.$$

As $\sin^2 2t_1 = \tau^2 - 2\sigma \cos 2t_1 \geq |u - v|^2 \geq \rho^2 \geq \tfrac{1}{8}\varepsilon^2$, we have $\cos^2 2t_1 \leq (1 - \tfrac{1}{8}\varepsilon^2) \leq (1 - \tfrac{1}{16}\varepsilon^2)^2$ and hence $f(N) \leq (1 - \tfrac{1}{16}\varepsilon^2)$. Thus we have $2^{-1/2} \leq f(N) \leq (1 - \tfrac{1}{16}\varepsilon^2)$. Since arc cos is decreasing in the interval $(2^{-1/2}, 1)$ and $f(N)$ is increasing $\cos^{-1} f(N)$ is a decreasing function of N. Further we observe that if $\cos u \leq 1 - \tfrac{1}{16}\varepsilon^2$, then $(1 - \tfrac{1}{2}u^2) \leq (1 - \tfrac{1}{16}\varepsilon^2)$ or $u \geq 8^{-1/2}\varepsilon$. Therefore, $\cos^{-1} f(N) \geq 8^{-1/2}\varepsilon$. The condition $2^{-1/2} \leq f(N) \leq (1 - \tfrac{1}{16}\varepsilon^2)$ translates into the inequality $8^{-1/2}\varepsilon \leq \cos^{-1} f(N) \leq \pi/4$.

Let L denote the number of consecutive integers N in the interval $I(y_1)$. Clearly $L \geq \frac{1}{2}\varepsilon^2 y_1^2$. As $\varepsilon^3 y_1^2 \geq 4$ we have $L \geq 2\varepsilon^{-1}$ or $\varepsilon/2 \geq 1/L$. Thus we have $1/L \leq \cos^{-1} f(N) \leq \pi/4$. Now we are in a position to check the conditions of the lemma. Let $g(N) = 2\psi(t_1)$, and $\cos^{-1} f(N) = 2t_1$. Since $\psi'(t_1) = 0$ an easy calculation shows that $g'(N) = 2t_1 = \cos^{-1} f(N)$. Therefore, the estimate $1/L \leq (g(N+1) - g(N)) \leq \pi/4$ is valid. By applying the lemma of Muckenhoupt we get $|\cos 2\psi(t_1)| > 1/(200)$ for at least $2/3$ of the integers in $I(y_1)$.

Lemma 2.6. *Let t_1 and t_2 be defined as in Lemma* 2.3. *Under the assumptions of Lemma* 2.5

$$(2.32) \qquad\qquad \sin 2t_2 \geq 1600 \sin 2t_1.$$

Proof. Following the same notations as in Lemma 2.5 we have $\cos 2t_1 = \sigma + \mu$ and $\cos 2t_2 = \sigma - \mu$. Since $u_1^2 \geq \frac{1}{2}(1 - \varepsilon^2)$ and $v_1^2 \geq \frac{1}{2}(1 - \varepsilon^2)$ we have $4u_1^2 v_1^2 \geq (1 - \varepsilon^2)^2$ or $2\sigma \geq (1 - \varepsilon^2)$. Also it is clear that $1 - \tau^2 \leq \varepsilon^2$. Therefore, as $\mu^2 > \sigma^2$ we have

$$(2.33) \quad \cos 2t_1 = \sigma + \mu \geq 2\sigma \geq (1 - \varepsilon^2) \quad \text{or} \quad \sin^2 2t_1 \leq 1 - (1 - \varepsilon^2)^2 \leq 2\varepsilon^2.$$

Again since $\mu^2 = 1 - \tau^2 + \sigma^2 \leq \sigma^2 + \varepsilon^2 \leq (\sigma + \varepsilon)^2$ we see that $\mu \leq (\sigma + \varepsilon)$ or $\mu - \sigma \leq \varepsilon$. Therefore,

$$(2.34) \qquad \cos^2 2t_2 = (\sigma - \mu)^2 \leq \varepsilon^2 \quad \text{or} \quad \sin^2 2t_2 \geq (1 - \varepsilon^2).$$

Hence $\sin 2t_2 \geq 1600 \sin 2t_1$ will hold true once we have $(1 - \varepsilon^2) \geq 2\varepsilon^2 (1600)^2$, i.e. if we have $\{1 + 2(1600)^2\}\varepsilon^2 \leq 1$ which is true by the choice of ε.

Having proved all the preliminary lemmas we can now complete the proof of Theorem 2.1.

Proof of Theorem 2.1. As we have already remarked we need to show that

$$\sup_y \int |F_n(x,y)| \, dx \geq CN.$$

Looking at just the main term in $\operatorname{Re} F_n(x,y)$ we have in view of Lemma 2.3

$$(2.35) \qquad |F_n(x,y)| \geq cN^{-1/2}\{\tfrac{1}{2}(\sin 2t_1)^{-1}|\cos 2\psi(t_1)| - 2(\sin 2t_2)^{-1}\}.$$

Let E denote the set of all x and y satisfying the conditions of Lemma 2.5. Using the results of Lemma 2.5 and Lemma 2.6 we get for at least $2/3$ of N in the interval I

$$(2.36) \qquad |F_n(x,y)| \geq 2cN^{-1/2}(\sin 2t_2)^{-1} \geq 2cN^{-1/2}.$$

Therefore, for those N we have $\int |F_n(x,y)| \, dx \geq \int_E |F_n(x,y)| \, dx \geq CN$ since the Lebesgue measure of E is $cN^{3/2}$ for some constant c. This completes the proof of the theorem.

Thus we see that in the three-dimensional case the Riesz means of order smaller than 1 are not effective for the L^1 summability. In the sections to come we will get good estimates for the kernel when the index is bigger than 1 and prove the uniform boundedness of the Riesz means.

3. ESTIMATION OF THE KERNEL WHEN $\alpha > (3n - 2)/6$

We will estimate the kernel of the Riesz means when $\alpha > (3n - 2)/6$ in this section. For that purpose we need a good expression for the kernel. We consider $n = 3$ for the sake of simplicity. As in the one-dimensional case our starting point is the Mehler kernel. Consider the kernel $G_r(t, x, y)$ defined as follows:

$$(3.1) \qquad G_r(t, x, y) = \sum_{k \geq 0} e^{-(2k+3)it} r^k \Phi_k(x, y).$$

In view of the Mehler's formula this series can be summed. Let $g(t)$ denote the inverse Fourier transform of the function $h(t)$ defined by $h(t) = (1 - |t|)^\alpha$ for $|t| \leq 1$ and 0 otherwise. Let $S_R(\alpha)(x, y)$ denote the kernel of the Riesz means $S_R(\alpha)$. Multiplying (3.1) by $Rg(Rt)$ and integrating we obtain the following formula:

$$(3.2) \qquad R \int g(Rt) G_r(t, x, y) \, dt = \sum_{K \leq R} (1 - K/R)^\alpha r^k \Phi_k(x, y)$$

where we have put $K = 2k + 3$. Thus the kernel $S_R(\alpha)(x, y)$ is given as a pointwise limit by

$$(3.3) \qquad S_R(\alpha)(x, y) = \lim_{r \to 1} R \int g(Rt) G_r(t, x, y) \, dt.$$

Next we are going to integrate by parts and then pass to the limit under the integral sign.

Recalling the definition of $A_r(t)$ and $B_r(t)$ from the previous section we write (3.3) as

$$(3.4) \qquad S_R(\alpha)(x, y) = \lim_{r \to 1} \pi^{-3/2} R \int g(Rt) A_r(t)^{-3/2} \exp\{B_r(t)\} \, dt.$$

With the same notations as before we have the equation

$$\frac{d}{dt}\{A_r(t)^{-3/2}\{B_r'(t)\}^{-1}\} = (4r^2)^{-1} A_r(t)^{-1/2} C_r(t) E_r(t)^{-1}$$
$$+ b(2r^2)^{-1} A_r(t)^{1/2} D_r(t) E_r(t)^{-2}.$$

Therefore, we can write $S_R(\alpha)(x, y) = \lim_{r \to 1}\{L_r(x, y) - M_r(x, y) + N_r(x, y)\}$ where $L_r(x, y)$, $M_r(x, y)$ and $N_r(x, y)$ are defined as follows:

$$(3.5) \qquad L_r(x, y) = \pi^{-3/2} \int Rg(Rt) \frac{d}{dt}\{A_r(t)^{-3/2}\{B_r'(t)\}^{-1} \exp\{B_r(t)\}\} \, dt,$$

$$(3.6) \qquad M_r(x, y) = \pi^{-3/2}(4r^2)^{-1} \int Rg(Rt) \exp B_r(t) A_r(t)^{-1/2} C_r(t) E_r(t)^{-1} \, dt,$$

$$(3.7) \quad N_r(x,y) = \pi^{-3/2} b (2r^2)^{-1} \int Rg(Rt) \exp B_r(t) A_r(t)^{1/2} D_r(t) E_r(t)^{-2} \, dt \, .$$

Denote the limits of $L_r(x,y)$, $M_r(x,y)$ and $N_r(x,y)$ as r tends to 1 by $L(x,y)$, $M(x,y)$ and $N(x,y)$ respectively. An easy calculation gives the following with $\varphi(t) = -b \operatorname{cosec} 2t + \frac{1}{2} a^2 \cot 2t$:

$$(3.8) \quad L(x,y) = \pi^{-3/2} \int Rg(Rt) \frac{d}{dt} \{ (\sin 2t)^{1/2} (a^2 - 2b \cos 2t)^{-1} e^{i\varphi(t)} \} \, dt \, ,$$

$$(3.9) \quad M(x,y) = \pi^{-3/2} \int Rg(Rt) (\sin 2t)^{-1/2} \cos 2t (a^2 - 2b \cos 2t)^{-1} e^{i\varphi(t)} \, dt \, ,$$

$$(3.10) \quad N(x,y) = \pi^{-3/2} \int Rg(Rt) (\sin 2t)^{-3/2} \varphi''(t) \varphi'(t)^{-2} e^{i\varphi(t)} \, dt \, .$$

Lemma 3.1. *Assume that x is different from y and $-y$. Then $S_R(\alpha)(x,y)$ is given by*

$$S_R(\alpha)(x,y) = L(x,y) - M(x,y) + N(x,y) \, .$$

Proof. We only need to show that we can pass to the limit under the integral sign. But that is easily done in exactly the same way as we did in the proof of Lemma 2.1.

Finally we need to compute the function $g(t)$. Since $n = 3$, we are interested in the convergence of the Riesz means of order α where $\alpha > 1$. We have

Lemma 3.2. *Assume that $1 < \alpha < 2$. Then g is a bounded function and for $t > 0$ it is given by $g(t) = \sum_{1 \le i \le 3} c_i g_i(t)$ where $g_1(t) = t^{-\alpha-1} e^{it}$, $g_2(t) = t^{-\alpha-1} e^{-it}$, $g_3(t) = O(t^{-2})$.*

Proof. The proof is similar to that of the corresponding lemma in [3]. Here we have to integrate by parts twice to make sure that the resulting integral will converge.

Now we are in a position to estimate the Riesz kernel. First we estimate the kernel when α is greater than $7/6$. In this case the estimates are obtained more or less in the same way as the estimates for the one-dimensional kernel are obtained. In the process we will see that many terms give estimates which are good even when $\alpha > 1$. We will separate out the bad terms and estimate them in the next section. Again the main tools are the method of stationary phase and integration by parts. First we consider the terms M and N which are very easy to estimate.

For the kernel $S_R(\alpha)(x,y)$ we have an easy estimate viz. $|S_R(\alpha)(x,y)| \le C R^{3/2}$. This can be proved by induction using the bounds on the one-dimensional Hermite functions. We give a proof of this fact in Lemma 4.5. First let us estimate the integrals M and N. $M(x,y)$ is given by

$$(3.11) \quad M(x,y) = R \int g(Rt) (\sin 2t)^{-1/2} \cos 2t (a^2 - 2b \cos 2t)^{-1} e^{i\varphi(t)} \, dt \, .$$

Since the function g is even it is enough to estimate the integral

$$(3.12) \qquad I(x,y) = R \int_{t \geq 0} g(Rt)(\sin 2t)^{-1/2} \cos 2t (a^2 - 2b \cos 2t)^{-1} e^{i\varphi(t)} \, dt.$$

This integral looks very much like the one that appeared in [3]. Using the periodicity of the function φ we can easily convince ourselves that we have only to estimate terms of the form

$$A_k = R \int_0^{\pi/4} g\{R(t + k\pi)\}(\sin 2t)^{-1/2} \cos 2t \, e^{i\varphi(t)} \, dt,$$

$$B_k = R \int_0^{\pi/4} g\{R(t + k\pi + \pi/4)\}(\sin 2t)^{-1/2} \cos 2t \, e^{i\varphi^*(t)} \, dt,$$

where $\varphi^*(t) = b \operatorname{cosec} 2t + \frac{1}{2}a^2 \cot 2t$. The following estimates are easily checked. For $0 \leq t \leq \pi/4$

$$(3.13) \qquad -\varphi'(t) \sin^2 2t = a^2 - 2b \cos 2t \geq \tfrac{1}{2}|x - y|^2,$$

$$(3.14) \qquad -\varphi^{*\prime}(t) \sin^2 2t = a^2 + 2b \cos 2t \geq \tfrac{1}{2}|x + y|^2,$$

$$(3.15) \qquad |\varphi''(t) \sin 2t| \leq 4|\varphi'(t)|,$$

$$(3.16) \qquad |\varphi^{*\prime\prime}(t) \sin 2t| \leq 4|\varphi^{*\prime}(t)|.$$

What really matters is the estimation of A_0 as the other terms are easily estimated. We will now prove the following estimate for A_0.

Lemma 3.3. *There is a constant C independent of x, y and R such that*

$$(3.17) \qquad |A_0| \leq C R^{3/2} (1 + R^{1/2}|x - y|)^{-7/2}.$$

Proof. For the sake of convenience let us replace x and y by $R^{1/2}x$ and $R^{1/2}y$ and consider

$$(3.18) \qquad D_0 = \int_{0 \leq t \leq \pi/4} g\{R(t)\}(\sin 2t)^{-1/2} \cos 2t (a^2 - 2b \cos 2t)^{-1} e^{iR\varphi(t)} \, dt.$$

We need to estimate D_0 only when $|x - y| \geq 1/R$. We will prove that $|D_0| \leq C R^{-2}|x - y|^{-7/2}$. If we combine this with the estimate $|S_R(\alpha)(x,y)| \leq C R^{3/2}$ we obtain $|D_0| \leq C R^{3/2}(1 + R|x - y|)^{-7/2}$. Split the integral into two parts viz. $D_0 = E_0 + E = \int_{0 \leq t \leq 1/R} + \int_{1/R \leq t \leq \pi/4}$. Integrating by parts,

$$(3.19) \qquad |E_0| \leq C R^{-2}|x - y|^{-7/2}.$$

The term E is a sum of three terms corresponding to the three terms in the expression for g. We will consider only the term corresponding to g_1 as the estimation of the other terms are similar. So we consider the integral

$$E_1 = R^{-\alpha-1} \int_{R^{-1}}^{\pi/4} t^{-\alpha-1}(\sin 2t)^{-1/2}\{a^2 - 2b \cos 2t\}^{-1} \cos 2t \, e^{iRt} e^{iR\varphi(t)} \, dt.$$

For $|x - y| \geq 1/2$ we integrate by parts to give the estimate $|E_3| \leq CR^{-2}|x - y|^{-4}$. When $1/R \leq |x - y| \leq 1/2$ we split the integral into two parts as follows. Let $2B = |x - y|$ and write $E_1 = \int_{1/R \leq t \leq B/2} + \int_{B/2 \leq t \leq \pi/4}$. If $\psi(t) = t + \varphi(t)$, we have $|\psi'(t)| \geq c|x - y|^2$ for $1/R \leq t \leq \frac{1}{2}B$. Therefore, an integration by parts gives the estimate $R^{-2}|x - y|^{-7/2}$ for the first integral. Integration gives the estimate $R^{-\alpha-1}|x-y|^{-\alpha-5/2}$ for the second integral. Since $1/R \leq |x - y|$ and $\alpha > 1$, we get the estimate $R^{-2}|x - y|^{-7/2}$. Thus we have shown $|E_1| \leq CR^{-2}|x - y|^{-7/2}$. Returning to the original variables completes the proof of the lemma.

For A_k we can show that $|A_k| \leq Ck^{-\alpha}R^{3/2}(1 + R^{1/2}|x - y|)^{-7/2}$. The estimation of the term $N(x,y)$ is similar. At some point we have to use the estimate

$$(3.20) \qquad |\varphi'''(t)| \leq \operatorname{cosec} 2t|\varphi''(t)| + |\varphi'(t)|$$

which can be easily checked. The integrals involving φ^* give estimates in terms of $x + y$. We can sum all the terms to get the following final estimate for M and N.

Proposition 3.1. *For some constant C independent of x, y and R we have*

$$|M(x,y)| \leq C\{R^{3/2}(1 + R^{1/2}|x - y|)^{-7/2} + R^{3/2}(1 + R^{1/2}|x + y|)^{-7/2}\}$$

under the condition that $\alpha > 1$. A similar estimate holds for the integral $N(x,y)$ also.

Let us now turn our attention to the estimation of L. Recall that L is given by the integral

$$(3.21) \qquad L(x,y) = R \int g(Rt)\frac{d}{dt}\{(\sin 2t)^{1/2}(a^2 - 2b\cos 2t)^{-1}e^{i\varphi(t)}\}\,dt.$$

Integration by parts reduces L to the following form.

$$(3.22) \qquad L(x,y) = R^2 \int g'(Rt)(\sin 2t)^{1/2}(a^2 - 2b\cos 2t)^{-1}e^{i\varphi(t)}\,dt.$$

Since g is an even function it is enough to consider the integral from 0 to infinity. As in the case of M and N we can write the above integral as an infinite sum of terms of the form

$$I_k(x,y) = R^2 \int_0^{\pi/4} g'\{R(t + k\pi)\}(\sin 2t)^{1/2}(a^2 - 2b\cos 2t)^{-1}e^{iR\varphi(t)}\,dt,$$

$$J_k(x,y) = R^2 \int_0^{\pi/4} g\{R(t + k\pi + \pi/4)\}(\sin 2t)^{1/2}(a^2 - 2b\cos 2t)^{-1}e^{iR\varphi^*(t)}\,dt.$$

I_0 is the only difficult term to estimate. Other I_k are easily estimated. The estimation of J_k are similar. The only difference is that the estimates will be in terms of $(x + y)$. We estimate only I_0.

Splitting the integral into two parts we have $I_0 = S_0 + S$ where $S_0 = \int_{0 \le t \le 1/R}$ and $S = \int_{1/R \le t \le \pi/4}$. Integration by parts easily gives the estimate $|S_0| \le CR^{3/2}(1 + R^{1/2}|x - y|)^{-7/2}$. To estimate S we use the expression for $g(t)$. Since $g(t) = \sum_{1 \le i \le 3} c_i g_i(t)$ we see that $g'(t)$ is a sum of many terms. Among all the terms only one term, namely S_1, is really difficult to estimate. This comes from the term $h_1(t) = t^{-\alpha-1} e^{it}$ of $g'(t)$. The contribution of the other terms are somewhat easy to estimate and so we will not consider them. One can easily show that those terms give an estimate of the form $R^{3/2}(1 + R^{1/2}|x - y|)^{-7/2}$. To estimate S_1 we use the method of stationary phase in the form of the following lemma.

Lemma 3.4 (Van der Corput). *Let φ be a real valued function and assume that $|\varphi^{(\kappa)}(t)| \ge c$. When $k = 1$ further assume that φ' is monotonic. Then we have*

$$\left| \int_{[a,b]} \omega(t) e^{iR\varphi(t)} \, dt \right| \le CR^{-1/k} \left\{ |\omega(b)| + \int_{[a,b]} |\omega'(t)| \, dt \right\}.$$

The estimation of S_1 is very similar to the estimation of the corresponding integral in the one-dimensional case. So, detailed proofs will be given only if there is a significant difference in the present case. Our aim is to prove the following proposition.

Proposition 3.2. *Assume that $7/6 < \alpha < 3/2$. Then we have the following estimate for S_1:*

$$|S_1| \le CR^{3/2}(1 + R^{1/2}|x - y|)^{-\alpha - 11/6}.$$

The proof follows in several steps. Recall that the integral S_1 is given by

$$S_1 = R^{-\alpha+1} \int_{R^{-1}}^{\pi/4} t^{-\alpha-1} (\sin 2t)^{1/2} (a^2 - 2b \cos 2t)^{-1} e^{iRt} e^{i\varphi(t)} \, dt.$$

For $|x - y| \le R^{-1/2}$ we use the obvious estimate $CR^{3/2}$. For $|x - y| \ge R^{-1/2}$ we will prove that $|S_1| \le CR^{-\alpha/2+7/12} |x - y|^{-\alpha-11/6}$. These two estimates will then prove the proposition. Getting the estimate $|S_1| \le CR^{-\alpha/2+7/12} |x - y|^{-\alpha-11/6}$ when $|x - y| \ge 2\delta$ is easy. Here δ is any small positive number. As in the one-dimensional case we first show that in the interval $0 \le t \le \pi/4$, $|\psi'''(t)| \ge c|x - y|^2$ where $\psi(t) = t + \varphi(t)$. After a preliminary integration by parts we apply Lemma 3.4. That will produce the required estimate.

Estimating S_1 when $R^{-1/2} \le |x - y| \le 2\delta$ is more difficult. Replacing x and y by $R^{1/2}x$ and $R^{1/2}y$ we consider the integral

$$K = R^{-\alpha} \int_{R^{-1}}^{\pi/4} t^{-\alpha-1} (\sin 2t)^{1/2} (a^2 - 2b \cos 2t)^{-1} e^{iR\psi(t)} \, dt$$

where $\psi(t) = t + \varphi(t)$. We will prove that $|K| \le CR^{-\alpha-1/3} |x - y|^{-\alpha-11/6}$. Let $B = 2|x - y|$ and

$$I = R^{-\alpha} \int_{R^{-1}}^{B/2} t^{-\alpha-1} (\sin 2t)^{1/2} (a^2 - 2b \cos 2t)^{-1} e^{iR\psi(t)} \, dt,$$

$$J = R^{-\alpha} \int_{B/2}^{\pi/4} t^{-\alpha-1} (\sin 2t)^{1/2} (a^2 - 2b \cos 2t)^{-1} e^{iR\psi(t)} \, dt,$$

so that $K = I + J$. Since $-\psi'(t) \sin^2 2t \geq B^2$ for $0 \leq t \leq B/2$ an integration by parts will give the estimate $|I| \leq CR^{-\alpha-1/2}|x-y|^{-\alpha-2}$. To estimate J we rewrite it as

$$J = -iR^{-\alpha-1} \int_{B/2}^{\pi/4} t^{-\alpha-1} (\sin 2t)^{1/2} (a^2 - 2b \cos 2t)^{-1} e^{iR\varphi(t)} d\{e^{iRt}\}.$$

Integrating by parts the boundary terms give the estimates $R^{-\alpha-1}|x-y|^{-2}$ and $R^{-\alpha-1}|x-y|^{-\alpha-5/2}$. Since $|x-y| \geq R^{-1}$ both estimates are bounded by $R^{-\alpha-1/2}|x-y|^{-\alpha-2}$. Many of the differentiated terms give the same estimate. It remains to consider the following term:

$$J' = R^{-\alpha} \int_{B/2}^{\pi/4} t^{-\alpha-1} (\sin 2t)^{-3/2} e^{iR\psi(t)} \, dt.$$

The estimation of J' is easy when $b \leq 0$. Assume that δ is small enough so that $8\delta^2 \leq 1$. Since $|x|^2 + |y|^2 \leq |x-y|^2 \leq 4\delta^2 \leq 1/2$ there is only one stationary point for the function ψ which is given by $\cos 2t_1 = b + m$ where $m^2 = (1 - a^2 + b^2)$. Observe that $m^2 \geq 1/4$ since $|x|^2 + |y|^2 \leq 1/2$. When $b > 0$ but $|x|^2 + |y|^2 \leq 1/2$, again there is only one stationary point. The proof of the following lemma is exactly the same as the proof of the corresponding lemma in the one-dimensional case [3] and so it will not be repeated here.

Lemma 3.5. *Assume that* $b \leq 0$ *or* $b > 0$ *with* $|x|^2 + |y|^2 > 1/2$. *Then for* $R^{-1} \leq |x-y| \leq 2\delta$ *we have*

$$|J'| \leq CR^{-\alpha-1/2}|x-y|^{-\alpha-2}.$$

Estimation of J' when $b > 0$ and $|x|^2 + |y|^2 > 1/2$ is troublesome. We have to treat several cases. First consider the case when $|x|^2 + |y|^2 > 4$. The estimation of J' is easy in this case. We prove the following lemma.

Lemma 3.6. *Assume that* $b > 0$ *and* $|x|^2 + |y|^2 > 4$. *Then for* $R^{-1} \leq |x-y| \leq 2\delta$ *we have*

$$|J'| \leq CR^{-\alpha-1/2}|x-y|^{-\alpha-2}.$$

Proof. We claim that $|\psi'(t)| \geq 1$. To prove the claim observe that $\psi'(t)$ attains a maximum at the point t_0 where $\psi''(t_0) = 0$ since $\psi'''(t) \leq 0$. The point t_0 is defined by

(3.23) $$\cos 2t_0 = (|x+y| - |x-y|)(|x+y| + |x-y|)^{-1}.$$

Let us calculate $\psi'(t_0)$. With $\lambda = \cos 2t_0$ we have

(3.24) $$-\psi'(t_0) \sin^2 2t_0 = \lambda^2 - 2b\lambda + a^2 - 1 = \lambda^{-1}\{\lambda^3 - 2b\lambda^2 + a^2\lambda - \lambda\}.$$

Since $\psi''(t_0) = 0$ we have $a^2\lambda - b\lambda^2 - b = 0$ or $a^2\lambda = b\lambda^2 + b$ so that $\{\lambda^3 - 2b\lambda^2 + a^2\lambda - \lambda\} = (b - \lambda)(1 - \lambda^2)$. Therefore, $\psi'(t_0) = 1 - b\lambda^{-1}$. Since

$$\lambda^{-1} = (|x + y| - |x - y|)^{-1}(|x + y| + |x - y|) = \frac{1}{4b}(|x + y| + |x - y|)^2$$

we get $\psi'(t_0) = \frac{1}{4}\{4 - 2a^2 - 2|x - y||x + y|\}$. Since $a^2 \geq 4$, $\psi'(t_0) \leq -1$ or $|\psi'(t_0)| \geq 1$. Hence the claim.

Having proved the claim now it is easy to prove the lemma. Since $\psi'(t)$ is monotone in each of the intervals $\frac{1}{2}B \leq t \leq t_0$ and $t_0 \leq t \leq \pi/4$ we can split J' into two parts and apply Lemma 3.4 to obtain the estimate $|J'| \leq CR^{-\alpha-1}|x - y|^{-\alpha-5/2}$. Since $|x - y| \geq R^{-1}$ this completes the proof.

Finally we consider the case $1/2 \leq |x|^2 + |y|^2 \leq 4$. As in the case of one dimension by expanding $\operatorname{cosec} 2t$ and $\tan t$ in powers of t we can write $\psi(t) = \psi_1(t) + \psi_2(t)$ where

$$(3.25) \qquad \psi_1(t) = \{1 + \tfrac{1}{6}|x - y|^2 - \tfrac{1}{2}(|x|^2 + |y|^2)\}t + \tfrac{1}{4}t^{-1}|x - y|^2,$$

$$(3.26) \qquad \psi_2(t) = \tfrac{1}{2}|x - y|^2 a(t) - \tfrac{1}{2}(|x|^2 + |y|^2)b(t),$$

where $a(t)$ and $b(t)$ are both $O(t^3)$. Again we write J' as a sum of two integrals. Let $\rho > 0$ be a small number. Define E and F by

$$E = R^{-\alpha} \int_{B/2}^{\rho B^{3/5}} t^{-\alpha-1}(\sin 2t)^{-3/2} e^{iR\psi(t)} \, dt,$$

$$F = R^{-\alpha} \int_{\rho B^{3/5}}^{\pi/4} t^{-\alpha-1}(\sin 2t)^{-3/2} e^{iR\psi(t)} \, dt,$$

so that $J' = E + F$. To estimate E we consider two cases. First assume that $1/2 \leq |x|^2 + |y|^2 \leq 2$. Then taking $\rho = 1$ or $\rho = 1/3$ E can be estimated as in the one-dimensional case. We have

Lemma 3.7. *Assume that $b > 0$ and $1/2 \leq |x|^2 + |y|^2 \leq 2$. Then the estimate*

$$(3.27) \qquad |E| \leq CR^{-\alpha-1/2}|x - y|^{-\alpha-2}$$

is valid for $R^{-1} \leq |x - y| \leq 2\delta$ provided δ is chosen sufficiently small.

Next we will estimate E when $2 \leq |x|^2 + |y|^2 \leq 4$. For that purpose let us make the following observations. $-\psi'(t)\sin^2 2t = (\lambda^2 - 2b\lambda + a^2 - 1)$ is a decreasing function of λ for $0 \leq \lambda \leq b$. If b is > 1 we immediately obtain $-\psi'(t)\sin^2 2t \geq |x - y|^2$ for $0 \leq t \leq \pi/4$. If $b \leq 1$ we will have

$$(3.28) \qquad -\psi'(t)\sin^2 2t \geq (a^2 - b^2 - 1).$$

We claim that $(a^2 - b^2 - 1) \geq \frac{1}{2}|x - y|^2$. We need to prove that $(a^2 - b^2 - 1) \geq \frac{1}{2}(a^2 - 2b)$ or $\frac{1}{2}a^2 \geq (1 - b + b^2)$. Since $b \leq 1$, $(1 - b + b^2) \leq 1$ and since $\frac{1}{2}a^2 \geq 1$ the claim is proved.

Lemma 3.8. *Assume that* $b > 0$ *and* $2 \leq |x|^2 + |y|^2 \leq 4$. *Then the estimate*

$$|E| \leq CR^{-\alpha-1/2}|x - y|^{-\alpha-2}$$

is valid for $R^{-1} \leq |x - y| \leq 2\delta$ *provided* δ *is chosen sufficiently small.*

Proof. We need to get another estimate for the second derivative of ψ. We claim that

(3.29) $$|\psi''(t)| \leq 16|\psi'(t)|^2|x - y|^{-2}\sin 2t, \quad \text{for } B/2 \leq t \leq B^{3/5}.$$

To prove the claim we recall that $\psi''(t) = 4\cosec^3 2t(a^2\cos 2t - b\cos^2 2t - b)$ vanishes at the point t_0 defined by

$$\cos 2t_0 = (|x + y| - |x - y|)(|x + y| + |x - y|)^{-1}.$$

First we will show that $\psi''(t) \geq 0$ in the interval $0 \leq t \leq B^{3/5}$. Since $(a^2\cos 2t - b\cos^2 2t - b)$ is decreasing and vanishing at t_0 it is enough to show that $t_0 \geq B^{3/5}$. As $4t_0^2 \geq \sin^2 2t_0$, we need to check if $\sin^2 2t_0 \geq 4B^{6/5}$. But

$$\sin^2 2t_0 = 4|x + y||x - y|(|x + y| + |x - y|)^{-2} \geq |x - y||x + y|^{-1} \geq 2|x - y|.$$

Therefore, it is enough to check if $|x - y| \geq 2B^{6/5}$ which is guaranteed if δ is small enough. Thus $(a^2\cos 2t - b\cos^2 2t - b)$ has a maximum of $|x - y|^2$ at $t = 0$ and therefore, we have the estimate

(3.30) $$|\psi''(t)| = 4\cosec^3 2t(a^2\cos 2t - b\cos^2 2t - b) \leq 4\cosec^3 2t|x-y|^4|x-y|^{-2}.$$

Since $|\psi'(t)\sin^2 2t| \geq \frac{1}{2}|x - y|^2$ we get

$$|\psi''(t)| \leq 4\cosec^3 2t|x - y|^{-2}4|\psi'(t)\sin^2 2t|^2$$

which proves the claim. Now using the fact that $|\psi'(t)\sin^2 2t| \geq \frac{1}{2}|x - y|^2$ and the estimates on the second derivative an integration by parts will prove the lemma.

Finally the estimation of F is easy. As in the one-dimensional case we can show that for t in the interval $(0, \pi/4), |\psi'''(t)| \geq b$. Since $a \geq 1/2$ and $|x - y| \leq 2\delta$, by choosing δ small enough we can ensure that $|\psi'''(t)| \geq c$ for some constant c. Now an application of Lemma 3.4 gives the estimate $|F| \leq CR^{-\alpha-1/3}|x - y|^{-3\alpha/5-3/2}$. Since $|x - y| \leq 2\delta$, this gives the estimate

(3.31) $$|F| \leq CR^{-\alpha-1/3}|x - y|^{-\alpha-11/6}.$$

If we put all the lemmas together and go back to the original variables we get the estimate $R^{-\alpha/2+7/12}|x - y|^{-\alpha-11/6}$ for S_1 as desired. Combining Propositions 3.1 and 3.2 we obtain the following theorem.

Theorem 3.1. *Assume that* $n = 3$. *For* $\alpha > 7/6$ *we have the following estimate:*

$$|S_R(\alpha)(x, y)| \leq C\{R^{3/2}(1 + R^{1/2}|x - y|)^{-\alpha-11/6} + R^{3/2}(1 + R^{1/2}|x - y|)^{-7/2}\}$$
$$+ C\{R^{3/2}(1 + R^{1/2}|x + y|)^{-\alpha-11/6} + R^{3/2}(1 + R^{1/2}|x + y|)^{-7/2}\}.$$

If we examine the estimates we see that we are using the condition $\alpha > 7/6$ only at two places. When $|x - y| \geq R^{1/2}$ we use the condition to make sure that the kernel is uniformly integrable. When $R^{-1/2} \leq |x - y| \leq R^{1/2}$ we obtained the estimate $R^{-\alpha/2+1/2}|x - y|^{-\alpha-2}$ for many terms. These estimates are good enough as long as α is greater than 1. Only in the estimation of F we used the condition $\alpha > 7/6$. In the next section we will estimate this term when $\alpha > 1$. We will also get an estimate for the kernel when $|x - y| \geq R^{1/2}$.

4. ESTIMATION OF THE KERNEL WHEN $\alpha > (n - 1)/2$

In this section we will get a good estimate for $S_R(\alpha)(x,y)$ when $\alpha > (n - 1)/2$. Recall that we are assuming $n = 3$. As we mentioned in the introduction the estimates we are going to get now are not so neat as the estimates we got in the previous section. However, the estimates are sufficient to prove that the Riesz means $S_R(\alpha)f$ converge to f in the norm for $\alpha > (n - 1)/2$. The following theorem is the main result of this section.

Theorem 4.1. *Assume that $n = 3$ and $\alpha > 1$. Then the kernel of the Riesz means $S_R(\alpha)(x,y)$ is uniformly integrable, i.e. there is a constant C independent of x, y and R such that*

$$(4.1) \qquad \int |S_R(\alpha)(x,y)|\,dx \leq C \quad and \quad \int |S_R(\alpha)(x,y)|\,dy \leq C.$$

Since $S_R(\alpha)(x,y)$ is symmetric in x and y it is enough to prove the first estimate. Replacing x and y by $R^{1/2}x$ and $R^{1/2}y$ we need to prove that

$$(4.2) \qquad R^{3/2} \int |S_R(\alpha)(R^{1/2}x, R^{1/2}y)|\,dx \leq C.$$

As we remarked at the end of the last section we have to estimate the integral

$$F = R^{-\alpha+3/2} \int_{\rho B^{3/5}}^{\pi/4} t^{-\alpha-1}(\sin 2t)^{-3/2} e^{iR\psi(t)}\,dt$$

for $R^{-1} \leq |x - y| \leq 2\delta$, where $2B = |x - y|$. Also we have to estimate the following integral for the range $|x - y| \geq 2\delta$:

$$I = R^{-\alpha+3/2} \int_{R^{-1}}^{\pi/4} t^{-\alpha-1}(\sin 2t)^{1/2}(a^2 - 2b\cos 2t)^{-1} e^{iR\psi(t)}\,dt.$$

Since there is rotational symmetry in the kernel we can assume that $y = (y_1, 0, 0)$. Also we write $x = (x_1, x')$ where x' is in R^2. Introducing polar coordinates in R^2, it is enough to show that

$$(4.3) \qquad \int |F(x_1, y_1, r)|\,r\,dx_1\,dr \leq C \quad and \quad \int |I(x_1, y_1, r)|\,r\,dx_1\,dr \leq C$$

where we have set $r = |x'|$. First we estimate F. Recall that we are in the case $x_1 y_1 \geq 0$. There are two cases to consider. First we consider the case when either x_1 or y_1 exceeds 1. We prove the following lemma in this case.

Lemma 4.1. *Assume that* $x_1 y_1 > 0$, $1/2 \le |x|^2 + |y|^2 \le 4$ *and either* x_1 *or* y_1 *exceeds* 1. *If* δ *is chosen sufficiently small, then for the region* $R^{-1} \le |x - y| \le 2\delta$ *we have the estimate*

$$|F| \le C\{R^{-\alpha+1}|x - y|^{-\alpha-2} + R^{1/2}(1 + R^{1/2}r)^{-2}|x_1 - y_1|^{-\alpha/2-1/4}r^{-1}\}.$$

Proof. Let t_0 be the point where the second derivative of ψ vanishes. As calculated earlier we have $\sin^2 2t_0 = 4|x + y||x - y|(|x + y| + |x - y|)^{-2}$. From this we obtain

$$(4.4) \qquad |x - y||x + y|^{-1} \le \sin^2 2t_0 \le 4|x + y|^{-1}|x - y|.$$

Since $1/2 \le (|x|^2 + |y|^2) \le |x+y|^2 \le 2(|x|^2 + |y|^2) \le 8$ we see that $c_1|x-y|^{1/2} \le \sin 2t_0 \le c_2|x - y|^{1/2}$. Since $|x_1 - y_1| \le \delta$ and x_1 or y_1 exceeds 1 we have $x_1 y_1 \ge c$ for some c in the region of interest. Therefore, for the third derivative of ψ we have the estimate $|\psi'''(s)| \ge x_1 y_1 \ge c$. For $t_0 \le t \le \pi/4$, $\psi''(t)$ is decreasing and we have the estimate

$$|\psi''(t)| = \int_{t_0}^{t} |\psi'''(s)|\, ds \ge c(t - t_0).$$

If $2t_0 \le t \le \pi/4$, we get $|\psi''(t)| \ge c(t - t_0) \ge ct_0 \ge c'|x - y|^{1/2}$. For $B^{3/5} \le t \le t_0$, $\psi''(t)$ is positive and so as above $\psi''(t) \ge c(t_0 - t)$. If $B^{3/5} \le t \le \frac{1}{2}t_0$, we get $|\psi''(t)| \ge c|x - y|^{1/2}$.

Now we can split our integral into three parts. In view of the estimate $|\psi''(t)| \ge c|x-y|^{1/2}$ the integrals taken from $B^{3/5} \le t \le \frac{1}{2}t_0$ and $2t_0 \le t \le \pi/4$ give the estimates $CR^{-\alpha+1}|x - y|^{-3\alpha/5-7/4}$ and $CR^{-\alpha+1}|x - y|^{-\alpha/2-3/2}$. Since $|x - y| \le 2\delta$ the above are dominated by $CR^{-\alpha+1}|x - y|^{-\alpha-2}$. So it remains to estimate the integral

$$G = R^{-\alpha+3/2} \int_{t_0/2}^{2t_0} t^{-\alpha-1}(\sin 2t)^{-3/2} e^{iR\psi(t)}\, dt.$$

To estimate G, we have to consider two cases. First of all let us calculate the first derivative of ψ. An easy calculation shows that $-\psi'(t)\sin^2 2t = \{\lambda^2 - 2x_1 y_1\lambda + x_1^2 + y_1^2 - 1\} + r^2$. From this it is clear that when $x_1 y_1 \ge 1$ we have $-\psi'(t)\sin^2 2t \ge (x_1 - y_1)^2 + r^2$ and when $x_1 y_1 < 1$ we have $-\psi'(t)\sin^2 2t \ge -(1 - x_1^2)(1 - y_1^2) + r^2$. In our present situation as $(1 - x_1^2)(1 - y_1^2)$ is negative we have $|\psi'(t)| \ge r^2$. An application of Lemma 3.4 will give the following estimate for G:

$$(4.5) \qquad |G| \le CR^{-\alpha+1/2}r^{-2}|x - y|^{-\alpha/2-5/4} \le CR^{-1/2}r^{-3}|x_1 - y_1|^{-\alpha/2-1/4}.$$

We will use this estimate for the range $R^{-1/2} \le r \le 2\delta$. When $0 \le r \le R^{-1/2}$, we use the fact that $|\psi'''(s)| \ge x_1 y_1 \ge c$ to get the estimate

$$(4.6) \qquad |G| \le CR^{-\alpha+3/2-1/3}|x - y|^{-\alpha/2-5/4} \le CR^{1/2}r^{-1}|x_1 - y_1|^{-\alpha/2-1/4}.$$

Combining the two estimates we get

$$|G| \leq Cr^{-1}R^{1/2}(1 + R^{1/2}r)^{-2}|x_1 - y_1|^{-\alpha/2-1/4}$$

and this completes the proof of the lemma.

Next we will estimate F when both x_1 and y_1 are less than or equal to 1. The estimate will be in terms of a function $k_1(x,y)$. Let $d^2 = (1 - x_1^2)(1 - y_1^2)$ and define $k_1(x,y)$ by

$$\begin{aligned}
k_1(x,y) &= (d^2 - r^2)^{-1/2}r^{-1}, \quad \text{for } 0 \leq r \leq d \\
(4.7) \qquad &= R^{2/3}\{1 + R^{2/3}(r^2 - d^2)\}^{-3/2}, \quad \text{for } d \leq r \leq 2\delta \\
&= 0, \quad \text{otherwise.}
\end{aligned}$$

Lemma 4.2. *Assume that $x_1y_1 > 0$, $1/2 \leq |x|^2 + |y|^2 \leq 4$ and both x_1 and y_1 are ≤ 1. Then for the region $R^{-1} \leq |x - y| \leq 2\delta$ we have the estimate*

$$|F| \leq C\{R^{-\alpha+1}|x - y|^{-\alpha-2} + k_1(x,y)|x_1 - y_1|^{-3\alpha/5-1/4}\}.$$

Proof. Observe that since both x_1 and y_1 are less than or equal to 1 we have $-\psi'(t)\sin^2 2t \geq -d^2 + r^2$. First of all let us consider the case when there are no stationary points, i.e. when $r > d$. For the region $R^{-1} \leq |x - y| \leq R^{-2/5}$ we just integrate to get the estimate $|F| \leq CR^{-\alpha+3/2}|x - y|^{-3\alpha/5-9/10}$. Since $R^{1/2} \leq |x - y|^{-5/4}$ we get

$$(4.8) \quad |F| \leq CR^{-\alpha+1}|x - y|^{-3\alpha/5-5/4-9/10} \leq CR^{-\alpha+1}|x - y|^{-\alpha-2}|x - y|^{(8\alpha-3)/10}.$$

Since $8\alpha - 3 \geq 0$ we obtain the estimate $|F| \leq CR^{-\alpha+1}|x - y|^{-\alpha-2}$. When $R^{-2/5} \leq |x - y| \leq 2\delta$ we proceed as follows. For $r^2 - d^2 \geq R^{-2/3}$ we apply Lemma 3.4. Since $|\psi'(t)| \geq r^2 - d^2$, we get

$$\begin{aligned}
|F| &\leq CR^{-\alpha+1/2}(r^2 - d^2)^{-1}|x - y|^{-3\alpha/5-3/2} \\
(4.9) \qquad &\leq CR^{-\alpha+1/2}(r^2 - d^2)^{-1}(\beta^2 + r^2)^{-3\alpha/10-1/8}(\beta^2 + r^2)^{-5/8} \\
&\leq CR^{-\alpha+1/2}(r^2 - d^2)^{-1}\beta^{-3\alpha/5-1/4}(\beta^2 + r^2 - d^2)^{-1/2}(\beta^2 + r^2)^{-1/8}
\end{aligned}$$

where $\beta = |x_1 - y_1|$. Since $|x - y|^{-1} \leq R^{2/5}$ we get the estimate

$$\begin{aligned}
(4.10) \qquad |F| &\leq CR^{-\alpha+1/2+1/10}(r^2 - d^2)^{-3/2}\beta^{-3\alpha/5-1/4} \\
&\leq CR^{-1/3}(r^2 - d^2)^{-3/2}\beta^{-3\alpha/5-1/4}.
\end{aligned}$$

For $0 \leq r^2 - d^2 \leq R^{-2/3}$ we apply Lemma 3.4 with $k = 3$. In view of the estimate $|\psi'''(t)| \geq c$ we obtain $|F| \leq CR^{-\alpha+3/2-1/3}|x - y|^{-3\alpha/5-3/2}$. Again using $|x - y|^{-1} \leq R^{2/5}$ we get the estimate

$$(4.11) \qquad |F| \leq CR^{-\alpha+3/2-1/3+1/2}\beta^{-3\alpha/5-1/4} \leq CR^{2/3}\beta^{-3\alpha/5-1/4}.$$

Combining estimates (4.10) and (4.11) we finally get the estimate

$$(4.12) \qquad |F| \leq R^{2/3}\{1 + R^{2/3}(r^2 - d^2)\}^{-3/2}\beta^{-3\alpha/5-1/4}.$$

We will now estimate F when $r \leq d$. Since $|x - y| \leq 2\delta$, by choosing δ sufficiently small we can ensure that $x_1 y_1 \geq c$ so that $|\psi'''(t)| \geq c$. There are two cases to consider. First assume that

$$(4.13) \qquad m = (d^2 - r^2)^{1/2} \leq R^{-1/6}|x - y|^{3/10}.$$

Using the above estimate $|\psi'''(t)| \geq c$ we get

$$
\begin{aligned}
(4.14) \qquad |F| &\leq CR^{-\alpha + 3/2 - 1/3}|x - y|^{-3\alpha/5 - 3/2} \\
&\leq CR^{-\alpha + 3/2 - 1/3}(d^2 - r^2)^{-1/2}R^{-1/6}|x - y|^{3/10}|x - y|^{-3\alpha/5 - 3/2} \\
&\leq C(d^2 - r^2)^{-1/2}r^{-1}|x_1 - y_1|^{-3\alpha/5 - 1/4}.
\end{aligned}
$$

This takes care of the case when $m \leq R^{-1/6}|x - y|^{3/10}$.

Next assume that $m > R^{-1/6}|x - y|^{3/10}$. In this case there are two distinct stationary points given by $\cos 2t_1 = b + m$ and $\cos 2t_2 = b - m$. Let t_0 denote the point where $\psi''(t) = 0$. Without loss of generality we can assume that $B^{3/5} \leq t_1 < t_0 < t_2 \leq \pi/4$. Let $E_1 = (B^{3/5}, t_1)$, $E_2 = (t_1, t_0)$, $E_3 = (t_0, t_2)$ and $E_4 = (t_2, \pi/4)$. Let F_j be the integral taken over E_j so that $F = F_1 + F_2 + F_3 + F_4$. We will first estimate F_2. Since $\psi'(t_1) = 0$, by Taylor's theorem, we have for $t_1 \leq t \leq t_0$

$$\psi'(t) = (t - t_1)\psi''(t_1) + \int_{t_1}^{t}(t - s)\psi''''(s)\,ds = (t - t_1)\psi''(t) + \int_{t_1}^{t}(s - t_1)(-\psi'''(s))\,ds$$

which gives $\psi'(t) \geq (t - t_1)\psi''(t)$. As $\psi''(t) \geq 0$ for $t_1 \leq t \leq t_0$ we have

$$|\psi'''(t)| = 6\cot 2t\,\psi''(t) + 8\,\mathrm{cosec}^2 2t(a^2 - 2b\cos 2t)$$

and therefore,

$$\psi''(t_1) - \psi''(t) \leq (t - t_1)\{6\cot 2t_1 \psi''(t_1) + 8\,\mathrm{cosec}^2 2t_1(a^2 - 2b\cos 2t_1)\}.$$

Since $\psi''(t_1) = m\,\mathrm{cosec}\, 2t_1$ we get the estimate

$$
\begin{aligned}
(4.15) \qquad \psi''(t_1) - \psi''(t) &\leq (t - t_1)\mathrm{cosec}\, 2t_1 \psi''(t_1)(6 + 16/m) \\
&\leq 22(1/m)(t - t_1)\,\mathrm{cosec}\, 2t_1 \psi''(t_1).
\end{aligned}
$$

Let $\varepsilon = \frac{1}{44}m\sin 2t_1$. Then for $0 \leq t - t_1 \leq \varepsilon$, we have $\psi''(t) \geq \frac{1}{2}\psi''(t_1)$. This also gives for $t \geq t_1 + \varepsilon$, $\psi'(t) \geq \psi'(t_1 + \varepsilon) \geq \varepsilon\psi''(t) \geq \frac{1}{2}\varepsilon\psi''(t_1)$, i.e. $\psi'(t) \geq \frac{1}{88}m^2$. Splitting the integral into two parts we apply Lemma 3.4 to both of them. For the integral

$$H = R^{-\alpha + 3/2}\int_{t_1}^{t_1 + \varepsilon}t^{-\alpha - 1}(\sin 2t)^{-3/2}e^{iR\psi(t)}\,dt$$

we get the estimate

$$|H| \leq CR^{-\alpha + 1}m^{-1/2}(\sin 2t_1)^{1/2}\left\{(t_1 + \varepsilon)^{-\alpha - 5/2} + \int_{t_1}^{t_1 + \varepsilon}t^{-\alpha - 7/2}\,dt\right\}.$$

Since $t_1 \geq B^{3/5}$ the above gives the estimate

$$(4.16) \qquad \begin{aligned} |H| &\leq C R^{-\alpha+1} m^{-1} |x-y|^{-3\alpha/5-6/5} \\ &\leq C (d^2 - r^2)^{-1/2} r^{-1} |x_1 - y_1|^{-3\alpha/5-1/4}. \end{aligned}$$

Since $\psi'(t) \geq \frac{1}{88} m^2 \geq \frac{1}{88} R^{-1/3} |x-y|^{3/5}$ in the interval $t_1 + \varepsilon \leq t \leq t_0$, for the remaining part K of F we get the following estimate:

$$(4.17) \qquad |K| \leq C R^{-\alpha+1/2} R^{1/3} |x-y|^{-3\alpha/5-3/2-3/5} \leq C R^{-\alpha+1} |x-y|^{-\alpha-2}.$$

This takes care of F_2. The estimation of F_1 is similar. Next consider the term F_3. Again we use a Taylor expansion of ψ'. Since $\psi'(t_2) = 0$ we have for $t_0 \leq t \leq t_2$

$$\psi'(t) = (t-t_2)\psi''(t_2) + \int_{t_2}^{t} (t-s)\psi'''(s)\,ds = (t_2-t)|\psi''(t)| + \int_{t}^{t_2} (t_2-s)(-\psi'''(s))\,ds$$

which gives $\psi'(t) \geq (t_2 - t)|\psi''(t)|$. Also we have

$$|\psi'''(t)| \leq 6 \cot 2t |\psi''(t)| + 8 \operatorname{cosec}^2 2t (a^2 - 2b \cos 2t)$$

and therefore

$$\psi''(t) - \psi''(t_2) = \int_{t}^{t_2} \{-\psi'''(s)\}\,ds = \int_{t}^{t_2} \{6 \cot 2s |\psi''(s)| + 16 \operatorname{cosec}^2 2s\}\,ds.$$

If $\frac{1}{2} t_2 \leq t \leq t_2$ we get

$$(4.18) \qquad \psi''(t) - \psi''(t_2) \leq (t_2 - t)\{6 \operatorname{cosec} t_2 |\psi''(t_2)| + 16 \operatorname{cosec}^2 t_2\}.$$

Since $\operatorname{cosec}^2 t_2 \leq 4 \operatorname{cosec}^2 2t_2$ this gives

$$(4.19) \qquad -|\psi''(t)| + |\psi''(t_2)| \leq 76(t_2 - t) m^{-1} |\psi''(t_2)| \operatorname{cosec} 2t_2.$$

If $\varepsilon = \frac{1}{152} m \sin 2t_2$ then for $0 \leq t_2 - t \leq \varepsilon$, we have $|\psi''(t)| \geq \frac{1}{2}|\psi''(t_2)| = \frac{1}{2} m \operatorname{cosec} 2t_2$. This gives for $t_0 \leq t \leq t_1 - \varepsilon$, the bound $|\psi'(t)| \geq cm^2 \geq cR^{-1/3}|x-y|^{3/5}$. Using these observations F_3 is estimated just like F_2. The estimation of F_4 is similar. Putting all the estimates together we complete the proof of the lemma.

This takes care of the term F. It is easy to check that

$$\int_{1/R \leq |x-y| \leq 2\delta} F(x_1, y_1, r) r\, dx_1\, dr \leq C.$$

Therefore, it remains to consider the integral I for the region $|x-y| \geq 2\delta$. As in the case of F we have to treat two cases, namely when there is no stationary point and when there are stationary points. Again the estimates will be in terms of a function $k(x, y)$ which involves d and r. We assume that $1 < \alpha \leq 7/6$. We first consider the case when $x_1 y_1 \geq 1$ or $x_1 y_1 < 1$ but $(1 - x_1^2)(y_1^2 - 1)$ is positive. This is the case with no stationary points. We prove the following lemma.

Lemma 4.3. *Assume that* $x_1y_1 \geq 1$ *or* $x_1y_1 < 1$ *but* $(1 - x_1^2)(y_1^2 - 1) \geq 0$. *Then for* $|x - y| \geq 2\delta$ *we have*

$$|I(x,y)| \leq CR^{1/2}(1 + R^{1/2}r)^{-5/4}(1 + \beta^2)^{-3/4}r^{-1}.$$

Proof. Before going into the proof we recall the following estimate on the second derivative of ψ:

(4.20) $|\psi''(t)| \leq 4|\psi'(t)| \operatorname{cosec} 2t + 4 \operatorname{cosec} 2t.$

As observed earlier $|\psi'(t)| \geq r^2$ in the present situation. We have to consider several cases. When $r \geq \delta$ an integration by parts gives the estimate $|I(x,y)| \leq CR^{-\alpha+1/2}r^{-2}(r^2 + \beta^2)^{-1}$. Since $r \geq \delta$ and $\alpha > 1$ we can rewrite the above estimate as $|I(x,y)| \leq CR^{-1/8}r^{-5/4}r^{-1}(1+\beta^2)^{-3/4}$. When $\beta \geq \delta$ and $R^{-1/2} \leq r \leq \delta$, we split the integral into two parts:

$$I = R^{-\alpha+3/2} \int_{R^{-1}}^{r^{1/2}} t^{-\alpha-1}(\sin 2t)^{-3/2}e^{iR\psi(t)}\,dt$$

$$+ R^{-\alpha+3/2} \int_{r^{1/2}}^{\pi/4} t^{-\alpha-1}(\sin 2t)^{-3/2}e^{iR\psi(t)}\,dt.$$

Integration by parts gives the estimate

$$CR^{-\alpha+1/2}r^{-2}(r^2 + \beta^2)^{-1}r^{-\alpha/2-1/4}$$

for the first part. Since $\beta \geq \delta$ and $R^{-1/2} \leq r$, we get the estimate

$$CR^{-1/8}r^{-5/4}r^{-1}(1 + \beta^2)^{-3/4}.$$

Applying Lemma 3.4 to the second part we get the estimate

$$CR^{-\alpha+1/2}r^{-2}(r^2 + \beta^2)^{-1}r^{-\alpha/2-1/4}$$

which again is dominated by $R^{-1/8}r^{-5/4}r^{-1}(1 + \beta^2)^{-3/4}$. Hence

$$|I(x,y)| \leq CR^{-1/8}r^{-5/4}r^{-1}(1 + \beta^2)^{-3/4}$$

in the present case also.

When $0 \leq r \leq R^{-1/2}$ and $\beta \geq \delta$, we rewrite the integral in the following way. Let us define the function

$$\sigma(t) = -x_1y_1 \operatorname{cosec} 2t + \tfrac{1}{2}(x_1^2 + y_1^2)\cot 2t$$

and write the integral I as follows:

$$I = R^{-\alpha+1/2} \int_{R^{-1}}^{\pi/4} t^{-\alpha-1}(\sin 2t)^{5/2}k(x,y,t)$$

$$\times m(x_1,y_1,t)e^{iRt}e^{iRr^2(\cot 2t)/2}d\{e^{iR\sigma(t)}\}$$

where $k(x,y,t) = (a^2 - 2b\cos 2t)^{-1}$ and $m(x_1,y_1,t) = (4x_1y_1 \sin^2 t + \beta^2)^{-1}$. Integrating by parts the boundary terms give the estimate

$$R^{-\alpha+1/2}\beta^{-2}(r^2 + \beta^2)^{-1}.$$

Many of the differentiated terms give the same estimate. When differentiation falls on e^{iRt} we get the estimate $R^{-\alpha+3/2}\beta^{-2}(r^2+\beta^2)^{-1}$. When differentiation falls on $\exp(iRr^2\cot 2t)$ we get the same estimate since $r \le R^{-1/2}$ and $\alpha < 7/6$. Thus we have obtained $|I(x,y)| \le CR^{-\alpha+3/2}\beta^{-2}(r^2+\beta^2)^{-1}$. Since $\beta \ge \delta$ and $r \le R^{-1/2}$ we get the estimate

$$(4.21) \qquad\qquad |I(x,y)| \le CR^{1/2}r^{-1}(1+\beta^2)^{-3/4}.$$

Thus the following estimates are obtained for $I(x,y)$:

$$(4.22) \qquad |I(x,y)| \le CR^{1/2}r^{-1}(1+\beta^2)^{-3/4} \quad \text{for } 0 \le r \le R^{-1/2},$$

$$(4.23) \qquad |I(x,y)| \le CR^{-1/8}r^{-5/4}r^{-1}(1+\beta^2)^{-3/4} \quad \text{for } r \ge R^{-1/2}.$$

Combining the two estimates we get

$$|I(x,y)| \le CR^{1/2}(1+R^{1/2}r)^{-5/4}(1+\beta^2)^{-3/4}r^{-1}.$$

This completes the proof of the lemma.

Next we want to estimate $I(x,y)$ when $0 \le x_1, y_1 \le 1$ still considering the region $|x - y| \ge 2\delta$. The estimate will be in terms of a function $k_0(x,y)$. Let $d^2 = (1 - x_1^2)(1 - y_1^2)$. The function $k_0(x,y)$ is defined to be 0 unless $0 \le x_1, y_1 \le 1$. When $0 \le x_1, y_1 \le 1$ the function is defined by

$$
\begin{aligned}
k_0(x,y) &= r^{-4}, \quad \text{for } r \ge 2, \ 0 \le \beta \le 1/2 \\
&= R^{1/2}\{1 + R^{1/2}(r^2 - d^2)\}^{-3/2}, \quad \text{for } d \le r \le 2, \ \delta \le \beta \le 1/2 \\
&= (d^2 - r^2)^{-1/2}, \quad \text{for } 0 \le r \le d, \ \delta \le \beta \le 1/2 \\
&= R^{1/2}\{1 + R^{1/2}(r^2 - d^2)\}^{-3/2}, \quad \text{for } d \le r \le 2, \ 0 \le \beta \le \delta \\
&= (d^2 - r^2)^{-1/2}, \quad \text{for } \delta \le r \le d, \ 0 \le \beta \le \delta.
\end{aligned}
$$

We will now prove the following lemma. The proof is somewhat similar to that of Lemma 4.2. Therefore, at times we will make the proof sketchy omitting the details.

Lemma 4.4. *Assume that $0 \le x_1, y_1 \le 1$ and $|x - y| \ge 2\delta$. Then we have the estimate*

$$|I(x,y)| \le C\{k_0(x,y) + R(1 + R|x - y|)^{-3/2}|x - y|^{-2}\}.$$

Proof. There are several cases to be treated. First of all observe that $\beta \le 1/2$ since $0 \le x_1, y_1 \le 1$. First we assume that $0 \le r \le \delta$ and $\delta \le \beta \le 1/2$ and consider the following integral:

$$I_1 = R^{-\alpha+3/2} \int_{R^{-1}}^{\beta/2} t^{-\alpha-1}(\sin 2t)^{1/2}(a^2 - 2b\cos 2t)^{-1}e^{iR\psi(t)}\,dt.$$

Since $1/R \le t \le \frac{1}{2}\beta$ in the range of integration we have $|\psi'(t)| \ge \frac{1}{4}|x - y|^2$. As $\beta \ge \delta$, and $\alpha > 1$ an integration by parts will give the estimate $|I_1| \le$

$CR^{-1/2}|x - y|^{-2}$. Since $|x - y| \leq 2$, the above estimate is dominated by $R(1 + R|x - y|)^{-3/2}|x - y|^{-2}$. Next consider the integral

$$I_2 = R^{-\alpha+3/2} \int_{\beta/2}^{\pi/4} t^{-\alpha-1}(\sin 2t)^{1/2}(a^2 - 2b \cos 2t)^{-1} e^{iR\psi(t)}\, dt.$$

Rewriting this integral as

$$J = -iR^{-\alpha+1/2} \int_{\beta/2 \leq t \leq \pi/4} t^{-\alpha-1}(\sin 2t)^{1/2}(a^2 - 2b \cos 2t)^{-1} e^{iR\varphi(t)} d(e^{iRt})$$

we observe that the boundary terms are bounded by $R^{-\alpha+1/2}|x - y|^{-2}$ which in turn are bounded by $R(1 + R|x - y|)^{-3/2}|x - y|^{-2}$ as before. When the differentiation falls on the integrand many terms give the same estimate. There remains only one term, viz. the one given by the integral

$$J = R^{-\alpha+3/2} \int_{\beta/2}^{\pi/4} t^{-\alpha-1}(\sin 2t)^{-3/2} e^{iR\psi(t)}\, dt$$

to be estimated. Now there are two cases to be considered.

First assume that $r > d$. Since $-\psi'(t) \sin^2 2t \geq r^2 - d^2 > 0$, there are no stationary points in this case. When $r^2 - d^2 \geq R^{-1/2}$ we can apply Lemma 3.4 to obtain the estimate $CR^{-\alpha+1/2}(r^2 - d^2)^{-3/2}$ since $\beta \geq \delta$ and $r \leq 2$. When $r^2 - d^2 \leq R^{-1/2}$, we can use the following estimate on the third derivative of $\psi: |\psi'''(t)| \geq |x - y|^2 \geq 4\delta^2$. Applying the same lemma we see that J is bounded by a constant times $R^{-\alpha+3/2-1/3}$ which is $\leq R^{1/2}$. Thus we have obtained the estimates

(4.24) $$|J| \leq CR^{1/2}, \quad \text{for } 0 \leq r^2 - d^2 \leq R^{-1/2},$$

(4.25) $$|J| \leq CR^{-1/4}(r^2 - d^2)^{-3/2}, \quad \text{for } r^2 - d^2 \geq R^{-1/2}.$$

Combining them we have $|J| \leq CR^{1/2}\{1 + R^{1/2}(r^2 - d^2)\}^{-3/2}$ when $r > d$. Next assume that $r \leq d$. In this case there are two stationary points in the interval of integration. They are given by $\cos 2t_1 = b + m$ and $\cos 2t_2 = b - m$ where $m^2 = d^2 - r^2 \geq 0$. Again we have to treat two subcases. When $m \leq R^{-1/6}$, we apply Lemma 3.4 with $k = 3$. If we use the lower bound $|\psi'''(t)| \geq |x - y|^2 \geq 4\delta^2$ we get the estimate $|J| \leq CR^{-\alpha+3/2-1/3}\beta^{-\alpha-5/2}$. Since $\beta \geq \delta$, and $\alpha \geq 1$ the above gives the estimate $|J| \leq CR^{1/6} \leq C(d^2 - r^2)^{-1/2}$. Let us now assume that $m > R^{-1/6}$. Without loss of generality we can assume that $\beta/2 \leq t_1 \leq t_2 \leq \pi/4$.

As in the proof of Lemma 4.2 we can find an $\varepsilon = cm \sin 2t_1$ such that in an ε neighbourhood of t_1 we have $|\psi''(t)| \geq \frac{1}{2}\psi''(t_1) = m/2 \operatorname{cosec} 2t_1$ and outside the ε neighbourhood the lower bound $|\psi'(t)| \geq m^2 \geq m^{1/2}R^{-1/4}$ holds. From these things we obtain

(4.26) $$CR^{-\alpha-1}\beta^{-\alpha-5/2}m^{-1/2} \leq C(r^2 - d^2)^{-1/4}$$

for the part of the integral taken from t_1 and $t_1 + \varepsilon$. For the integral from $t_1 + \varepsilon$ to t_0 where t_0 is the point at which the second derivative vanishes we get the estimate

$$(4.27) \qquad CR^{-\alpha - 3/4} \beta^{-\alpha - 5/2} m^{-1/2} \leq C(r^2 - d^2)^{-1/4}.$$

The integral from $\beta/2$ to t_1 gives a similar estimate. We can estimate the integral near t_2 in a similar fashion. Putting all the estimates together we have $|J| \leq C(r^2 - d^2)^{1/2}$ in the case when $\delta \leq \beta \leq 1/2$ and $0 \leq r \leq \delta$.

Next we consider the case when $\delta \leq r \leq 1/2$ and $0 \leq \beta \leq \delta$. The estimation is very similar to the previous case. We split the integral at the point $t = r/4$ and then proceed as before. Finally the case when $r \geq 2$ is easy. Since the first derivative of ψ is bounded away from 0 we immediately get the estimate $|I(x, y)| \leq r^{-4}$. This completes the proof of the lemma.

From Lemmas 4.3 and 4.4 it is easy to see that $I(x, y)$ is uniformly integrable when $\alpha > 1$. Thus, we have estimated the kernel in 3 dimensions modulo the estimate $|S_R(\alpha)(x, y)| \leq CR^{3/2}$. The estimation in the general case is similar to this. We start with the n-dimensional Mehler kernel. Before passing to the limit in r we have to integrate by parts many times. For example, when $n = 5$ we integrate by parts twice; when $n = 6$ we integrate by parts thrice and so on. The kernel will have many terms and most of them are easily estimated. To estimate the main term we can repeat the arguments of the three-dimensional case. Let us now complete the estimation of the Riesz kernel by proving the following estimate which we have already used.

Lemma 4.5. *There is a constant $C > 0$ independent of x, y and R such that*

$$|S_R(\alpha)(x, y)| \leq CR^{n/2}.$$

Proof. Recall that the kernel $S_R(\alpha)(x, y)$ can be expressed in terms of the kernel $S_R(0) = S_R$ as given by the following formula:

$$(4.28) \qquad S_R(\alpha)(x, y) = \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha)} \int_{0 \leq t \leq 1} (1 - t)^{\alpha - 1} S_{Rt}(x, y)\, dt.$$

Therefore, any estimate we obtain for the kernel S_R will automatically hold true for the kernel $S_R(\alpha)(x, y)$ also. So we need to prove $|S_R(x, y)| \leq CR^{n/2}$. Since $S_R(x, y)$ is given by

$$(4.29) \qquad S_R(x, y) = \sum_{0 \leq k \leq R} \Phi_k(x, y)$$

where $\Phi_k(x, y) = \sum_{|\nu| = k} \Phi_\nu(x) \Phi_\nu(y)$ we can actually prove more. Using induction we will prove the following estimate:

$$(4.30) \qquad \sum_{|\nu| \leq R} |\Phi_\nu(x)| |\Phi_\nu(y)| \leq CR^{n/2}.$$

It is enough to prove this estimate for $n = 1$. The following estimates for the Hermite functions are well known (see e.g. [25]). There exists positive constants C and d independent of x and n such that for every $n \geq 0$

(4.31)
$$|\varphi_n(x)| \leq C(N^{1/3} + |x^2 - N|)^{-1/4}, \quad \text{if } x^2 \leq 2N$$
$$\leq C \exp(-dx^2), \quad \text{if } x^2 > 2N,$$

where as usual $N = 2n + 1$. Using these estimates we can easily prove the lemma.

Consider the sum $\sum_{0 \leq k \leq n} |\varphi_k(x)\varphi_k(y)|$. Since this is symmetric in x and y, we can assume that x and y are both positive and $y < x$. Suppose first that $x^2 \leq 1$. Then $y^2 \leq x^2 \leq K$ where $K = 2k + 1$ for all $k = 0, 1, \ldots, n$. Therefore,

$$(4.32) \quad \sum_{0 \leq k \leq n} |\varphi_k(x)\varphi_k(y)| \leq C \sum_{0 \leq k \leq n} (K^{1/3} + |x^2 - K|)^{-1/4}(K^{1/3} + |x^2 - K|)^{-1/4}.$$

Applying the Schwarz inequality we get

$$\sum_{0 \leq k \leq n} |\varphi_k(x)\varphi_k(y)| \leq C \left\{ \sum_{0 \leq k \leq n} (K^{1/3} + |x^2 - K|)^{-1/2} \right\}^{1/2}$$
$$\times \left\{ \sum_{0 \leq k \leq n} (K^{1/3} + |y^2 - K|)^{-1/2} \right\}^{1/2}.$$

Since $K > 1$, we have

(4.33)
$$\sum_{0 \leq k \leq n} (K^{1/3} + |x^2 - K|)^{-1/2} \leq \sum_{0 \leq k \leq n} (1 + |x^2 - K|)^{-1/2}$$
$$\leq \sum_{|x^2 - K| \leq N} (1 + |x^2 - K|)^{-1/2} \leq CN^{1/2}.$$

Therefore, $\sum_{0 \leq k \leq n} |\varphi_k(x)\varphi_k(y)| \leq CN^{1/2}$.

Getting the estimate when $1 < x^2 < 2N$ is similar. We split the sum into three parts:

$$(4.34) \quad \sum_{0 \leq k \leq n} |\varphi_k(x)\varphi_k(y)| = \sum_{2K \leq y^2} + \sum_{y^2 \leq 2K \leq x^2} + \sum_{x^2 \leq 2K}.$$

Each term can be estimated as before using estimates (4.31) without much difficulty. This completes the proof of the lemma.

Using the estimates of $S_R(\alpha)(x, y)$ we can prove all three theorems stated in the introduction. When $\alpha > (3n - 2)/6$ we can obtain the Fejér-Lebesgue and Riemann's localisation theorems. When $\alpha > (n - 1)/2$ we can show that the Riesz means are uniformly bounded on L^p. Consequently $S_R(\alpha)f$ will converge to f in the norm as R tends to infinity.

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